THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICAN

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV

1927

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 1, JANUARY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE AMERICAN MATHEMATICAL MONTHLY

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association offers a new form of service to its members by the maintenance of an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office will not handle detailed recommendations, after the manner of a teachers' agency, but will supply certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim will be to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

TENTH ANNUAL MEETING OF THE MISSOURI SECTION

The tenth annual meeting of the Missouri Section of the Mathematical Association of America was held at the Junior College of Kansas City on November 13, 1926. Chairman G. E. Wahlin called the meeting to order at 9 A. M.

The attendance was twenty-one including the following eleven members of the Association:

E. F. Allen, B. Cosby, B. F. Finkel, R. R. Fleet, W. A. Luby, Arria Murto, A. D. Pierson, P. R. Rider, G. E. Wahlin, R. A. Wells, W. D. A. Westfall.

At a brief business session Professors Dunkel and Roever and the Secretary were appointed a committee to draft resolutions on the death of Professor C. A. Waldo, who was chairman of the committee which organized the Missouri Section and first chairman of the section. The following officers were elected for 1927: Chairman, KATHRYN WYANT, University of Missouri; Vice-Chairman, W. H. ROEVER, Washington University; Secretary-Treasurer, P. R. RIDER, Washington University.

The following papers were presented:

1. "Causes of the present popular attitude toward mathematics" by Professor R. A. WELLS, Park College.

2. "Desirable courses for students intending to do graduate work in mathematics" by Professor W. D. A. WESTFALL, University of Missouri.

3. "Unified courses in mathematics" by Professor BYRON INGOLD, Culver Stockton College.

4. "Some mathematical questions in Missouri history" by Miss KATHRYN WYANT, University of Missouri.

5. "Old and new concepts of mathematics" by Professor B. F. FINKEL, Drury College.

6. "Ten years of the Missouri section" by Professor P. R. RIDER, Washington University.

Abstracts of these papers are given below, the numbers corresponding to the numbers in the list of titles.

1. A study of the attitude of college students and of the educational public reveals the existence of much indifference and considerable antagonism to the study of mathematics. One frequently hears such expressions as, "I cannot learn mathematics," and "I do not care for mathematics." Attempts are continually being made to give it a less important place in the scheme of education. Professor Wells stated that the causes of this condition seem to be:

(a) Ignorance of the real nature of the subject matter of mathematics, caused usually by the ignorance or carelessness of teachers of elementary mathematics,

(b) The persistent agitation carried on by persons, willing to be known as educational experts, who have the public ear and who are continually trying to convince the public that there is no use in studying mathematics beyond the simple processes of arithmetic,

(c) Carelessness on the part of writers of text books on elementary mathematics shown in the use of inaccurate English and in the lack of regard for the technical use of mathematical terms.

The remedy seems to be for those who are interested in the subject of mathematics and who know what it really means to seize every opportunity to go before the educational public and give them some real information as to what mathematics is and why it should have a place in the scheme of education. Mathematics does not deserve a place in the curriculum because of its disciplinary value, or because of its practical application, or because of its training in the use of formal logic or in symbolic thinking, although these are all important reasons for studying it. But it does deserve a place in the scheme of education because of what it is. Mathematics is an important science with a perfectly definite subject matter and that subject matter is of vital interest to every human being. That is the reason why mathematics should have a place in the course of study. Teachers of mathematics should keep this idea before

the public and they should do it as persistently and insistently as the other group have worked in trying to discredit the study of mathematics.

2. The changing conditions in graduate schools that make greater homogeneity in the preparation of first year graduate students were considered by Professor Westfall. A tentative outline for a uniform undergraduate mathematical curriculum was proposed.

3. Professor Ingold's discussion involved three steps; first, setting forth arguments favorable to the giving of unified courses in undergraduate work; second, conditions unfavorable to such procedure; and third, an attempt at balancing these arguments with the final conclusion that courses be given separately under their respective headings. In the opinion of the author much depends upon the familiarity of the instructor with the correlated sciences and his ability to answer the eternal "why and wherefore" of the student concerning mathematical formulas of the sciences.

4. The first country newspaper to be published west of the Mississippi River was the *Missouri Intelligencer*. The paper was a weekly printed in Old Franklin, a little town on the Missouri River. Since a stage made a round trip between St. Louis and Franklin each week, bringing passengers and mail, the paper was able to keep the people of this newly settled territory in touch with the Eastern States.

It was in September, 1823, that one who signed himself "The Missouri Mathematician" sent several riddles and a problem in Diophantine Analysis to the Intelligencer. The problem was answered the next March by "A Subscriber from Howard County," who says he had "formerly, nearly half a century ago, taken great delight in studying mathematics." His solution, however, was not a general one but consisted of a set of numbers, each of which was greater than 10,000,000. The numbers check in the problem as "The Subscriber from Howard County" interpreted it, but it seems as if he had the wrong meaning of the problem. At the close of his discussion of this problem, "The Subscriber from Howard County" proposed two other questions, the first about the weight of an infinite cylinder the second another Diophantine problem.

Two months later "The Missouri Mathematican" answered these, but "for want of proper engraving" his solution to the Diophantine problem was never published. His answer to the question concerning the infinite cylinder was a harsh criticism of the proposer, stating that "an attempt to solve this question would appear like insanity." Two more questions were then proposed by "The Missouri Mathematician," one concerning the dimensions of a triangular field under certain conditions, the other a Diophantine problem concerning the dimensions of a certain trapezium.

In her paper, Miss Wyant gave a short history of Diophantine Analysis, a sketch of the local conditions in Missouri in 1823, and solutions to some of the problems proposed by the two men who, though interested in mathematics, seem to have left no trace of their identity other than "The Missouri Mathematician" and "The Subscriber from Howard County."

5. Professor Finkel's paper brought out a comparison of the old and new conceptions of such terms as number, mathematics, infinity, infinitesimal, derivative, etc. The paper dwelt at some length on the various conceptions of the postulates of Euclid, Saccheri, Lobatschewsky, Riemann, and the deductions drawn from them.

6. The Secretary sketched briefly the history of the organization of the Missouri Section and summarized the activities of the section during the ten years since its founding on November 18, 1916.

PAUL R. RIDER, *Secretary-Treasurer.*

DIOPHANTINE PROBLEMS IN WEIGHING

BY H. A. SIMMONS, Northwestern University

1. **Introduction.** The purpose of this paper is to solve certain systems of linear Diophantine equations by use of a lever and a system of n positive integral weights. A more precise statement of our problem will be given after we have made a few definitions and exhibited our method of procedure.

Definition 1. The symbol L_{jk}, j and k being any two positive integers, will stand for a lever whose left arm is j units in length and whose right arm is k units, there being places for hanging weights at distances of $1, 2, 3, \cdots, k$ units from the fulcrum on the right arm, and at distances of $1, 2, 3, \cdots, j$ units on the left, as shown in Fig. 1.

$$\underline{-j \quad \cdot \quad \cdot \quad \cdot \quad -3 \quad -2 \quad -1 \qquad 0 \qquad 1 \qquad 2 \qquad 3 \quad \cdot \quad \cdot \quad \cdot \quad k}$$
$$\Lambda$$

FIG. 1

Definition 2. The symbol σ_n will stand for a system of n positive integral weights $w_1, w_2, w_3, \cdots, w_n$ where $w_1 \leqq w_2 \leqq w_3 \leqq \cdots \leqq w_n$; s_n will stand for the sum of these n weights.

Definition 3. Restricting the application of the word "object" throughout the paper to a thing whose weight in pounds can be expressed by a positive integer, we shall say that an object is "weighed" on L_{jk} if after it is placed at a division point $-p$ on the left arm of L_{jk}, where $1 \leqq p \leqq j$, the weights of σ_n are so placed at one or more division points q of L_{jk}, where $-j \leqq q \leqq k$, as to bring the lever into equilibrium. The symbols $1_0, 2_0, 3_0, \cdots$, wherever

they are used, will stand for objects of weights 1 lb., 2 lbs., 3 lbs., $\cdots$, which are to be weighed; the accented positive integers $1', 2', 3', \cdots$, wherever they are used, will stand for weights of 1 lb., 2 lbs., 3 lbs., $\cdots$, which belong to σ_n. These subscripts and accents will be omitted when the meaning intended is clear without them.

Definition 4. A system σ_n will be said to have a *consecutive weighing capacity* with respect to L_{jk} if it suffices to weigh on L_{jk} every object from 1_0 to $(ks_n)_0$ inclusive; otherwise σ_n will be called a non-consecutive system.

Example. If $j=k=1$ and $\sigma_n=\sigma_2$, the weights $w_1=1'$, $w_2=3'$ constitute a consecutive system σ_2 which suffices to weigh on L_{11} objects of all weights from 1_0 to $[k(w_1+w_2)]_0=4_0$ inclusive, but the weights $w_1=1'$, $w_2=4'$ constitute a non-consecutive system σ_2 which does not suffice to weigh 2_0.

Definition 5. Of two consecutive systems of n weights each, the system whose weights have the larger sum will be said to have the *greater consecutive capacity*.

With the above definitions holding, we proceed to find a system σ_n which the lever suggests as a possible consecutive system of large capacity; later we shall discuss completely the nature of the capacity of this system σ_n.

Since $w_1=1'$ is the only single weight which suffices to weigh 1_0 on L_{11} and since $w_1=1'$ suffices to weigh 1_0 on L_{jk}, whatever be the positive integral values of j, k, we select $w_1=1'$. When w_1 is placed at the points $1, 2, \cdots, k$ of L_{jk}, it suffices to weigh at -1 the objects $1_0, 2_0, \cdots, k_0$, respectively. The next object to be weighed is $(k+1)_0$. We place[1] it at -1, and we place $w_1=1'$ at $-j$. The moment thus created on the left arm of L_{jk} is $j'+(k+1)_0=j+k+1$. Hence we select $w_2=(k+j+1)'$. Now with w_2 at 1, we weigh w_0. We further observe that if w_2 is placed at p where p is a positive integer $\leqq k$, w_1 can be used at the division points of L_{jk} to weigh at -1 objects of all weights from $pw_2-j=(p-1)w_2+k+1$ to pw_2+k inclusive. Therefore, by using all values of p from 1 to k, we weigh at -1 all objects from $1\cdot w_2-j=(k+1)_0$ to $kw_2+k=k(w_2+w_1)$. Adding these objects to those which w_1 alone suffices to weigh at -1, we form a consecutive set of objects running from 1_0 to $[k(w_1+w_2)]_0=[k[(j+k+1)^2-1]/(j+k)]_0$ inclusive. Furthermore, if we select $w_3=[(j+k+1)^2]'$ and carry one step further the method of procedure which we have just used, we find that w_1, w_2, w_3 suffice to weigh at -1 all objects from 1_0 to $(ks_3)_0=[k[(j+k+1)^3-1]/(j+k)]_0$ inclusive. These facts lead us to ask two questions:

---

[1] That one cannot place $(k+1)_0$ at $-p$, $p>1$, and $w_1=1'$ at $-j$, and then select $w_2=j'+p(k+1)_0=j+pk+p$, for every pair of positive integers j, k is due to the fact, found in Theorem 4 of this paper, that if $k>1$ the weights $w_1=1'$ and $w_2=j+pk+p$ constitute a non-consecutive system σ_2.

(*A*) *If we select the system* σ_n *whose weights are*

$$w_1 = 1', \quad w_2 = (j + k + 1)', \quad w_3 = [(j + k + 1)^2]', \cdots, \quad w_n = [(j + k + 1)^{n-1}]', \tag{1}$$

with sum

$$s_n = \left[\frac{(j + k + 1)^n - 1}{j + k}\right]', \tag{2}$$

can we weigh on L_{jk} *all integral weights from* 1_0 *to* $(ks_n)_0$ *inclusive?*

(*B*) *If the system* σ_n *of* (1) *is consecutive with respect to* L_{jk}, *does it have a greater weight sum* s_n, *given by* (2), *than has any other system of n weights which is consecutive with respect to* L_{jk}?

Through questions (A) and (B) we approach our main problem, which may be stated as follows. *Among all systems* σ_n *whose weighing capacities with respect to* L_{jk} *are consecutive* (*see Definition* 4), *to find that one whose weight sum is the greatest.*

Stated in Diophantine form, the problem is: *to find the set of n positive integers* $w_1, w_2, w_3, \cdots, w_n$ *of largest sum* s_n *for which there exists a consistent system of linear equations in* $w_1, w_2, w_3, \cdots, w_n$ *of the form*

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \qquad (i = 1, 2, \cdots, ks_n), \tag{3}$$

where the a's and b's are integers satisfying the conditions

$$-j \leqq a_{ip} \leqq k, \qquad (p = 1, 2, \cdots, n), \tag{4}$$

$$-j \leqq b_i \leqq -1, \tag{5}$$

j and k being positive integers.

Example. If $n=3$ and $j=k=1$, the largest s_3 is 13 and σ_3 consists of the weights $w_1=1'$, $w_2=3'$, $w_3=9'$. The 13 equations in (3) may then be written schematically as follows:

$$\begin{array}{ll}
1 \cdot 1 + 0 \cdot 3 + 0 \cdot 9 - 1 = 0, & -1 \cdot 1 + 1 \cdot 3 + 0 \cdot 9 - 2 = 0, \\
1 \cdot 1 + 1 \cdot 3 + 0 \cdot 9 - 4 = 0, & -1 \cdot 1 - 1 \cdot 3 + 1 \cdot 9 - 5 = 0, \\
1 \cdot 1 - 1 \cdot 3 + 1 \cdot 9 - 7 = 0, & -1 \cdot 1 + 0 \cdot 3 + 1 \cdot 9 - 8 = 0, \\
1 \cdot 1 + 0 \cdot 3 + 1 \cdot 9 - 10 = 0, & -1 \cdot 1 + 1 \cdot 3 + 1 \cdot 9 - 11 = 0, \\
1 \cdot 1 + 1 \cdot 3 + 1 \cdot 9 - 13 = 0, &
\end{array}$$

$$\begin{array}{l}
0 \cdot 1 + 1 \cdot 3 + 0 \cdot 9 - 3 = 0, \\
0 \cdot 1 - 1 \cdot 3 + 1 \cdot 9 - 6 = 0, \\
0 \cdot 1 + 0 \cdot 3 + 1 \cdot 9 - 9 = 0, \\
0 \cdot 1 + 1 \cdot 3 + 1 \cdot 9 - 12 = 0.
\end{array}$$

Insofar as is known to the author, this is the first time the above problem has been considered, though simple special cases of it have received some attention. We solve the above problem and certain modifications of it completely in this paper. In § 2, we give an inductive answer to question (A). In § 3, we develop a set of lemmas which are used in sections 4 to 8 inclusive to answer question (B). In § 9, we solve the problem which results when condition (4) above is replaced by

$$0 \leqq a_{ip} \leqq k, \qquad (p = 1,2, \cdots, n), \quad (6)$$

all other features of the above problem being unaltered. In § 10, we find the maximum consecutive capacity of a system of positive integral weights of which there are n types of weights $w_1, w_2, \cdots, w_n$ and m_r weights of type w_r $(r=1, 2, \cdots, n)$, with $m_r>1$ for at least one value of r, each weight being applicable as specified in condition (4). In § 11, we modify this problem merely by using condition (6) instead of (4) and we solve the resulting problem completely. For every problem which we treat, we obtain a unique solution, and do not avoid any case that arises, special or not. In § 12, we state the most general Diophantine problem that is solved in this paper, and then propose three problems to the reader.

Throughout the paper, we use the method of attack which is suggested by results of experiments with systems of weights σ_n and levers L_{jk}, though we admit that the Diophantine form of the problem is also interesting. The proofs of our more difficult theorems are made by use of *mathematical induction* together with a *set of lemmas*. In the lemmas certain inequalities develop which play a rôle in the proofs here somewhat like the rôle played by the inequalities used by Professor D. R. Curtiss[1] in solving the Kellogg Problem (*1922*, 380–387).

2. **Answer to** (A). We know from § 1 that the answer to this question is in the affirmative for $n=1, 2, 3$. We wish to prove that this is true for all positive integral values of n. Suppose it is true for $n=m$, a positive integer. Then from (1) the weights are

$$w_1 = 1', \ w_2 = (j + k + 1)', \cdots, \quad w_m = [(j + k + 1)^{m-1}]', \qquad (7)$$

which suffice to weigh on L_{jk} objects of all integral weights from 1_0 to

$$(ks_m)_0 = \left[k \cdot \frac{(j + k + 1)^m - 1}{j + k} \right]_0 \qquad (8)$$

inclusive. The next object to be weighed is $w_0 = (ks_m+1)_0$, and the next weight to be added to the set (7) is $w_{m+1} = [(j+k+1)^m]' = [(j+k)s_m+1]'$,

[1] The author is pleased to acknowledge his indebtedness to Professor Curtiss for his numerous helpful suggestions in regard both to the method of attack and to the form of this paper.

by (8). We observe now that if w_{m+1} is placed at p, where p is a positive integer $\leqq k$, the weights of (7) can be so placed at division points of L_{jk} as to weigh at -1 objects of all integral weights from $pw_{m+1}-js_m=(p-1)w_{m+1}+ks_m+1$ to $pw_{m+1}+ks_m$ inclusive. Therefore, by using all values of p from 1 to k, we weigh at -1 all integral weights from $1\cdot w_{m+1}-js_m=ks_m+1=w_0$ to $kw_{m+1}+ks_m=ks_{m+1}$. Adding these weights to those which the set (7) alone suffices to weigh at -1, we form a consecutive set of weights running from 1_0 to

$$(ks_{m+1})_0=\left[k\cdot\frac{(j+k+1)^{m+1}-1}{j+k}\right]_0 \tag{9}$$

inclusive. But (9) is the result which we obtain if in (8) we replace m by $m+1$. The induction is therefore complete, and the answer to (A) is in the affirmative for every positive integral value of n.

3. **Lemmas used in discussing (B).** *Definition* 6. The symbol Σ_n will stand for the system of weights which our problem, § 1, requires and it will be called the *maximum consecutive* system with respect to L_{jk}; S_n will stand for the sum of the weights of Σ_n.

We state and prove four lemmas below which will enable us to solve the problem of § 1. The reader should note that each of these lemmas expresses a property of Σ_n, though the first two of them suppose the *consecutive*, but not the *maximum consecutive,* property of Σ_n.

LEMMA I. *A consecutive system σ_n, $n>1$, contains*[1] *a $1'$ weight, $w_1=1'$.*

With $n>1$, suppose σ_n does not contain a $1'$ weight. Then $w_1\geqq 2'$ and w_2, which exists since $n>1$, must be at least as heavy as w_1. Therefore $w_2\geqq 2'$, and $s_n\geqq 4'$. We ask now how an object $W\equiv(ks_n-1)_0$, which is $\geqq 3_0$, can be weighed. We shall show that

(*a*) W must be weighted at -1;

(*b*) W cannot be weighed at -1 when $w_1\geqq 2'$, $n>1$.

To prove (*a*), let us suppose that W can be weighed at $-p$, where p is a positive integer $\geqq 2$. Placing W at $-p$, we create on the left arm of L_{jk} a moment of $pW=p(ks_n-1)$. Now the maximum moment that σ_n suffices to create on the right arm of L_{jk} is $ks_n=W+1$. But $pW>W+1$ since $W\geqq 3$ and $p\geqq 2$. Hence W cannot be weighed at $-p$; W must therefore be weighed at -1, which was to be proved.

---

[1] At present the reader may convince himself that if $n=1$, the system σ_1 having $w_1=1'$ has not the maximum consecutive capacity for all pairs of values of j, k by observing that if $j\geqq 2$ and $k=1$, a $2'$ weight has a greater consecutive capacity than has a $1'$ weight. A complete treatment of the case $n=1$ will be given in §5.

To prove (b), we observe that W cannot be weighed at -1 if any one of the weights of σ_n is not at k; for, if any weight of σ_n is placed on L_{jk} at p, where p is an integer $<k$, the moment created on the right arm of L_{jk} will not exceed ks_n-2 since $w_1 \geqq 2'$. But if all weights of σ_n are placed at k, the moment ks_n $(>ks_n-1)$ will be created on the right arm of L_{jk}. Hence (b) is true. The proof of Lemma I is now complete.

LEMMA II. *Let σ_n be a consecutive system; then every object of weight w which satisfies the relation $(ks_n/2)<w\leqq ks_n$ must be weighed at -1 by σ_n.*

For if an object of weight w were placed at -2, or at $-p$ where $p>2$, it would create on the left arm of L_{jk} a moment greater than ks_n, which is the maximum moment that can be formed on the right arm of L_{jk} with the weights of σ_n.

LEMMA III.

$$S_n \geqq \frac{(j+k+1)^n-1}{j+k}\,.$$

For Σ_n (Definition 6) must have at least as great a consecutive capacity with respect to L_{jk} as has the set (1) and therefore S_n must be $\geqq$ the sum (2) of the weights (1).

LEMMA IV. *Let $r=j+k+1$ and let m be an integer >1; then the following conditions on the weights of Σ_n are necessary (but not sufficient).*

$$w_{i+1} \leqq (r-1)(w_1+w_2+\cdots+w_i)+1 \text{ if } n>i+1,$$
$$(i=1,2,\cdots,\ m-1;\ m<n)\,; \qquad (10_i)$$

$$w_{p+1} \leqq (r-1)(w_1+w_2+\cdots+w_p)+1 \text{ if } n>p,$$
$$(p=1,2,\cdots,m-1;\ m<n+1)\,. \qquad (11_p)$$

We first prove (10_1).[1] Assume that (10_1) is not true. Then, since $w_1=1'$ (Lemma 1), $w_2>(r-1)+1$ if $n>2$. We shall prove that this assumption leads to a contradiction by showing that

(a) $W_1\equiv kS_n-(j+k)w_1-1$ must be weighed at -1;

(b) W_1 cannot be weighed at -1 if $w_2>(r-1)w_1+1=j+k+1$ and $n>2$.

To prove (a), we need only to show that with $n>2$

$$W_1=kS_n-j-k-1>kS_n/2\ ;\ i.e.,\ kS_n/2>j+k+1, \qquad (12)$$

so that W_1 must be weighed at -1 (Lemma 2). But since $n>2$, therefore $\geqq 3$, the last form of (12) follows directly from the two relations

$$S_n \geqq \frac{(j+k+1)^n-1}{j+k}\,;\ k\cdot\frac{(j+k+1)^n-1}{2(j+k)}>j+k+1\,;$$

[1] The notation (10_1) is used to indicate what formula (10_i) becomes when i is replaced by 1.

the former relation being true by Lemma III, and the latter holding for every pair of positive integers j, k. The relation (12) is therefore true; hence (a) is true for (10_1).

We observe now that if we place upon n in (10_1) only the condition $n>1$, which is the hypothesis on n in (11_1), the modified relation (10_1) is not true for all pairs of positive integers j, k and all choices of n. For example, it is not true for the case $L_{jk}=L_{21}$, $n=2$, where $w_1=1'$, $w_2=6'$. But if we use both hypotheses of (11_1), that $n>1$ and $k>1$, we can prove (a) for (11_1) as follows. Certainly

$$W_1 = kS_n - j - k - 1 > kS_n/2 \text{ if } kS_n/2 > j + k + 1, \quad (n > 1, \quad k > 1). \tag{13}$$

But the last inequality of (13) follows from the two relations

$$S_n \geqq \frac{(j+k+1)^2-1}{j+k}; \quad k\,\frac{(j+k+1)^2-1}{2(j+k)} > j+k+1;$$

the former relation being true by Lemma III, and the latter holding when, and only when, the hypothesis $k>1$ is satisfied. Therefore the inequalities (13) hold, and, by Lemma II, (a) is true for (11_1).

To prove (b) for (10_1), we observe that W_1 cannot be weighed at -1 without the use of w_2 and all weights of Σ_n which are $\geqq w_2$ at k, since $w_2>(r-1)w_1+1$ when $n>2$ by our hypothesis, which we are to prove false. But with all of these weights at k, the lightest weight that Σ_n suffices to weigh at -1 on L_{jk} is $kS_n-(j+k)w_1=kS_n-r+1$, which is heavier than W_1. We have thus arrived at a contradiction. This proves (b) for (10_1), and thus completes the proof of (10_1).

The proof of (b) for (11_1) is essentially the same as that for (10_1), the two proofs differing only in that $n>2$ in (10_1) while n is merely >1 in (11_1). Hence (b) for (11_1) is true. This completes the proof of (11_1).

Adding (10_1), (11_1) to our hypotheses on Σ_n, we can prove (10_2), (11_2), respectively; then, with (10_1), (10_2) and (11_1), (11_2) holding, we can prove (10_3) and (11_3), respectively; etc. We shall prove these statements by assuming that (10_1), (10_2), $\cdots$, (10_{m-2}) hold and that (11_1), (11_2), $\cdots$, (11_{m-2}) hold and then showing that on these assumptions (10_{m-1}) and (11_{m-1}), respectively, hold.

We first prove (10_{m-1}). Assume that (10_1), (10_2), $\cdots$, (10_{m-2}) are true, but that (10_{m-1}) is not true. Then, since $w_1=1'$ (Lemma I), $w_m>(r-1)(1+w_2+\cdots\cdot w_{m-1})+1$ when $n>m$. We shall prove that this assumption leads to a contradiction by showing that

(c) $W_{m-1}\equiv kS_n-(r-1)(w_1+w_2+\cdots+w_{m-1})-1$ must be weighed at -1;

(d) W_{m-1} cannot be weighed at -1 if $w_m>(r-1)(1+w_2+\cdots+w_{m-1})+1$ and $1<m<n$.

To prove (c), we need only to show that when $1<m<n$,

$$W_{m-1} > kS_n/2\ ;\quad i.e.,\quad kS/2_n > (r-1)(1+w_2+\cdots+w_{m-1})+1, \tag{14}$$

so that W_{m-1} must be weighed at -1 (Lemma II). But the last inequality of (14) follows directly from two relations: *first*, with $n \geqq m+1$,

$$S_n \geqq \frac{(j+k+1)^{m+1}-1}{j+k},$$

(Lemma III); *second*, as we shall presently prove,

$$k \cdot \frac{(j+k+1)^{m+1}-1}{2(j+k)} > (r-1)(1+w_2+\cdots+w_{m-1})+1. \tag{15}$$

To prove (15), we first make use of our assumption that (10_1), (10_2), $\cdots$, (10_{m-2}) hold and the fact that $w_1 = 1'$, and obtain the following relations:

$$\begin{aligned} &w_1 = 1\ ; \\ &w_2 \leqq (r-1)w_1 + 1, \text{ so that } w_2 \leqq r\ ; \\ &w_3 \leqq (r-1)(w_1+w_2)+1 \leqq (r-1)(r+1)+1, \text{ so that } w_2 \leqq r^2\ ; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &w_{m-1} \leqq (r-1)(1+r+r^2+\cdots+r^{m-3})+1, \text{ so that } w_{m-1} \leqq r^{m-2}. \end{aligned} \tag{16}$$

Then from (16) we have

$$S_{m-1} \leqq \frac{r^{m-1}-1}{r-1} = \frac{(j+k+1)^{m-1}-1}{j+k}\cdot \tag{17}$$

We now prove (15) by showing that it holds when every weight of the set $w_1, w_2, \cdots, w_{m-1}$ has its maximum value (see (16)), *i. e.*, when their sum (17) is $[(j+k+1)^{m-1}-1]/(j+k)$. Replacing $w_1+w_2+\cdots+w_{m-1}$ in (15) by this sum, we obtain

$$k\frac{(j+k+1)^{m+1}-1}{2(j+k)} > (j+k+1)^{m-1},$$

which may be written in the form $(j+k+1)^{m-1}[k(j+k)^2+2(k-1)(j+k)+k] > k$. Since this last inequality is true for every pair of positive integers j, k the relation (15) holds, and therefore (14) is true; hence (c) is true for (10_{m-1}).

We observe now that if we place upon n in (10_{m-1}) the condition $n > m-1$, which is the hypothesis on n in (11_{m-1}), the modified relation (10_{m-1}) is not true for all pairs of positive integers j, k and all choices of n. For example, it is not true for the case $n=m=3$, $L_{jk}=L_{31}$, where $w_1=1'$, $w_2=5'$, $w_3=39'$.

But if we use both hypotheses of (11_{m-1}), that $n>m-1$ *and* $k>1$, we can prove (*c*) for (11_{m-1}) as follows. Certainly

$$W_{m-1} > kS_n/2 \text{ if } kS_n/2 > (r-1)(1+w_2+\cdots+w_{m-1})+1. \tag{18}$$

But the last inequality of (18) follows from two inequalities: *first*, $S_n \geqq [(j+k+1)^m-1]/(j+k)$ when $n>m-1$ and $k>1$, which is true by Lemma III; *second*, as we presently prove

$$k\cdot\frac{(j+k+1)^m-1}{2(j+k)} > (r-1)(1+w_2+\cdots+w_{m-1})+1.$$

To prove the last inequality, we replace $1+w_2+\cdots+w_{m-1}$ in its right member by the maximum value of this sum (see (17)), and simplify the result. We thus obtain the inequality $(j+k+1)^{m-1}[k(j+k+1)-2(j+k)]>k$, which is true since $k>1$. Therefore (18) holds; hence (*c*) is true for (11_{m-1}).

To prove (*d*) for (10_{m-1}), we observe that $W_{m-1}=kS_n-(j+k)(1+w_2+\cdots+w_{m-1})-1$ cannot be weighed at -1 without the use of w_m and all weights of Σ_n which are $\geqq w_m$ at k, since $w_m>(j+k)(1+w_2+\cdots+w_{m-1})+1$ when $n>m$ by our hypothesis, which we are to prove false. But with all of these weights at k, the lightest object that Σ_n suffices to weigh at -1 on L_{jk} is $kS_n-(j+k)(1+w_2+\cdots+w_{m-1})$, which is heavier than W_{m-1}. We have thus arrived at a contradiction. This proves (*d*) for (10_{m-1}) and completes the proof of (10_{m-1}).

The proof of (*d*) for (11_{m-1}) differs from that just given for (10_{m-1}) only in that $n>m$ in (10_{m-1}) while n is merely greater than $m-1$ in (11_{m-1}). Therefore (*d*) is true for (11_{m-1}); hence (11_{m-1}) is true. This completes the proof that the conditions (10_1), (10_2), $\cdots$, (10_{m-1}) and (11_1), (11_2), $\cdots$, (11_{m-1}) are necessary. That they are not sufficient is obvious. Hence Lemma IV is true.

4. **Cases to be discussed in answering (B).** The statements of Lemma I and Lemma IV above imply that the case $n=1$ requires special consideration. In Lemma IV, the relations (10_1), $\cdots$, (10_{m-1}) imply also that $n=2$ is a special case; while relations (11_1), $\cdots$, (11_{m-1}) imply that $k=1$ is a special case. We shall answer (B) completely, without any overlapping in our argument, by treating the following cases: $n=1$, in § 5; $k=1$, $n=2$, in § 6; $k=1$, $n>2$, in § 7; $k>1$, $n>1$, in § 8.

5. **Discussion of (B) in the case $n=1$. Theorem I.** That $w_1=1'$ has a consecutive capacity with respect to every lever L_{jk} was shown just after Definition 5, § 1. But the example cited in the footnote to Lemma I, § 3, namely $w_1=2'$, $j\geqq 2$, $k=1$, in which w_1 suffices to weigh 1_0 at -2 and 2_0 at -1 shows that a $1'$ weight does not have the maximum consecutive capacity for one weight with respect to every lever L_{jk}. This example leads us to seek

all pairs of special values of j, k for which a single weight $w_1 > 1'$ has a consecutive capacity. Having investigated the cases $w_1 = 1'$ and $w_1 = 2'$, we need only to consider pairs of positive integers j, k for which $w_1 > 2'$ has a consecutive capacity. Such a w_1 must suffice to weigh on L_{jk} all integral weights from 1_0 to $(kw_1)_0$ inclusive. In particular, then, w_1 must suffice to weigh $(kw_1 - 1)_0$. Since $w_1 \geqq 3'$, $kw_1 - 1 > kw_1/2$. Hence, by Lemma II, § 3, $(kw_1 - 1)_0$ must be weighed on L_{jk} at -1. Now the only weights which a consecutive system $\sigma_1 = w_1$ suffices to weigh at -1 are $w_1, 2w_1, 3w_1, \cdots, (k-1)w_1, kw_1$. But $kw_1 - 1$ does not appear in this set of weights, since the largest weight here which is less than kw_1 is $(k-1)w_1$, while $(k-1)w_1 < kw_1 - 1$ by the hypothesis that $k \geqq 1$ and $w_1 \geqq 3'$. Therefore $(kw_1 - 1)_0$ can not be weighed at -1. Hence there does not exist a $w_1 > 2'$ which has a consecutive capacity with respect to L_{jk}. We have therefore proved the following theorem.

THEOREM I: *The only case in which $w_1 > 1'$ has a consecutive capacity with respect to L_{jk} is when $k = 1, j \geqq 2$. Then $w_1 = 2'$ has not merely a consecutive capacity, but the maximum consecutive capacity for one weight on L_{jk}. In all other cases, $w_1 = 1'$ constitutes the only consecutive system σ_1 and therefore the maximum consecutive system Σ_1.*

6. **Discussion of (B) when $k=1$ and $n=2$. Theorem II.** By trial we find that if $j = k = 1$, the weights of the system Σ_2 are $w_1 = 1'$, $w_2 = 3'$; if $j = 2$, $k = 1$, the weights of Σ_2 are $w_1 = 1'$, $w_2 = 6'$. But when $j = 3, 4, 5, 6$, the respective systems Σ_2 are $(1', 9')$, $(1', 11')$, $(1', 13')$, $(1', 15')$. These facts lead us to expect that if $k = 1$, $n = 2$, and $j > 2$, then the weights of the system Σ_2 are $w_1 = 1'$ and $w_2 = (2j+3)'$. That $w_2 \leqq (2j+3)'$ follows from the fact that if $w_2 \geqq (2j+4)'$, the object $(w_2 - j - 1)_0$ cannot be weighed by w_1, w_2. Now if $w_2 = (2j+3)'$, can we weigh on L_{jk}, $j > 2$, objects of all weights from 1_0 to $(2j+4)_0$ inclusive? or since $w_1 = 1'$ and since the weights $w_2 - j$, $w_2 - j + 1$, $w_2 - j + 2, \cdots, w_2$, $w_2 + w_1 = w_2 + 1$ can be weighed at -1, can the weights from 2_0 to $(j+2)_0$ inclusive be weighed on L_{j1}? We answer this question as follows. The weights which $w_1 = 1$ and $w_2 = 2j + 3$ suffice to weigh on L_{j1} at $-2, -3, \cdots, -j$ are included in the following $(j-1)$ sets of weights, each set being weighed[1] at the point indicated.

$$\frac{j+3}{i}, \quad \frac{j+4}{i}, \quad \frac{j+5}{i}, \cdots, \frac{2j}{i}, \cdots, \frac{2j+4}{i} \text{ at } -i, \qquad (i = 1, 2, \cdots, j). \qquad (19i)$$

That these sets contain all integral weights from 2 in (19_j) to $j+2$ in (19_2) follows from two facts. *First*, (19_m), where m is an integer such that $2 \leqq m \leqq j-1$, con-

[1] Of course we are not interested in the application of the term "weighed" to the non-integral weights of the $(j-1)$ sets.

tains every integral weight u satisfying the relation $(j+3)/m \leqq u \leqq (2j+4)/m$. *Second,* since $(2j+4)/(m+1) > (j+3)/m$ under our hypothesis that $j \geqq 3$, with $2 \leqq m \leqq j-1$, there exists no integral weight which is heavier than $(2j+4)/(m+1)$ and lighter than $(j+3)/m$. This proves that we may choose the maximum value, $2j+3$, for w_2. We therefore have the following theorem.

THEOREM II. *When $k=1$ and $n=2$, the system Σ_2 is obtained by selecting the weights w_1, w_2, as follows: $w_1=1$, $w_2=3$ if $j=1$; $w_1=1$, $w_2=6$ if $j=2$; $w_1=1$, $w_2=2j+3$ if $j>2$.*

7. **Discussion of (B) when $k=1$ and $n>2$. Theorem III.** Putting $m=n-1$ in the relations (10_i), and determining w_1 by Lemma I, we obtain

$$w_1 = 1, \quad w_2 \leqq r, \quad w_3 \leqq r^2, \cdots, \quad w_{n-1} \leqq r^{n-2}. \tag{20}$$

In (20), we have upper bounds for $w_2, w_3, \cdots, w_{n-1}$. Suppose all of the equality signs of (20) held. Then the $(n-1)$ weights $w_1, w_2, \cdots, w_{n-1}$ would be

$$w_1 = 1, \quad w_2 = r, \quad w_3 = r^2, \cdots, \quad w_{n-1} = r^{n-2}; \tag{21}$$

they would have the sum $S_{n-1} = [(j+2)^{n-1}-1]/(j+1)$, as $k=1$; and they would, by § 2, suffice to weigh all objects from 1_0 to $(kS_{n-1})_0 = (S_{n-1})_0$ inclusive. The next object to be weighed would then be $(S_{n-1}+1)_0$. Suppose this object and the $(n-1)$ weights (21) were placed at $-j$. They would create on the left arm of L_{j1} a moment of $M = j(2S_{n-1}+1)$. That M is an upper bound for w_n follows from two facts: (i) if w_n were taken $>M$, w_n could not be used with the weights (21) to weigh $(S_{n-1}+1)_0$, and hence w_n would form with the weights (21) a non-consecutive system; (ii) if in (20) not all of the equality signs held, the sum of the $(n-1)$ weights $w_1, w_2, \cdots, w_{n-1}$ would be a positive integral weight S where $S<S_{n-1}$. Then, as is easily seen by the type of argument which was used in (i), the weight w_n could not exceed $j(2S+1)$, and would therefore be $<M$.

We wish now to show that the system of weights

$$w_1 = 1, \quad w_2 = r, \quad w_3 = r^2, \cdots, \quad w_{n-1} = r^{n-2}, \quad w_n = j(2S_{n-1}+1), \qquad (r = j+2), \tag{22}$$

obtained by adding w_n to the system (21), has a consecutive capacity with respect to L_{j1}. The objects which can be weighed at -1 by the system (22) are those that can be weighed by the weights (21) alone and those which the system (22) suffices to weigh when w_n is placed at 1; the two sets of objects being all those

$$\begin{aligned} &\text{from } 1_0 \text{ to } (S_{n-1})_0 \text{ inclusive,} \\ &\text{and from } [j(S_{n-1}+1)]_0 \text{ to } [(2j+1)S_{n-1}+j]_0 \text{ inclusive,} \end{aligned} \tag{23}$$

respectively. The remaining objects which the system (22) suffices to weigh on L_{j1} are included in the following $(j-1)$ sets of weights, each set being weighed at the point indicated.

$$\frac{j(S_{n-1}+1)}{i},\ \frac{j(S_{n-1}+1)+1}{i},\ \cdots,\ \frac{2j(S_{n-1}+1)-2}{i},\ \cdots,$$
$$\frac{(2j+1)S_{n-1}+j}{i} \text{ at } -i, \qquad (i=2,3,\cdots,j). \qquad (24i)$$

In order to show that the integral weights in $(24_2), \cdots, (24_j)$ together with those in (23) form a consecutive set running from 1 to $(2j+1)S_{n-1}+j$ inclusive, we must show that the sets $(24_2), \cdots, (24_j)$ contain all integral weights from $S_{n-1}+1$, which occurs in (24_j), to $j(S_{n-1}+1)-1$, which occurs in (24_2). That this is true can be proved by the method which was used in making a similar proof just after equations (19_i). Hence system (22) is a consecutive system of $n>2$ weights, and indeed the maximum consecutive one for the case $k=1$, $n>2$ since in (22) each w_p $(p=1, 2, \cdots, n)$ has attained its upper bound. We have therefore the following theorem.

THEOREM III. *When $k=1$ and $n>2$, the system Σ_n is obtained by selecting the weights* (22): $w_1=1$, $w_2=r$, $\cdots$, $w_{n-1}=r^{n-2}$, $w_n=j(2S_{n-1}+1)$, $(r=j+2)$, *where S_{n-1} is the sum of the first $(n-1)$ weights.*

8. **Discussion of (B) when $k>1$ and $n>1$. Theorem IV.** Putting $m=n$ in the relations (11_p), and determining w_1 for the case $n>1$ by Lemma I, § 3, we obtain

$$w_1=1,\ w_2 \leqq r,\ \cdots,\ \ w_{n-1}\leqq r^{n-2},\ w_n \leqq r^{n-1},\ (r=j+k+1,\ k>1). \quad (25)$$

Now from (25) and Lemma III, we have the respective inequalities

$$S_n \leqq \frac{(j+k+1)^n-1}{j+k},\ k>1\ ;\ S_n \geqq \frac{(j+k+1)^n-1}{j+k},\ k\geqq 1.$$

As $k>1$ in this discussion, these two inequalities must hold simultaneously; hence $S_n=[(j+k+1)^n-1]/(j+k)$, the value given for s_n in equation (2). This equality of course implies that the inequality signs in (25) must be discarded. Then the weights of Σ_n become identical with the set (1), and give an affirmative answer to (B). We have therefore proved the following theorem.

THEOREM IV. *When $k>1$ and $n>1$, Σ_n is obtained by selecting the weights* (1), *which suffice to weigh at* -1 *every positive integral weight which they suffice to weigh on L_{jk}.*

9. **Weights of system applicable only on the right arm of L_{jk}. Theorem V.** With this restriction, expressed above in condition (6), on the use of weights, let us seek the system Σ_n.

Reasoning here as we reasoned just before question (A), § 1, we find that the system σ_n defined by the weights

$$w_1 = 1, \ w_2 = k + 1, \cdots, \quad w_n = (k + 1)^{n-1} \tag{26}$$

suffices to weigh on L_{jk} all objects from 1_0 to $[(k+1)^n - 1]_0$ inclusive. The question here which corresponds to (B) of § 1 is: Does the system (26) have the maximum consecutive capacity with respect to L_{jk}? To answer this question, we proceed as we did in §§ 3–8 inclusive.

To make the discussion here which corresponds to that of § 3, we first observe that Lemma I and Lemma II of § 3 hold here. In place of Lemma III, we have

LEMMA IIIa. $$S_n \geqq \frac{(k + 1)^n - 1}{k}.$$

In place of Lemma IV, we have

LEMMA IVa. *Let m be an integer >1; then the following conditions on the weights of Σ_n are necessary (but not sufficient).*

$$w_{i+1} \leqq k(w_1 + w_2 + \cdots + w_i) + 1 \text{ if } n > i + 1,$$
$$(i = 1, 2, \cdots, \quad m - 1; \ m < n); \tag{27_i}$$
$$w_{p+1} \leqq k(w_1 + w_2 + \cdots + w_p) + 1 \text{ if } n > p,$$
$$(p = 1, 2, \cdots, \quad m - 1; \ m < n + 1). \tag{28_p}$$

We omit the proofs of Lemma IIIa and Lemma IVa because they are easily made by the methods of § 3.

We shall solve our problem completely, without any overlapping in our argument, by treating separately the following cases. Case 1: $n=1$. Case 2: $k=1, j=1, n>1$. Case 3: $k=1, j>1, n>1$. Case 4: $k>1, n>1$.

Case 1. $n=1$. We obtain the result here which is given in Theorem I, since w_1 there could not be used except at the points $1, 2, \cdots, k$.

Case 2. $k=1, j=1, n>1$. By Lemma I, $w_1=1$. If $n=2$, we find by trial that $w_2=2$. If $n>2$, we put $m=n-1$ and $k=1$ in relation (27_i) and obtain

$$w_1 = 1, \ w_2 \leqq 2, \ w_3 \leqq 2^2, \cdots, \ w_{n-1} \leqq 2^{n-2}. \tag{29}$$

In (29) we have upper bounds for $w_2, \cdots, w_{n-1}$. It is necessary then to consider two possibilities: (a) when each of the $(n-2)$ weights attains its upper bound; (b) when one or more of the $(n-2)$ weights does not attain its upper bound.

(a) In this case the $(n-1)$ weights in (29) become $w_1=1, w_2=2, \cdots, w_{n-1}=2^{n-2}$, whose sum is $S_{n-1}=2^{n-1}-1$. Then Lemma IIIa implies that $w_n \geqq 2^{n-1}$. Suppose $w_n>2^{n-1}$. Then $(w_n-1)_0$ cannot be weighed; for w_n-1 is an integer satisfying the inequality $S_{n-1}<w_n-1<w_n$. This contradiction

shows that the inequality $w_n > 2^{n-1}$ does not hold. Therefore $w_n = 2^{n-1}$, and our system of weights is given by

$$w_1 = 1,\ w_2 = 2, \cdots,\ w_{n-1} = 2^{n-2},\ w_n = 2^{n-1}. \tag{30}$$

(b) Here the sum of the $(n-1)$ weights (29) is less than $2^{n-1}-1$. By Lemma IIIa, then, $w_n > 2^{n-1}$. But if this is true, $(w_n - 1)_0$ cannot be weighed. Hence no system σ_n satisfying (b) is the system Σ_n.

In case 2, therefore, Σ_n is given by (30).

Case 3. $k=1, j>1, n>1$. By considering possibilities (a) and (b) here, we can show by argument of the type just used that if $n>2$ the inequality $w_n > 2^{n-1}$ can hold in one, and only one, way, namely when $w_n = 2^{n-1}+1$. By trial we find that if $n=2$, the weights $w_1=1$, $w_2=3$ constitute Σ_2, so that this case is not an exception. We thus conclude that Σ_n is given by

$$w_1 = 1,\ w_2 = 2,\ w_3 = 2^2, \cdots,\ w_{n-1} = 2^{n-2},\ w_n = 2^{n-1} + 1. \tag{31}$$

Case 4. $k>1, n>1$. Putting $m=n$ in relation (28_p), and determining w_1 for this case by Lemma I, § 3, we obtain

$$w_1 = 1,\ w_2 \leqq k+1,\ w_3 \leqq (k+1)^2, \cdots,\ w_n \leqq ((k+1)^{n-1}. \tag{32}$$

Now from (32) and Lemma IIIa, we have the respective inequalities

$$S_n \leqq \frac{(k+1)^n - 1}{k},\ k > 1\ ;\ S_n \geqq \frac{(k+1)^n - 1}{k},\ k \geqq 1.$$

As $k>1$ in this discussion, these two inequalities must hold simultaneously; hence $S_n = [(k+1)^n - 1]/k$, the sum of the weights (26). This equality, of course, implies that every inequality sign in (32) must be discarded. We thus conclude that Σ_n for case 4 is given by (26).

Noting that the result in case 2 (see (30)) is a special case of the result for case 4, we have the following theorem.

THEOREM V. *Let the weights of the system Σ_n be applied only at the division points* 1, 2, $\cdots$, *k on the right arm of L_{jk}. Then, when $n=1$, Theorem* I, § 5, *applies here. When $k=1, j=1$, and $n>1$, or when $k>1$, and $n>1$, the weights of the system Σ_n are given by* (26). *When $k=1, j>1$, and $n>1$, the weights of Σ_n are given by* (31).

10. **Repetitions. Weights of system applicable on either arm of L_{jk}. Theorem VI.** *Definition* 7. Let $\Sigma_{m_r n}$ denote the maximum consecutive[1] system of positive integral weights of which there are n types $w_1, w_2, \cdots, w_n$ and m_r

[1] The term "consecutive" here, as in all of this paper, is applied only to a system of weights which suffices to weigh on L_{jk} all objects from 1_0 to k times the sum of the weights of the system, inclusive; the objects being placed at division points b_i which satisfy condition (5), §1.

weights of type w_r $(r=1, 2, \cdots, n)$, where $w_1 \leqq w_2 \leqq \cdots \leqq w_n$; let S_{m_rn} denote the sum of the weights of Σ_{m_rn}. Further, let σ_{m_rn} and s_{m_rn} denote a consecutive (but not necessarily the maximum consecutive) system and its sum.

We begin our search for Σ_{m_rn} as we first sought Σ_n in preceding sections, namely by finding a system of weights which the lever suggests as a possible consecutive system of large capacity.

To weigh 1_0, we select $w_1=1$. As there are m_1 weights w_1, they suffice to weigh on L_{jk} all objects from 1_0 to $(km_1)_0$ inclusive. The next object to be weighed is $(km_1+1)_0$. Placing it at -1 and the m_1 weights $w_1=1$ at $-j$, we create on the left arm of L_{jk} a moment of $(km_1+1)+jm_1=(j+k)m_1+1$. Hence we select $w_2=(j+k)m_1+1$. Continuing this reasoning by a method well illustrated above, we are led to believe that the system of weights which has m_r weights of the rth type below $(r=1, 2, \cdots, n)$,

$$w_1 = 1,\ w_2 = (j+k)m_1w_1 + 1, \cdots,$$
$$w_n=(j+k)(m_1w_1 + m_2w_2 + \cdots + m_{n-1}w_{n-1}) + 1, \quad (33)$$

is consecutive. Application of mathematical induction, as in § 2, shows that this is true, and indeed that the system σ_{m_rn} whose n types are given in (33) suffices to weigh *at* -1 all objects from 1_0 to $[k(m_1w_1+m_2w_2+\cdots + m_nw_n)]_0$ inclusive.

In order to abbreviate a discussion which parallels that of §§ 3–8 above, we state here, without proof, the four lemmas which correspond to those established in § 3:

LEMMA Ib. *A consecutive system* σ_{m_rn} ($m_r>1$ *for at least one value of* r) *contains a weight* $w_1=1$.

LEMMA IIb. *Every object of weight* w *where* $ks_{m_rn}/2<w\leqq ks_{m_rn}$ *must be weighed by* σ_{m_rn} *at* -1.

LEMMA IIIb. $S_{m_rn}\geqq s_{m_rn}$, *where* s_{m_rn} *is the sum of the weights in the system whose* n *types are given in* (33).

LEMMA IVb. *Let* s *be an integer* >1; *then the following conditions on the weights of* Σ_{m_rn} *are necessary* (*but not sufficient*).

$$w_1 = 1 \text{ (if } m_1n > 1) \ ; \ w_{i+1} \leqq (j+k)(m_1w_1 + \cdots + m_iw_i) + 1 \text{ if } n > i+1,$$
$$(i = 1,2, \cdots, \ s-1;\ s < n)\ ; \quad (34_i)$$

$$w_1 = 1 \text{ (if } kn > 1) \ ; \ w_{p+1} \leqq (j+k)(m_1w_1 + \cdots + m_pw_p) + 1 \text{ if } n > p,$$
$$(p = 1,2, \cdots, \ s-1;\ s < n+1). \quad (35_p)$$

We are supposing in this section that $m_1n>1$, as we have treated the case $m_1=n=1$ in §§ 1–8. If $n=1$, then $w_1=1$; otherwise the m_1 (>1) weights w_1

would not suffice to weigh $(km_1-1)_0$. It is easy to see now from (34_i) and (35_p) that our problem calls for consideration of the following cases: (*a*) when $n=2$ and $k=m_2=1$; (*b*) when $n>2$ and $k=m_n=1$; (*c*) when $km_n>1$. We consider these cases presently.

(*a*) *When* $n=2$ *and* $k=m_2=1$. We find by proceeding as in § 6 that Σ_{m_rn} is obtained by selecting the following types of weights:

$$\text{with } w_1 = 1, \text{ then } w_2 = m_1(2j+1)+2 \text{ if } m_1 \leqq j-2;$$
$$\text{or } w_2 = j(2m_1+1) \text{ if } m_1 \geqq j-2. \tag{36}$$

(*b*) *When* $n>2$ *and* $k=m_n=1$. By putting $s=n-1$ in relations (34_i), we obtain upper bounds for the first $(n-1)$ types of weights $w_1, w_2, \cdots, w_{n-1}$. Then proceeding as we did just before Theorem III, § 7, we find that by letting each of these types attain its upper bound, we can choose $w_n=j(2S_{m_rn-1}+1)$, where S_{m_rn-1} is the sum of all weights of the first $(n-1)$ types. We find then that Σ_{m_rn} has the following n types of weights

$$w_1 = 1, \; w_2 = (j+k)m_1w_1+1, \cdots, \; w_{n-1} = (j+k)(m_1w_1+m_2w_2+\cdots$$
$$+ m_{n-2}w_{n-2})+1, \; w_n = j(2S_{m_rn-1}+1). \tag{37}$$

(*c*) *When* $km_n>1$. Putting $s=n$ in relations (35_p) and proceeding as in § 8, we find that the system Σ_{m_rn} for this case has the n types given in (33).

The results of this section are summarized in the following theorem.

THEOREM VI. *If* $n=1$, *the system* Σ_{m_r1} (*cf. Definition* 7, § 10) *consists of* m_1 (>1) *weights*, w_1, *each equal to* 1. *If* $n=2$ *and* $k=m_2=1$, Σ_{m_r2} *is obtained by selecting its two types of weights as described in* (36). *If* $n>2$ *and* $k=m_n=1$, Σ_{m_rn} *has the n types of weights which are given by* (37). *If* $km_n>1$, *then* Σ_{m_rn} *is obtained by selecting the n types of weights which are given by* (33); *in this case* Σ_{m_rn} *suffices to weigh at* -1 *every positive integral weight which it suffices to weigh on* L_{jk}.

The following corollary to Theorem VI is easily deduced.

COROLLARY. *If* $m_1=m_2=\cdots=m_n=u$, $u>1$, *so that* m_r *has the same integral value* >1 *for each of the n values of r, the system* $\Sigma_{m_rn}\equiv\Sigma_{un}$ *is obtained by selecting u weights of each of the following n types* (*cf.* (33)):

$$w_1 = 1, \; w_2 = (j+k)u+1, \; w_3 = [(j+k)u+1]^2, \cdots,$$
$$w_n = [(j+k)u+1]^{n-1}.$$

Furthermore, Σ_{un} *suffices to weigh at* -1 *every positive integral weight which it suffices to weigh on* L_{jk}.

11. **Repetitions. Weights of system applicable only on the right arm of L_{jk}.** Theorem VII. To obtain the system $\Sigma_{m_r n}$ in this case, we proceed as in § 10. We find that the lever suggests the n weights

$$w_1 = 1, \ w_2 = km_1w_1 + 1, \ w_3 = k(m_1w_1 + m_2w_2) + 1, \cdots,$$
$$w_n = k(m_1w_1 + m_2w_2 + \cdots + m_{n-1}w_{n-1}) + 1 \qquad (38)$$

as the types of a possible consecutive system $\sigma_{m_r n}$ of large capacity. It can be shown by mathematical induction that this system is consecutive, and indeed that it suffices to weigh at -1 all integral weights from 1_0 to k times the sum of its weights.

Lemma Ib and Lemma IIb of § 10 hold here. In place of Lemma IIIb and Lemma IVb of § 10, we have the following:

LEMMA IIIc. *$S_{m_r n} \geqq s_{m_r n}$, where $s_{m_r n}$ is the sum of the weights of the system $\sigma_{m_r n}$ whose n types of weights are given by* (38).

LEMMA IVc. *Let s be an integer >1; then the following conditions on the weights of the system $\Sigma_{m_r n}$ are necessary (but not sufficient).*

$$w_1 = 1 \text{ (if } m_1 n > 1) \ ; \ w_{i+1} \leqq k(m_1w_1 + \cdots + m_iw_i) + 1 \text{ if } n > i + 1,$$
$$(i = 1, 2, \cdots, \ s - 1; \ s < n):$$
$$w_1 = 1 \text{ (if } kn > 1) \ ; \ w_{p+1} \leqq k(m_1w_1 + \cdots + m_pw_p) + 1 \text{ if } n > p,$$
$$(p = 1, 2, \cdots, \ s - 1; \ s < n + 1).$$

Proceeding now as in § 10, we obtain the following theorem.

THEOREM VII. *Let the weights of the system $\Sigma_{m_r n}$ be applied only at the division points on the right arm of L_{jk}. Then when $n=1$, $\Sigma_{m_r n}$ consists of m_1 (>1) weights w_1 each equal to* 1. *If $n \geqq 2$ and $k = m_n = 1$, the first $(n-1)$ types of weights of $\Sigma_{m_r n}$ are given by the equations*

$$w_1 = 1, \quad w_2 = m_1 + 1, \cdots, \quad w_{n-1} = (m_1w_1 + m_2w_2 + \cdots + m_{n-2}w_{n-2}) + 1,$$

while $w_n = S_{m_r n-1} + 1$ if $j=1$ and $w_n = S_{m_r n-1} + 2$ if $j>1$, where $S_{m_r n-1}$ is the sum of all weights of the first $(n-1)$ types. When $km_n > 1$, the system $\Sigma_{m_r n}$ has the n types of weights which are given by (38); *in this case $\Sigma_{m_r n}$ suffices to weigh at -1 every positive integral weight which it suffices to weigh on L_{jk}.*

The following corollary to Theorem VII is easily obtained.

COROLLARY. *If $m_1 = m_2 = \cdots = m_n = v$, $v > 1$, so that m_r has the same integral value for each of the n values of r, the system $\Sigma_{m_r n} \equiv \Sigma_{vn}$ is obtained by selecting v weights of each of the following n types (cf.* (38)):

$$w_1 = 1, \ w_2 = kv + 1, \ w_3 = (kv + 1)^2, \cdots, \ w_n = (kv + 1)^{n-1}.$$

Furthermore, Σ_{vn} suffices to weigh at -1 every positive integral weight which it suffices to weigh on L_{jk}.

12. **Problems solved. Problems proposed to the reader.** Theorems I, II, III, IV, and VI together constitute a solution of the following general Diophantine problem.

Let $w_1, w_2, \cdots, w_n$ be n types of positive integers. Select a positive number m_r of integers of type w_r $(r=1, 2, \cdots, n)$, and denote the sum of all the integers selected by $S_{m_r n}$. It is required to find the largest $S_{m_r n}$ for which there exists a consistent system of linear equations in $w_1, w_2, \cdots, w_n$ of the form

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \quad (i = 1,2, \cdots, kS_{m_r n}),$$

where the a's and b's are integers satisfying the conditions

$$-jm_r \leqq a_{ir} \leqq km_r \ (r = 1,2, \cdots, n), \ -j \leqq b_i \leqq -1,$$

j and k being positive integers.

Theorems V and VII together constitute a solution of the problem which results when the condition on the a's above is replaced by

$$0 \leqq a_{ir} \leqq km_r, \qquad (r = 1,2, \cdots, n),$$

all other features of the above problem remaining the same.

At the suggestion of Professor D. R. Curtiss, we propose here three problems which we hope will interest the reader. The author attempted to solve Problem 1 before he made any progress on the present paper. The last two problems are due to Professor Curtiss.

Problem 1. Let $w_1, w_2, \cdots, w_n$ be n positive integers. It is required to find the largest positive integer p for which there exists a consistent system of linear equations of the form

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \ (i = 1,2, \cdots, p),$$

where the a's and b's are integers satisfying the conditions

$$-j \leqq a_{iq} \leqq k, \ (q = 1,2, \cdots, n), \ -j \leqq b_i \leqq -1,$$

j and k being positive integers.

Problem 2. Given n objects of weights $a_1, a_2, \cdots, a_n$, which can be weighed (cf. Definition 3, § 1) on L_{jk} by use of n, but not less than n, positive integral weights $w_1, w_2, \cdots, w_n$: it is required to find the largest weight w that can occur in any set of n weights which suffice to weigh the n objects $a_1, a_2, \cdots, a_n$; in other words, to find the largest w that can occur in a solution of the system of linear equations in $w_1, w_2, \cdots, w_n$,

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \qquad (i = 1,2, \cdots, n),$$

where the coefficients a_{iq} and b_i are as described in Problem 1 except that here for each i there must exist at least one q such that $a_{iq} \neq 0$.

Example. If $j=k=1$, if $n=3$, and if the three objects which cannot be weighed by two weights are a_1, a_2, a_3, then three w's which suffice to weigh a_1, a_2, a_3 are $w_1=a_2+a_3$, $w_2=a_1+a_2+a_3$, $w_3=a_1+a_2+2a_3$. If $a_1<a_2<a_3$, we assert that $w_3=a_1+a_2+2a_3$ is the largest w that can occur in a set of three w's which suffice to weigh a_1, a_2, a_3 on L_{11}.

Problem 3. Assuming as in Problem 2 that no $(n-1)$ of the w's suffice to weigh the n objects $a_1, a_2, \cdots, a_n$ on L_{jk}, find the number of sets of n weights w which will accomplish the result.

Example. If $j=k=1$ and if $a_1=8'$, $a_2=15'$, $a_3=53'$, we assert that there are more than 50, but less than 300, sets of three w's which suffice to weigh a_1, a_2, a_3 on L_{11}. Let the reader find how many such sets of w's exist. Further, with this choice of j, k, n, let him find the maximum number of solutions w_1, w_2, w_3 of Problem 3 for all positive integral w_1, w_2, w_3.

THE CLOCK PARADOX OF THE THEORY OF RELATIVITY

By LUISE LANGE, Chicago, Illinois

At superficial thought the very first consequence of the Lorentz transformation may appear paradoxical: that of two systems in uniform relative motion each one finds its own clocks to go faster than those of the other. Yet, of course, there is no paradox in this. As this situation is of basic importance for our further discussion we shall begin by briefly recalling it. Each of the two systems S and S' is thought of as fitted with a set of identical clocks placed at certain measured distances and synchronized in the following way: at the moment when stations $x=0$ and $x'=0$ pass each other a light signal is sent out from that point at times $t=t'=0$, and observers in S and S' at x and x' respectively are instructed to set their clocks on $t=x/c$ and $t'=x'/c$ on receiving the signal. Now an observer O' stationed at $x'=0$ passes $x=vt$ at time t. According to the Lorentz equations[1] his clock then shows

$$t' = (1 - v^2c^{-2})^{-1/2}(t - vc^{-2}x) = (1 - v^2c^{-2})^{-1/2}(t - v^2c^{-2}t) = t(1 - v^2c^{-2})^{1/2},$$

so that observers in S will declare it to run $(1-v^2c^{-2})^{1/2}$ times slower than theirs. O' however could answer that he does not agree with their view of simultaneity on which their conclusion concerning his time-lag rests; that according to his own standards clock C_1 at $x=vt$ did not show $t=0$ when he, O', passed $x=0$. For that place, which he with his velocity v reaches at

[1] In case these should not be familiar, they may be found derived in every treatise on the theory of relativity.

$t' = (1 - v^2c^{-2})^{1/2}t$, is located, relative to S', at $x' = v(1 - v^2c^{-2})^{1/2}t = (1 - v^2c^{-2})^{1/2}x$; and for $t' = 0$, $x' = (1 - v^2c^{-2})^{1/2}x$, the L. E. give

$$t = (1 - v^2c^{-2})^{-1/2}(t' + vc^{-2}x') = (1 - v^2c^{-2})^{-1/2}vc^{-2}(1 - v^2c^{-2})^{1/2}x = vc^{-2}x = v^2c^{-2}t,$$

so that this clock C_1 during his journey from $x = 0$ to $x = vt$ went on only by $t - v^2c^{-2}t = t(1 - v^2c^{-2})$, that is $(1 - v^2c^{-2})^{1/2}$ times *slower* than his own.

Thus it is the different view of simultaneity of distant events which causes opposite inferences to be drawn from the same clockreadings in the two systems. Yet in thus comparing the clocks in S and S' it is really only *one* clock of the one system which is always compared to a *set of synchronized clocks* in the other. A direct comparison of two individual clocks in uniform relative motion is indeed intrinsically impossible because, as they cannot meet more than once, they cannot be compared more than once by an observer in the neighborhood of both; hence all he can do is either —as explained before—combine with a second observer, in which case synchronism must be established between the two distant places by assuming that light travels with velocity c in either direction relative to them. Or if the first observer can continue to keep the moving clock in sight and thus compare it to his own he must yet subtract the time of light transmission, again on the same assumption. Hence not the march of clocks in two such systems is being compared, but the judgment of two observers concerning the time between two events (e. g. the departure of O' at $x = 0$ and his arrival at $x = vt$), where this judgment involves more than the reading of the clocks to be compared.

Now one point is important: among the determinations of the time between two given events made from various systems there is one privileged, namely that measured by the observer present at both events (in our example O' who, in contrast to observers in S, is present at both his departure and arrival); for he uses for the measurement nothing but one clock, whereas all other observers obtain their result by combining the readings of two distant clocks defined to be synchronous; this time is also the only one which is *experienced* as time, a steady flow of one consciousness accompanying the movement of the clock-hands, whereas judged from S this time interval is merely a computed magnitude, one which by the nature of the case cannot be lived through. The supposed comparison described above was therefore one between "experienced" or "proper time" in the one system and "computed" in time in the other, not between "time" unqualified.

With these distinctions brought out we approach the starting point of the so-called paradox: It being felt that no comparison of the proper times of *two systems meeting only once* is possible, efforts were made to make the theory answer this question for two systems which do return to each other a second

time, such that two observers can directly compare the time between the two encounters.

In 1911 P. Langevin[1] formulated and solved the problem in the since well-known way: Peter stands on the earth, Paul, enclosed in a bullet, is fired away with tremendous velocity, but due to a suitable encounter with some star his path is directed back to the earth. On both journeys from and to the earth his clock is slow in comparison to those at rest to the earth; (as say the L. E.); hence in returning his time is behind earth-time in the ratio $(1-v^2c^{-2})^{-1/2} : 1$. If he stayed away two years according to his time that elapsed on the earth is, say, two hundred years, if the velocity with which he performed his cosmic journey was such that $(1-v^2c^{-2})^{1/2}=0.01$. While Paul has aged only slightly during his absence, the sixth generation of Peter's descendants are already populating the planet. (This result is frequently spoken of as Langevin's Paradox, but unjustly; for there is nothing self-contradictory in it, contrary as it may be to customary notions. It turned paradox only in the later development.)

As one notices, the accelerations in the movement of the traveler have been left out of account; it was treated as consisting of two uniform motions in opposite directions. Apparently to bring out this point with greater clearness, another French physicist[2] changed the formulation as follows: Paul leaves Peter at point A at time $t=t'=0$ with a velocity such that $(1-v^2c^{-2})^{1/2}=\frac{1}{2}$. He reaches point B at earth-time $t=2$, his own clock then reading $t'=1$ (inferred from the L. E.). Now he "suddenly jumps into another system animated with a uniform velocity $-v$ and thus travels back. Everything happens as on the first part of the trip, as the numerical value of v is unaltered; coming together again they state that the time which was "measured and experienced" as four hours by Peter was "measured and experienced" as only two hours by Paul; time for the observer traveling back and forth flowed half as fast as for the one standing still.

Yet an objection arose: why say that Peter stood still and that Paul moved, in a theory which asserts the equivalence of systems in relative motion, with the implication that every result must be reciprocal for the systems involved? Taking the bullet as system of reference it is the earth with Peter which moves back and forth; hence it is the earth-time that lags behind, and thus at their second encounter Paul finds Peter not in his grave, but practically unaltered since he lived through only 0.02 year.—But, most distressing of all, *both views are equally justified*, the time lived through by either one is both longer and shorter than that of the other. This is Langevin's paradox.

[1] Paper presented at a meeting at Bologne, 1911.

[2] In a letter to H. Bergson, quoted in *Durée et Simultanéité*, pp. 246 ff. its author is spoken of by Bergson as "one of the most distinguished physicists," but his name is not revealed.

Though much was made of this paradox, notably by philosophers,[1] relativist-physicists soon pointed out that it arises not as a consequence, but by one or several misapplications of the theory of relativity.

We consider it first in the form of the traveler who "changes trains," but to deal with motions free throughout from accelerations we avoid the jump from one system to the other; instead, we leave Paul attached to his first system and place another observer, James, on the one moving oppositely, who sets his clock in accordance with Paul's when he passes him, on $t''=t'=1$. Now we have to distinguish three different uniform systems, S, S', S''. The systems S' and S'' move relatively to S with velocities $\pm v$, and therefore S'' moves relatively to S' with velocity

$$u=\frac{-v-v}{1-\frac{v}{c^2}(-v)}=\frac{-2v}{1+\frac{v^2}{c^2}},$$

and, using again the numerical value $(1-v^2c^{-2})^{1/2}=\frac{1}{2}$, this gives $7u+8v=0$.

The conclusion now that time in the double-system $S'-S''$ flows twice as slow as in S was based on the consideration that, as both move with the same numerical velocity relatively to S, the time elapsing in either of them in going from A to B and back from B to A is the same. Yet it is only judged by the standards of simultaneity in S that the clocks in S' and S'' run synchronously; whereas judged from S' the proper time between two events in S'' is $(1-u^2c^{-2})^{-1/2}$ or 7 times slower than the time computed to have elapsed between them in S'-time, and *vice versa*. Therefore observer James judges: For Paul to travel from A to B took 1 hour S'-time, which is 7 hours S''-time; back from B to A in my system took 1 hour in S''-time, hence the two trips $\overline{AB}$ and $\overline{BA}$ together 8 hours; and Paul figures similarly in S'-time. Thus there is the usual reciprocal overestimation of the proper times in relatively moving systems, but no paradox, as no two of the three observers meet more than once.

Next, what difference does it make if one of the two systems undergoes an acceleration in reversing its velocity?[2] The answer is: it destroys the equivalence of the two systems such that they cannot any more be at will considered as moving or at rest. For so far we are dealing exclusively with the restricted theory of relativity which asserts the equivalence only of systems in uniform relative motion. Accelerations from its standpoint are absolute, they imply forces, they bring into effect phenomena of inertia. Paul, if he did not jump,

---

[1] Two interesting though strikingly different interpretations are given by J. Petzold, *Die Stellung der Relativitätstheorie in der geistigen Entwicklung der Menschheit*, 2nd. ed., Leipzig, 1923, and by H. Bergson, *Durée et Simultanéité, à propos de la théorie d'Einstein*, 2nd. ed., Paris, 1923.

[2] If it is the observer who "jumps" from one system to the other this involves accelerations of no less consequence than if the system itself reverses its velocity.

suffered a severe bump against the front-wall of his enclosure if the reversal was sudden, or a perpetual pull towards it if it lasted some time, whereas nothing of the kind happened to Peter, hence Paul *does* move and thus it is *his* time which flows slower, not, as the paradox asserts, Peter's as well.

Yet this argument is not quite satisfactory, for the two trips back and forth were uniform, during these periods the two systems therefore remained equivalent and they are the only ones considered in the computation. Hence one could proceed as follows: refer the two uniform motions to Paul at rest but in-between attribute to him a sudden change from the one system of reference to the other. The question is, does that bring back the paradox? Is it in this description again Peter's time which flows slower? It does not, as a short consideration shows. The first uniform motion of Peter's with velocity $-v$ lasts 1 hour according to Paul's clock, while only $\frac{1}{2}$ hour in Peter's time as judged by Paul, namely

$$t = (1 - v^2c^{-2})^{-1/2}(t' + vc^{-2}x') = (1 - v^2c^{-2})^{-1/2}(1 + vc^{-2}(-v)) = (1-v^2c^{-2})^{1/2} = \tfrac{1}{2}.$$

The same happens during the second uniform motion with velocity $+v$. But, in performing the change from the first to the second system of reference Paul's view of temporality, in particular of simultaneity of distant events changes. From the point of view of the first system the "jump" which according to his own time took place at 1:00, corresponded to 12:30 ($\frac{1}{2}$ hour after the zero point) on Peter's clock; from the point of view of the second system however it is otherwise. For this system (S'') the Lorentz transformation with regard to Peter's world (S) assumes the form

$$t - t_1 = (1 - v^2c^{-2})^{-1/2}\left[(t'' - t_1'') - vc^{-2}(x'' - x_1'')\right]$$

where $t_1=2$ (S-time at place of jump), $t_1''=1$, $x_1''=0$, $t''=1$, $x''=-v$ (Peter's place referred to S'' at time of jump); for the assumed value $(1-v^2c^{-2})^{1/2}=\frac{1}{2}$ this gives $t=3\frac{1}{2}$. Thus in this description Peter's clock runs during the first uniform motion from 12:00 to 12:30, then during the jump leaps ahead to 3:30, and goes on from 3:30 to 4:00 during the second uniform motion. The result therefore is the same as before: what was 4 hours for Peter was 2 hours for Paul, not *vice versa*.

This is satisfactory inasmuch as it denies the paradox and seems to reveal a physically objective slower flow of time in the non-inert system, that is one independent of the system of reference. Yet there appears to be a doubtful point left. The computations made from the two different viewpoints show a strange dissymmetry. The first ran: Peter at rest, Paul moving back and forth losing time. The second: Paul at rest, Peter moving away losing time; Paul changing system of reference, Peter *gaining* time; Paul at rest, Peter losing time. The period of acceleration, of decisive influence in the second

treatment, has been entirely disregarded in the first. It was condensed into an "instant" and then neglected. But why could not this "instant" of Peter's —for it is in his time that we are figuring—be anything but an instant in Paul's time, when it has just been shown that the instantaneous change of system of reference of Paul's filled, in his computation, $3\frac{1}{2}$ hours of Peter's time? Why not figure out? There it lies: the theory does not provide the means to deal with this point. Its only tools are the Lorentz equations, and these speak of transformations among uniformly moving systems. How can they be applied to essentially non-uniform motions? Of course one is tempted to use differential methods, to consider the accelerated motion as consisting of indefinitely many uniform ones, successive velocities differing only by an indefinitely small amount Δv. But that does not remove the difficulty; for this increase Δv has to be acquired in a time small of at least second order,—so to speak in between two small intervals of uniform motion Δt—which makes these accelerations both infinitely great and infinitely frequent. Therefore to define as is sometimes done,[1] the proper time of an accelerated system as

$$\int_{x_1 t_1}^{x_1 t_2} (1 - [v(t)]^2 c^{-2})^{1/2} dt$$

is a very doubtful procedure. The transition from a differential $dy = f(x)dx$ to the integral $y = \int f(x)dx$ presupposes that the summation of any two elements leads to an element of the same kind or quality; here however one adds the proper time of one inert system to the proper time of another inert system; to say that their sum is equal to the proper time of a non-inert system means to disregard the possible influence of the accelerations as much as Langevin did in his treatment.[2]

Hence, inasmuch as this point is not cleared up, we have to admit that the restricted theory of relativity does not give the answer to the question: What are, in comparison, the proper times of two systems meeting twice and is the result obtained independent of the system of reference?

[1] See e.g. A. Kopff, *Mathematical Theory of Relativity*, English ed. 1923, p. 51, von Laue, *Theory of Relativity*, French ed. 1924, p. 80, J. Becquerel, *Exposé Elémentaire de la Théorie d'Einstein*, Paris 1922, p. 56.

[2] It is interesting to note how various authors get around this point. Kopff (*op. cit.*) attaches to his proper time integral a little inconspicuous footnote saying that all effects due to acceleration have been disregarded, and in another connection that accelerations are assumed to have no influence on the march of a clock (p. 125). Why so, he does not say. M. Born, *Theory of Relativity*, (French ed. 1923, p. 249) suggests that the uniform part of the motion in Langevin's proposition may be made so long that the period of acceleration may be considered negligible, which would (*a*) restrict the result to such cases where this assumption is fulfilled, and (*b*) even there is doubtful as we shall point out later. Von Laue, (*op. cit.* p. 63) speaks of "quasi-stationary accelerations, that is such that the inner status of the clock depends always on its instantaneous velocity, not on its acceleration," which, in this connection, might be called a *petitio principii*, for the question is just whether there exist such convenient quasi-stationary accelerations.

What about the general theory? In it all systems in an arbitrary state of motion are equivalent. Paul bumping against his enclosure may indeed be considered at rest and Peter who does not experience any disturbance as in accelerated motion, if namely one assumes a suitable field of gravitation oppositely directed to the previous acceleration in which the formerly accelerating forces keep Paul at rest, while Peter "falls free." The observable phenomena under these circumstances are indeed the same as before: The relative acceleration, Paul's being drawn towards the one wall of his bullet (which from the first point of view we would have ascribed to his inertia, from the second to his weight), and Peter's undisturbedness. Can this general theory deal better with our problem? It is claimed that it can; e. g. Kopff (*op. cit.*) remarks (p. 125) that it "provides the explanation for the clock paradoxes which turn up in the restricted theory of relativity." Einstein gave the solution as follows:[1] Two clocks A and B move so as to meet twice. In the first description of the process A is considered at rest, B as acted upon by forces which, after it has moved away[2] from A with uniform velocity $+v$ accelerate it in the opposite direction until it has assumed a velocity $-v$ which with it travels back to A uniformly. The question as to the comparative proper times in the two systems is treated as by Langevin; in neglecting the period of acceleration a time-lag of B during the two uniform motions is derived, $t'=(1-v^2c^{-2})^{1/2}t$, which is approximately $t'=t-\frac{1}{2}v^2c^{-2}t$.

In the second description B is considered at rest; A first moves away uniformly with velocity $-v$. Then a homogeneous field of gravitation in direction $\overrightarrow{AB}$ is introduced in which the previous forces keep B at rest while A falls free in it until it has assumed the velocity $+v$. The field vanishes, and A moves uniformly back to B. During the uniform parts of the motion A lags behind B; if they lasted t' in B-time they are only $t=(1-v^2c^{-2})^{1/2}t'$ in A-time. But during the reversing of the velocity A is at a place of higher gravitational potential than B; according to the theory, that makes A run faster than B so long as the field lasts. If in B-time it lasts $\Delta t'$, then in A-time it lasts $\Delta t'(1+\Delta\phi/c^2)$, where $\Delta\phi$ is the difference in gravitational potential. Now $\Delta\phi=h\cdot\gamma$, where h is the distance of A and B (measured in system B), and γ is the gravitational acceleration. Also $h=v\cdot t'/2$, ($t'/2$ being the time of each uniform motion in B-time), and $\Delta t'\cdot\gamma=2v$. Hence, substituting, $\Delta t'\cdot\Delta\phi/c^2=v^2/c^2\cdot t'$. Thus A *gains* during the reversal of its velocity by

[1] Einstein, Dialog über die Einwände gegen die Relativitätstheorie, in *Die Naturwissenschaften*, vol. 16, 1918; treated at more detail in Kopff. *op. cit.*, Born, *op. cit.*

[2] In the above presentation we treat the two clocks as having at the beginning and end a uniform relative velocity v. In the treatment referred to they are at relative rest at the beginning and end, which necessitates two more periods of acceleration. As their influence, however, is not accounted for we cut them out in this way, which does not affect the problem but slightly simplifies the presentation.

$v^2/c^2 \cdot t'$. This overcompensates the previous time-loss. If one uses the approximation $(1-v^2c^{-2})^{1/2}=1-\frac{1}{2}v^2c^{-2}$ the total gain of A over B at its return is $\frac{1}{2}c^2c^{-2} \cdot t'$, which is the same as that obtained on p. 28 by considering A at rest. Thus the paradox is removed; the time-lag of B is shown to be physically objective, that is independent of the system of reference.

Much as this method of treating the problem appears to differ from that given on pp. 26 the underlying idea is essentially the same: though the motion is referred to that system which is subject to the influence of forces yet the reversing of velocity has to be accounted for in such a way as to leave unaltered the physical phenomena making their appearance during this period. Accounting for them either way, by saying that the observer changes his system of reference, or that a gravitational field appears causing the appearance of his weight, the effect is the same: during this period time in the other system leaps ahead so as to overcompensate the time-loss during the uniform parts of the motion.

Yet just because of this similarity we do not feel any better satisfied than before, for our previous difficulty has not been touched at all. Now as before the description with A (Peter) as system of reference obtains the time-lag for B (Paul) only under disregard of the period of acceleration. Einstein in his paper does not mention this point at all, his interest being seemingly centered on showing how the case can also be dealt with by means of a gravitational field; Kopff, reproducing the argument and apparently feeling the gap, limits himself to the remark "these forces may have no influence on the movement of the clock B" (*loc. cit.*). Why *may* they not? These forces represent the only physical differentiation of the two clocks; it is due to them only that the unsymmetrical result as regards the two clocks is acceptable. Projected onto the other end by means of a fictitious gravitational field they swing the time-balance in favor of A, and yet, we are asked to *assume* that they have no influence on the clock on which they act. Moreover it appears that also Born's motivation for this omission (see footnote p. 27) is untenable; for the longer the *uniform* part of the motion the greater is the time-gain of A during the acceleration in the gravitational field. Why is not the same possible for B during its neglected acceleration? And again, why assume instead of compute? Apparently because also the general theory does not provide the means to deal with the influence of an acceleration not in a gravitational field. Nor will it do to say that only one description (with B as system of reference) is necessary to ascertain what happens, both versions referring to the same physical process; for, from the standpoint of the paradox, this simply means to beg the question.

Thus our survey has led us to the conclusion that the question regarding the difference in duration experienced in two relatively moving systems

between two encounters is as yet unanswered. It may be, as many relativists assert, that in worlds of different state of motion the *derôlement* of time is different, like on motion pictures reeled through slower and faster, with the inert systems being the fastest ones, but we do not see that the complete proof for it has as yet been given.

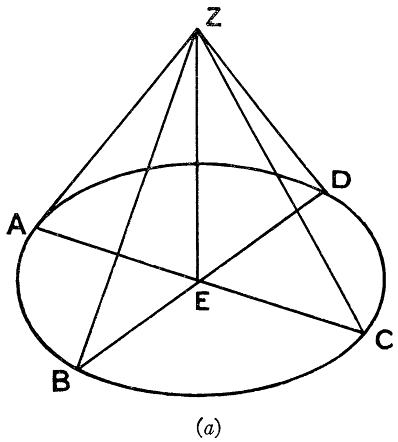
THE CIRCLE IN EUCLID'S TREATMENT OF OPTICS

By C. T. RUDDICK, University of Pennsylvania

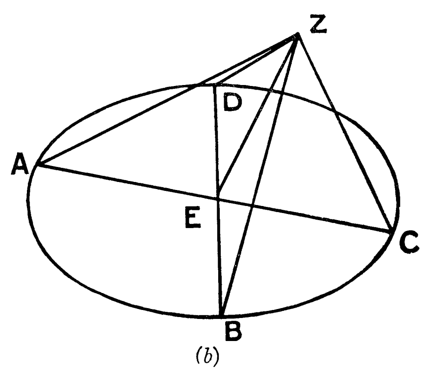
The study of physical optics yields many problems of interest to the mathematical investigator. This is especially true of the first known work on the subject, that of Euclid, which has been preserved in the "Collections" of Pappus. Here are brought to light some properties of the circle as it *appears* to the eye in different positions, particularly in perspective.

It is the purpose of this paper to discuss some of the more general results obtained, and to reproduce in condensed form the treatment of the apparent ellipse formed by a circle in perspective. The work is essentially that of Euclid and Pappus, but arranged and greatly condensed to allow briefness and clarity of presentation. The propositions themselves are literal translations by Dr. J. H. Weaver, then of the University of Pennsylvania, now of Ohio State, from the parallel Greek and Latin, edition Hultsch of Pappus's "Collections."

The basis of the work is the principle that the apparent length of a line depends upon the angle which it subtends at the eye. From this the first proposition follows at once.



(*a*)



(*b*)

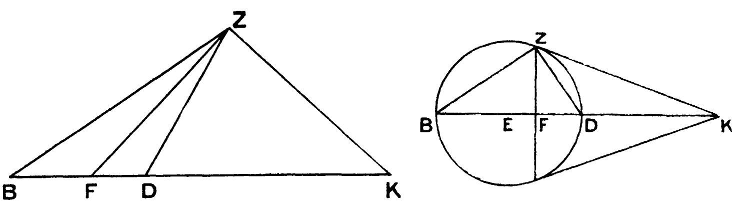
FIG. 1

PROP. A. "Let there be a circle whose center is E, and from E let EZ be drawn perpendicular to the plane of the circle, then if the eye is placed in the line EZ, the diameters of the circle appear equal." (Fig. 1a)

This is of course obvious, since at any point on EZ the diameters will subtend equal angles, from a simple theorem of geometry.

"But let EZ be non-perpendicular, and let it be equal to a semidiameter, then if the eye is placed at Z, the diameters will appear equal." (Fig. 1b). This is equally manifest, since for any diameter BED, BE, ED, EZ are equal, hence the angle at Z is always right. Since the locus of the point Z is of course the surface of a hemisphere, it appears that all the diameters of the base of a hemisphere will seem equal to the eye placed at any point on the surface.

PROP. B. "If there is a circle, and from the center of it a straight line is erected, which is neither perpendicular to the plane nor equal to half a diameter of the circle, and at the end of this the eye be placed, the diameters of the circle will appear unequal." Two particular diameters appear as maximum and minimum, and the others are equal only in pairs. The proofs are based on an extensive array of lemmas, to the end that the angles subtended at the eye are proved unequal. The circle, viewed from such a position, appears of course to be an ellipse, and some of the peculiar properties of this ellipse will now be discussed. Much of the work to follow depends on a single lemma, which appears in Pappus thus: "Let $BK : KD = BF : FD$, and $\angle BZF = \angle FZD$, and KZ be drawn, then the angle FZK is right." Fig. 2a.



(a) FIG. 2 (b)

The theorem really amounts to the statement that the internal and external bisectors of the vertex angle of a triangle, along with the adjacent sides, cut the base in a range of points whose cross-ratio is harmonic. It should be noticed that tangents to a circle at the extremities of a chord perpendicular to the diameter effect the same division on the diameter. Fig. 2b.

Now (Fig. 3), consider the circle ABC, with the eye placed at Z. Drop ZH perpendicular to the plane of the circle, and draw $HDEB$ through the center E, intersecting the circumference in B, D.
Draw BZ, ZD, and bisect $\angle BZD$ with ZF, meeting the plane in F. Draw $AFC \perp BD$ at F, and tangents KA, KC, meeting BD in K.

Then these things are true: 1. ABC appears to be an ellipse, of which the center is at F, and the conjugate axes are AC and BD. Ordinates drawn to BD will be, and *appear* to be, parallel to AC. 2. Ordinates to AC are drawn from K, but *appear* parallel to BD.

1. Draw AZ, ZC, then $\angle AZF = \angle FZC$ and AF appears to equal FC. Also $\angle BZD = \angle FZD$ by construction, hence BF appears to equal FD.

Now any line LFM appears halved at F. Draw $LNXK$, KM, MPX, MZ, XZ, NZ, LZ, KZ. Because of the tangents KA and KC, as above, $BK : KD = BF : FD$ and $\angle BZF = \angle FZD$ by construction so $\angle FZK$ is right. Since plane $FZK \perp AZC$ then $ZK \perp$ plane AZC, and $\angle NZK$ is right. And since $\angle LFC = \angle CFX$, being measured by equal arcs, and $\angle CFK$ is right, then $LK : KX = LN : NX$.

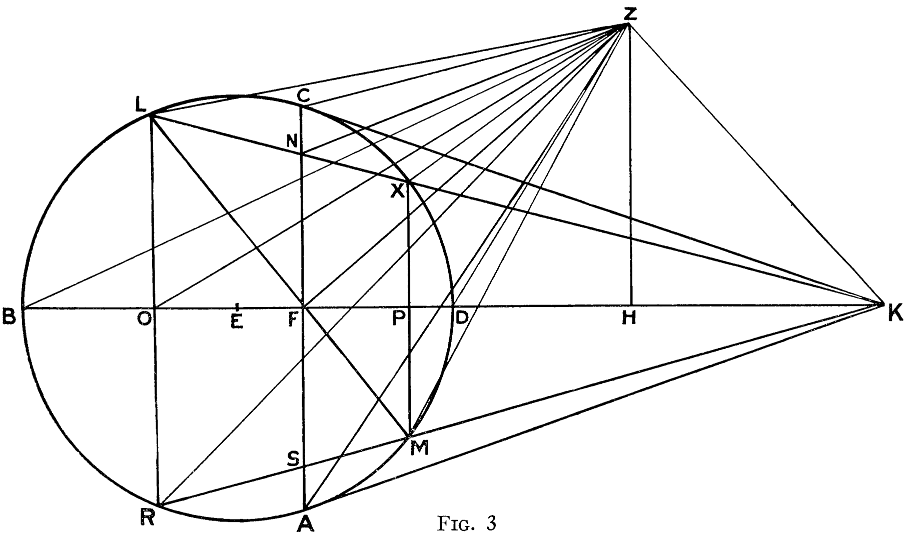


FIG. 3

Therefore, as a converse of the theorem stated above, $\angle LZN = \angle NZX$, and LN appears to equal NX. Also, because $\angle LZN = \angle NZX$ then $LZ : ZX = LN : NX$ and $ZM = ZX$, since $XM \parallel AC$, so $LZ : ZM = LN : NX$. But $LN : NX = LF : FM$, since $NF \parallel MX$, therefore $LZ : ZM = LF : FM$ and $\angle LZF = \angle FZM$. Thence LF appears to equal FM, and any line through F appears to be halved by it.

2. Now lines through the circle from K appear to be parallel. Draw $LO \perp BK$ and produce to R. Draw FZ, ZP, ZR. Since $LK : KX = LN : NX$ and $LN : NX = LZ : ZX$ and $LK : KX = LR : XM$, as shown above, then $LR : XM = LZ : ZX$. And, since $LZ = ZR$ and $ZX = ZM$, this becomes $LR : XM = ZR : ZM$.

Hence the isosceles triangles LZR and XMZ are similar, since their sides are all proportional, so that $\angle LZR$ equals $\angle XZM$. From this it is evident that $\angle LZO$ equals $\angle XZP$, therefore OL appears to equal PX. So LX appears parallel to BD, since perpendiculars between them appear equal.[1]

[1] It seems probable, from the method of propositions fifty-two, fifty-three, and fifty-four of this section of the sixth book of the "Collections," that this latter part of the discussion may be the work of Pappus himself, and not a part of the "Optics" of Euclid.

The question at once arises as to how two lines can appear parallel when we can see them meet at the point K. The investigation may be carried further by asking if all perpendiculars between LX and BD, or, more generally, between LK and BK, will appear equal. Taking the line FN for example, a new circle may be drawn through the four points N, X, M, S. Looking at this circle with the eye still at Z, a new point, say F', will appear to be the center, and since $\angle FZK$ is right, a new point K' will result, and lines from K' through this circle should appear parallel. But the line NXK does not pass through K', so NF will not appear to equal XP. The same reasoning applies to any other perpendicular.

So the appearance of such a line is uncertain. Experiment seems to show that it does appear parallel. Common sense suggests that it would appear curved instead of straight. The answer may lie in the construction of the eye. If it is capable of viewing the ensemble as a whole, the line perhaps should appear curved. But if the eye functions discontinuously, rather than continuously, it would seem that what has been demonstrated is true. That is, looking first at one and then at the other of the perpendiculars, they would appear equal, and the visual image retained would be one of parallel lines. Thus the phenomenon would reduce to an optical illusion.

ON PRODUCTS WHOSE DIGITS ARE CYCLICAL PERMUTATIONS OF THE DIGITS OF THE MULTIPLICAND

By ROBERT E. MORITZ, University of Washington

1. Introduction. It has been known for some years that the number 142857 has a property shared by no other number expressed in the common scale of notation, namely this, that it is the only number for which a multiplier of the same number of digits can be found such that the partial products consist of the numbers formed by cyclically permuting the digits of the number 142857.

$$\begin{aligned} 142857 \times 1 &= 142857 \\ 142857 \times 3 &= 428571 \\ 142857 \times 2 &= 285714 \\ 142857 \times 6 &= 857142 \\ 142857 \times 4 &= 571428 \\ 142857 \times 5 &= 714285 \end{aligned}$$

The following paper deals with a general theorem from which the property in question follows as a very special case. The theorem, based on the most elementary algebraic relations, reveals many surprising results as, for instance, that the sixteen numbers 0588, 2352, 9411, $7647 \times k$, where k has the values

1, 2, 3, $\cdots$, 16, are formed by the cyclical permutations of the digits of the multiplicand.

2. Let s, p, and q be any three integers satisfying the relation

$$s^q \equiv 1 \pmod{p}. \tag{1}$$

Furthermore, let $d_1, d_2, \cdots, d_q$ represent the successive digits in the scale of notation, radix s, and $r_1, r_2, \cdots, r_q$ the corresponding remainders resulting from the division of the successive powers of s by p, thus

$$\begin{aligned} s &= d_1 p + r_1, \\ s^2 &= (d_1 p + r_1)s = (d_1 s + d_2)p + r_2, \\ s^3 &= (d_1 s^2 + d_2 s + d_3)p + r_3, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ s^k &= (d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k)p + r_k, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ s^q &= (d_1 s^{q-1} + d_2 s^{q-2} + \cdots + d_{q-1}s + d_q)p + 1 \end{aligned} \tag{2}$$

Note: $r_q = 1$, because of the assumed relation (1).

From (2) we have

$$d_1 s^{q-1} + d_2 s^{q-2} + \cdots + d_{q-1}s + d_q = (s^q - 1)/p \tag{3}$$

and

$$r_k = s^k - (d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k)p. \tag{4}$$

Let N be the number represented by either side of equation (3). Then, from (3) and (4), the number $N \cdot r_k$ is equal to

$$(d_1 s^{q-1} + d_2 s^{q-2} + \cdots + d_{q-1}s + d_q)s^k \\ - (s^q - 1)(d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k)$$

which reduces to

$$d_{k+1}s^{q-1} + d_{k+2}s^{q-2} + \cdots + d_q s^k + d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k$$

a number whose digits, in the s-scale of notation, are the k-th cyclical permutation of the digits of (3).

The foregoing result may be expressed in the form of a theorem as follows:

Theorem. Let $d_1, d_2, \cdots, d_q$ *denote the successive digits, in the s-scale of notation, of a number* N, *obtained by dividing* $s^q - 1$ *by one of its divisors.*[1] *Let* r_k, $k < q$, *be the remainder obtained on dividing* s^k *by this same divisor. Then the product* $N \cdot r_k$, *when expressed in the s-scale of notation, will have for its successive digits the k-th cyclical permutation of the digits of* N.

[1] When the quotient has less than q digits the deficiency must be supplied with zeros. Thus $(10^3 - 1)/111 = 9 = 0 \cdot 10^2 + 0 \cdot 10 + 9$; that is $d_1 = 0$, $d_2 = 0$, $d_3 = 9$.

3. Examples. In the denary scale of notation, $10^6-1=142857\times 7$, and 10^3 divided by 7 leaves a remainder 6; hence we have $142857\times 6=857142$ whose digits are the third cyclical permutation of the digits 142857.

Again, $10^6-1=076923\times 13$, and 10^5 divided by 13 leaves the remainder 4; hence $076923\times 4=307692$ whose digits are the fifth cyclical permutation of the digits 076923.

In the duodenary scale of notation, using t and e to denote ten and eleven,

$$(10^{14}-1)/15 = 0\ 857\ 921\ 4e3\ 642\ 9t7$$

$$\begin{array}{llll}
r_1 = 10, & 0\ 857\ 921\ 4e3\ 642\ 9t7 \times 10 & = 8\ 579\ 214\ e36\ 429\ t70 \\
r_2 = 8, & 0\ 857\ 921\ 4e3\ 642\ 9t7 \times 8 & = 5\ 792\ 14e\ 364\ 29t\ 708 \\
r_3 = e, & 0\ 857\ 921\ 4e3\ 642\ 9t7 \times e & = 7\ 921\ 4e3\ 642\ 9t7\ 085 \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
r_{13} = t, & 0\ 857\ 921\ 4e3\ 642\ 9t7 \times t & = 7\ 085\ 792\ 14e\ 364\ 29t \\
r_{14} = 1. & &
\end{array}$$

It is obvious that the problem of finding the total number of products possessing the property set forth in our theorem resolves itself to the problem of the complete factorization of s^q-1 for given values of s and q. In the denary scale of notation, when $q=5$, $s^q-1=99999$, the distinct factors of which, not counting unity or the number itself, are 3, 9, 41, 123, 271, 369, 813, 2439, 11111, 33333. To each factor correspond theoretically $q-1=4$ products having the property in question. But whenever r_k is some power of 10 the result is trivial. The remaining significant results are

$$\begin{array}{llll}
N = (10^5-1)/41 & = 02439, & r_2 = 18, & r_3 = 16, \quad r_4 = 37\ ; \\
N = (10^5-1)/123 & = 00813, & r_3 = 16, & r_4 = 37\ ; \\
N = (10^5-1)/271 & = 00369, & r_3 = 187, & r_4 = 244\ ; \\
N = (10^5-1)/369 & = 00271, & r_3 = 262, & r_4 = 37\ ; \\
N = (10^5-1)/813 & = 00123, & r_3 = 187, & r_4 = 244\ ; \\
N = (10^5-1)/2439 & = 00041, & r_4 = 244. &
\end{array}$$

Of special interest is the case in which p is an odd prime and s a primitive root of p. Then $s^{p-1}-1\equiv 0$, mod. p, and the successive powers of s when divided by p leave as remainders some permutation of the numbers $1, 2, \cdots, p-1$. If then we write $(s^{p-1}-1)/p=Q$, our theorem shows that the multiples $1Q, 2Q, \cdots, (p-1)Q$, when expressed in the s-scale of notation, have for digits the $p-1$ cyclical permutations of the digits of Q, the digits of nQ being the k-th cyclical permutation of the digits of Q, where k is the index belonging to n.[1]

[1] This result was arrived at by quite a different method by Bachmann, Zeitschrift für Mathematik und Physik, vol. 36 (1891), pp. 381–383.

In the denary scale of notation the highest digit is 9; p must therefore be less than 10. Now the only prime number less than 10, of which 10 is a primitive root, is 7; hence our corollary yields only one number which, when expressed in the denary scale of notation, possesses the afore-mentioned property. This num-is $(10^6-1)/7=142857$.

When 12 is the radix of the scale of notation, there are two numbers Q such that every cyclical permutation of the digits of Q gives rise to a multiple of Q, for 12 is a primitive root of both 5 and 7. The two numbers are therefore $(12^4-1)/5=4147=2497$ in the duodenary scale, and $(12^6-1)/7=426569=186t35$ in the duodenary scale. The partial products of 2497 by 2431 and of $186t35$ by 546231, the radix being 12, are respectively as indicated below.

$$\begin{array}{l}
2497\\
\underline{2431}\\
4972\\
\ \ 9724\\
\ \ \ \ 7249\\
\underline{\ \ \ \ \ \ 2479}
\end{array}
\qquad\qquad
\begin{array}{l}
186t35\\
\underline{546231}\\
86t351\\
\ \ 6t3518\\
\ \ \ \ t35186\\
\ \ \ \ \ \ 35186t\\
\ \ \ \ \ \ \ \ 5186t3\\
\underline{\ \ \ \ \ \ \ \ \ \ 186t35}
\end{array}$$

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

A PERFECT NON-DENSE POINT SET

By RAYMOND GARVER, University of Rochester.

1. Introduction. If G is a linear point set, and P is a point such that every neighborhood of it contains points of G other than P itself, P is called a limit point (or an accumulation point) of the set G. If G contains all its limit points, it is said to be a closed set. If every point of G is a limit point of the set, G is said to be dense in itself. A set with both of these properties is a perfect set.

Now if a part of G is contained in a closed interval (a, b), and if every subinterval of (a, b) contains points of G, we say that G is everywhere dense in (a, b). If G has this property it is also dense in itself, provided that no points of G lie outside (a, b).[1] However, the converse is not true, as can be shown by examples. In fact, we can go further and define a set which is both perfect and

[1] See Cantor, Mathematische Annalen, vol. 23, p. 473.

non-dense; that is, *no* interval exists in which G is everywhere dense. The first such example was constructed by Cantor,[1] and others have been given. However the following example, which is similar to Cantor's, seems to illustrate the various properties of the set a little more plainly than do the others.

2. A Perfect Non-dense Set. The set G is defined to be the set of points (or real numbers) given by $.\,a_1a_2 \cdots a_n \cdots$, where each a_i is either 0 or 9. The set is thus contained in the interval (0, 1) and can easily be shown to be a perfect, non-dense set.

(a) *The set is dense in itself.* That is, any $P = .\,a_1a_2 \cdots a_n \cdots$ is a limit point of the set. For if we consider a neighborhood of P which extends at least a distance of $1/10^n$ to each side of P, we see that the point $Q = .\,a_1a_2 \cdots a_nb_{n+1}b_{n+2} \cdots$, where each b_j is either 0 or 9, but at least one b_j is different from the corresponding a_j in P, lies in this neighborhood, belongs to the set G, and is distinct from P.

(b) *The set is closed.* For consider a point $R = .\,a_1a_2 \cdots a_nc_{n+1} \cdots$, where R does not belong to G, and c_{n+1} is the first digit not a zero or nine. (If $c_{n+1} = 8$, not all the succeeding digits are to be nines, or if $c_{n+1} = 1$, not all the succeeding digits are to be zeros; in either of these cases R would belong to the set.) By definition, G contains no points between $(.\,a_1a_2 \cdots a_n0999 \cdots = .\,a_1a_2 \cdots a_n1)$ and $.\,a_1a_2 \cdots a_n9$. Hence we can take a neighborhood of R so small that it will certainly contain no point of G; that is, R can not be a limit point.

(c) *The set is non-dense in* (0, 1). We consider the sub-intervals of (0, 1) which contain no points of G in their interior. The largest is the interval $(.09999 \cdots, .90000 \cdots)$ or $(.1, .9)$, of length .8. Next are two of equal size, $(.00999 \cdots, .09000 \cdots) = (.01, .09)$ and $(.90999 \cdots, .9900 \cdots) = (.91, .99)$. In general, no points of G lie between $(.\,a_1a_2 \cdots a_n0999 \cdots, = .\,a_1a_2 \cdots a_n1)$ and $.\,a_1a_2 \cdots a_n9$. There are 2^n such intervals, since each of $a_1, \cdots a_n$ may be either 0 or 9, and each of the intervals is of length $8/10^{n+1}$. The infinite sum of all these intervals is that of a geometric series whose first term is .8 and common ratio .2; it is therefore 1, the length of the interval. This set of non-overlapping intervals is therefore everywhere dense in (0, 1), and the set of points G must certainly be non-dense.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Statistical Methods, Applied to Economics and Business. By F. C. MILLS. New York, Henry Holt and Company, 1924. xvi+604 pages.

This book is essentially for the student of economics and business and the fundamental facts are written up in considerable detail, presumably on the

[1] See Cantor, Mathematische Annalen, vol. 21, p. 590.

hypothesis that much of the matter, even though elementary, is new to the reader. The first sixty pages develop the classes of business activity and the idea of the quantitative character of many economic and business problems. Attention is then called to statistical methods and problems of internal and external administration. Fifty pages are devoted to graphic presentation, including the standard rules proposed by the Joint Committee on Standards for Graphic Presentation. The next eighty-five pages are devoted to frequency distributions and all the usual ideas concerning such distributions together with the various types of averages are described. Only twenty pages are given to the development of dispersion, standard deviation, skewness and probable error and other measures of variation. It would hardly seem that the student would get much of an idea of the important constants characterizing frequency distributions from this account. Index numbers of prices occupy over eighty pages and the author goes into much detail concerning index numbers and their application to prices. In connection with the discussions concerning the various formulas for calculating index numbers and the selection of a so-called "ideal" index, one cannot help but feel that the choice in this and other texts is based upon the personal preferences and notions of the authors rather than upon exact methods of reaching the best results. Perhaps this must be so in the very nature of the concept. In a later chapter index numbers of physical volumes are also developed and here again each writer has his own method of defining the index. It is quite interesting, but not very convincing, to read over the various papers by economists with the general title "A new index, etc." They have invented a greater variety of index number formulas than Heinz has of pickles. Almost one hundred pages are given to the analysis of time series through the measurement of trend, and seasonal and cyclical fluctuations. The usual ideas of moving averages (which by the way is nothing more or less than the smoothing of rough series by various graduation formulas) are developed in much detail and the resulting graduated curves are then subjected to fitting by some mathematical formula, usually one for a linear, quadratic, cubic or logarithmic curve.

Over one hundred and fifty pages are devoted to the measurement of relationship and for this purpose the theory of linear correlation is developed and applied to various types of economic statistics. Applications of correlations are made in the next chapter to the measurements of relationship between time series and in the following chapter the elements of non-linear correlations are developed including an account of the correlation ratio. Having developed the necessary concepts in the preceding chapters the author then makes an application to the problem of estimation together with the standard error of the estimate. The index of correlation based on logarithmic values of the variables also receives attention in this chapter. A short chapter on multiple and partial correlation is also included.

The chapters which have been put at the end dealing with elementary probabilities and the normal curve of error, statistical induction and the problem of sampling seem to the reviewer to be out of place. If they had appeared towards the beginning of the book the student would undoubtedly have a much clearer knowledge of what is behind the various formulas which he has been using throughout the book so freely. The fundamental conception of statistics as embodied in the relations between the theory of probability and random sampling should be developed early in any course in mathematical statistics before important applications are taken up. No student of any applied subject can make intelligent applications of his formulas until he has first grasped these fundamentals. It is a curious fact that most writers in non-mathematical departments of investigation plainly avoid this basic mode of treatment or development of the subject, most likely in order to dodge the mathematics. It is unfortunate that this situation persists in so many texts, because it cannot help but turn out a crop of statisticians who have acquired a lot of confidence in the manipulation of formulas which they do not understand, a confidence which apparently entitles them to deal with the applications in various lines of investigation—including economics and business. Appendix A is a good illustration of this point. It is entitled "The Method of Least Squares as Applied to certain Statistical Problems." Of course, if the student has not digested this Appendix, he cannot possibly have a clear understanding of many of the applications which the author has made in his book. This is particularly true in the matter of correlation and the measurement of relationships. The author has developed this subject fairly well and it is unfortunate that it is not a part of the body of the book and required of all students rather than something relegated to the end to be omitted if found too difficult—in other words if there is too much mathematics.

Appendix B is a glossary of symbols used in formulas in the development of mathematical statistics as applied to frequency distribution constants, correlations constants, curve fitting constants, and so on. It seems to the reviewer that all these symbols should have been defined and developed more fully in the body of the book instead of being shoved into a glossary or appendix at the end. A very good list of references appears at the end of the volume and also at the end of each chapter. In view of the fact that so much of the mathematics is left unexplained, it might be helpful to add to this list some of the more important elementary texts on algebra, calculus, mechanics and the theory of probability. This at least would show the reader where he could find some enlightenment on the fundamentals.

There is plenty of material in this book to keep a class busy throughout the year. It is full of ideas—perhaps there are too many ideas all brought together without sufficient emphasis on those which are more important and far reach-

ing. The reviewer leaves the book with the impression that the student will be taught *how* to do a good many things in this subject, but he is even more impressed with the fact that he will not know *why* he is doing many of them. In short the interpretation which he will give to his results will often not be based upon a clear understanding of the principles upon which they are built. It is very desirable to use personal judgment and intuition in matters of the kind treated in this book, however, it is also very easy to go too far in this direction and to be over confident in the interpretation of the final results as obtained from a formula or process which is not understood. Nothing short of a sound knowledge of mathematics through the calculus will enable the student to understand the statistical tools described in this book and if he undertakes advanced statistical investigations he must go very much beyond the calculus before he will really have an intelligent understanding of what he is doing.

The literary style is good and the descriptions of the mere processes are straight forward and clear. The charts and tables are set up in desirable form and the book printed on excellent paper. In many respects the text is encyclopedic and will serve as a first class reference book of processes applied to various kinds of statistics. The author is to be congratulated on having made a useful addition to statistical literature.

JAMES W. GLOVER.

Advanced Calculus. By F. S. WOODS. New York, Ginn and Company, 1926. ix+397 pages. Price $4.60

The proper content of an advanced calculus course will long be a debatable question. There are many advocates who maintain that it should consist solely of further applications of differentiation and integration. There are some who believe it ought to be a repetition of the first course with added emphasis at certain points but little additional material (it should not be advanced at all, but *elementary* calculus should be made advanced). There are fewer who believe that a second course in calculus should deal primarily with the fundamentals in which the student should be grounded such as limits, continuity, convergence, summation, etc.

The present textbook by Woods proceeds in the middle of the road between the first and third points of view, and only incidently takes care of the second. Such fundamental concepts as continuity, uniform convergence, uniform continuity, dominant functions, existence theorems for implicit functions and for differential equations are not avoided, but given extensive and detailed consideration. There is no dodging of issues here, though by some strange turn, the fundamental definition of a limit of a function of a single variable in (δ, ϵ) terms is overlooked, though the corresponding one for continuity of a function at a point is given. The class of functions treated are in the main continuous

with continuous derivatives, except at a finite number of points. Frequently the function is analytic and the power series development is used effectively. The student is receiving important grounding which will stand him in good stead in his scientific future.

It is assumed by the author that the student has had a thorough training in formal integration and the use of a table of integrals and that he has had practice in setting up the simple and multiple integrals connected with length, area, volume, mass, center of mass, moment of inertia, pressure, etc., usually given as applications of integration in a first course. In the present text the properties of these definite integrals are discussed and proofs given in many instances. This extends also to Green's theorem in two and three dimensions and to Stokes' theorem, including the introduction of vector notation and definitions. The application of partial derivatives to the differential geometry of space curves and surfaces is beautifully done. This is the outstanding feature of the many excellent presentations of the entire book.

Four chapters are devoted to differential equations with special reference to the Legendre and Bessel equations. An entire chapter is given over to the very important and interesting properties of Bessel's functions, including their asymptotic expansions in a later chapter.

The Gamma and Beta functions are treated briefly. Analytic functions of a complex variable receive a chapter of twenty-eight pages. The book closes with a final chapter on Elliptic integrals and functions.

The sets of problems at the close of each chapter are numerous, averaging a little over forty for each of the sixteen chapters of the book. Very few if any of these problems are of the "turn the crank" type. Their content, for the most part, is such as to lead the student on to a further development of the subject.

It is an unpleasant task for the reviewer to call attention to a departure of the author at one point from the high standard set by the text as a whole. One of the most striking examples of the persistence of an erroneous tradition in mathematics is the incorrect form of Duhamel's theorem which continues to find a place in our text books. On page 22 of the present book under review, we find this theorem as follows:

"If the sum of n positive[1] infinitesimals has a limit as n increases indefinitely and each infinitesimal approaches zero, that limit is unaltered by replacing each infinitesimal by its principal part."

In 1903, W. F. Osgood showed that this theorem is false. However, the tradition will not down. For I have a copy of Duhamel's *Elêments de Calcul Infinitesimal* (the source of this theorem) before me and find, on page 35, the following

[1] The author in reality means non-negative.

THÉORÈME: *La limite de la somme de quantités positives n'est pas changée, lorsqu'on remplace ces quantités par d'autres dont les rapport avec elles ont respectivement pour limite l'unité.* We find Gibson in an excellent text, *Elementary Treatise on the Calculus*, edition of 1919, has the identical statement of the incorrect form of the theorem in all its pristine beauty. Again this statement has made a recent appearance in Cohen's *Differential and Integral Calculus*, 1925, on page 275. Despite the fact that over twenty years ago Osgood gave a correct form for the theorem and others[1] have added further criticism, the false theorem appears in texts by eminent mathematicians.

Woods and others make a point of the non-negative character of the infinitesimals, whereas the "proof" and the examples by Woods on pages 22 and 23 show that it is the *boundedness* which is the essential element and not the likeness of sign of all the terms. It is this very quality which brings to grief Woods' immediate application of the theorem at the top of page 25. For example, if the arc under consideration in this application is a semi-circle with a horizontal diameter as base, all the required conditions are fulfilled, yet equations (4), (5) and (6) on page 25 become illusory. Except for certain definitions, sections 12 and 13 might very well be omitted from the present book without impairing its great value.

It is nevertheless true that the lack of an effective theorem of this kind is a distinct disadvantage. For example in section 59, page 140, on change of variable in a simple definite integral, various apologies are found necessary to justify the steps, such as "it is *usually* possible to split the interval (a, b) into regions in which the infinitesimals are always positive or always negative."

One may turn to W. F. Osgood's *Advanced Calculus* which appeared late in the year 1925 for comparison on this point. Despite the fact that Duhamel's theorem is not to be found in the index of Osgood's book, there are not less than eight applications of the theorem, (and at least a dozen other places where it might be applied) chiefly to problems in center of mass, attraction, surface integrals, polar coordinate integrals, spherical coordinate integrals, work, and change of variable in double and triple integrals. But Osgood does not introduce uniform convergence into his text, hence the application of his form of the theorem is ineffective as he himself states in several places.

The reviewer is so firmly convinced of the importance of a simple and correct form of the theorem (a simplified form of a theorem due to R. L. Moore) together with allied concepts, for the present and future development of the elementary parts of analysis, that he believes it ought to have a place in our elementary textbooks, certainly in advanced calculus books. Here, if you please, a tool can be placed in the hands of the student which in the beginning

[1] For complete references see my paper *A simple form of Duhamel's theorem and some new applications*, this MONTHLY, vol. 29 (1922), pp. 239–249.

may be used as a "crank" to obtain results, but which with greater mathematical maturity assumes a more and more definite meaning in the fundamental rôle which it plays in physical applications and in some advanced fields of analysis.

Other minor criticisms might be offered such as a bad and misleading figure on page 135 to represent a discontinuous function, and some inaccuracies of statement which will not be given detailed attention. But despite the vehemence of the reviewer's criticism of sections 12 and 13, he desires above all to point out that Woods has offered a distinct contribution to American mathematical textbooks in this excellent work on Advanced Calculus.

H. J. ETTLINGER.

An Introduction to Statistical Analysis. By G. G. CHAMBERS. New York, F. S. Crofts & Co., 1925. 254 pages.

According to the author this book represents the course in statistics which has been given for four semesters to first-year students in the University of Pennsylvania. It is supposed to be covered (apart from Chapter II or Chapter III) by the freshman in one semester of three hours per week. The applications are chiefly for students of biology and education. The book is extremely elementary, in fact too much so for university students. Practically the whole of the first chapter of twenty-two pages is employed in conveying to the student simple ideas of measurement, such as the length of a rod, or the weight and height of an individual. The final result obtained is that these measurements cannot be made exactly and that there will be a fluctuation from the true value which depends upon the smallness of the unit employed in taking the measures. The whole chapter could be condensed into two pages. This spirit of detail pervades the whole book and probably is a concession to the popular notion of the inability of most people to grasp elementary mathematical ideas and formulas. It would not be so surprising if the support of this notion came from a biologist or teacher of education but it is disappointing to find it coming from the pen of a mathematician. The second chapter deals with the most elementary examples on computation with approximations which one would expect to find in the text for a business college or in a "lightning calculator." The student is not supposed to know anything about logarithms so there is a chapter on this subject and another one on drawing simple graphs with much attention given to the rules proposed by the Joint Committee on Standards for Graphic Presentation. In chapter five a compilation of several pages of tables from original measurements is given and used as a basis for carrying out statistical problems developed in the remaining part of the book. Averages are discussed in chapter six and in chapter seven the ideas of dispersion and standard deviation are developed in much verbal detail. The chapter on cor-

relation is by far the best one in the book but the impression conveyed on reading it over—as well as the rest of the book for that matter—is that the author has left nothing for the student to think out for himself. Rules are given for each operation step by step just as one would direct a cook in preparing a dish. The final chapters on types of distributions and the theory of sampling would be more effective near the beginning of the book. The ideas are fundamental and should come to the attention of the student early in the course. The appendix contains a list of symbols and formulas given elsewhere in the book, also a five place table of common logarithms of numbers. The answers to every problem in the book are given in the appendix. The book is printed on good paper and the typing, style, tables and charts are well done.

The chief trouble with a book of this character is that after the student has completed it he is quite likely to believe that he knows much more about the subject than he really does. He ought to cover what is given in this book in four weeks instead of four months. The reviewer believes that statistics should not be approached in this manner in a university class in the department of mathematics. As a matter of fact the student should have freshman mathematics before he undertakes a course in elementary statistics which deals with measures of variation.

JAMES W. GLOVER.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) the titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Proceedings of the National Academy of Sciences, volume 12, no. 9, September 1926: "Postulates in the history of science" by G. A. Miller, 537–539; "On the zeros of functions associated with a linear system of the second order" by H. J. Ettlinger, 540–543; "Concerning irreducibly connected sets and irreducible continua" by H. M. Gehman, 544–547.

Science Progress, volume 20, no. 82, October 1926: "John Napier and the invention of logarithms" by G. A. Miller, 307–310.

Tokohu Mathematical Journal, volume 27, nos. 1, 2, July 1926: "Large primes having four consecutive quadratic residues" by A. A. Bennett, 53–57; "Applications of elliptic functions to the method of electrical images" by C. N. Wall, 176–188.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also

NOTE BY THE EDITORS: For the numerical example of the problem, other solutions are possible. For example, using the first pile alone, a cube may be obtained with the edge 10.573×5286; using the second alone, with the edge $10.573\times 30\times 653\times 71$; using both piles, with the edge $10.573\times 5\times 653\times 5286\times 71$.

3156 [3152; 1925, 481] Proposed by Otto Dunkel, Washington University.

From two fixed points, A and B of a given conic, two chords AC and BD are drawn intersecting on the fixed chord IJ of the same conic and determining another chord CD. Determine the envelope of the chord CD and a method for locating points on the envelope without the use of equations.

II. SOLUTION BY NATHAN ALTSHILLER-COURT, University of Oklahoma.

The four points A, B, C, D, determine a complete quadrangle inscribed in the given conic (O). Its diagonal points $E\equiv(AB, CD)$, $F\equiv(AC, BD)$, $G\equiv(AD, BC)$ determine a triangle which is self-polar with respect to (O), and since F lies on the line IJ, the polar EG of F will meet IJ in the conjugate F' of F with respect to (0) and will pass through the pole P of IJ. The pencil of rays $E-FGAD$ having E for its center is harmonic; hence the points $L\equiv(IJ, EAB)$ and $M\equiv(IJ, ECD)$ are harmonically separated by the points F, F'. Thus the line $CD\equiv ME$ joins the harmonic conjugate M, of the fixed point L with respect to the couple of conjugate points FF', to the projection E of F' from P upon the given line AB.

Starting with the point F of IJ we obtain the line ME. Now if we start with the point F', the roles of the points F, F' will be interchanged, while the points L, M will remain the same. Hence the line ME' joining M to the projection E' of F from P upon AB is another position of the variable line $CD\equiv ME$.

The lines PF, PF' are conjugate with respect to (0) because they project from the pole P of the line IJ two conjugate points F, F' of this line. As the point F varies on the fixed line IJ, the couple of conjugate points F, F' will describe an involution of points on IJ, and the point M will describe on IJ a range projective to this involution. Hence:

$$(M\cdots)\ \overline{\wedge}\ (FF'\cdots)\ \overline{\overline{\wedge}}\ P(FF',\cdots)\ \overline{\overline{\wedge}}\ (E'E,\cdots).$$

We have thus a projective one-to-two correspondence between the points of the two given lines, IJ and AB. Let L' be the conjugate of L with respect to the given conic (0) on the line IJ, and $H\equiv(PL', AB)$. The harmonic conjugate of the point L with respect to the couple LL' coincides with L; hence when the variable couple FF' of the line IJ coincides with the couple LL', the corresponding couple $E'E$ of AB coincides with the couple LH. Thus the common point L of the two bases IJ, AB is a united element of the two projective forms $(M\cdots)$ and $(E'E,\cdots)$. Consequently: *The two lines ME, ME' envelope a conic (M) tangent to the line AB.*

The lines ME, ME' determine on the conic (M) an involution of tangents of which the line IJ is the polar, and the point P is the pole. When the point F coincides with the point I, the conjugate F' of F with respect to (0) also coincides with I, hence the harmonic conjugate M of L with respect to F, F' will in its turn coincide with I. Thus from the point I of IJ only one tangent may be drawn to the conic (M). Similarly for the point J. Therefore the lines PI, PJ are the tangents to (M) at I, J. Thus: *The envelope (M) has a double contact with the given conic, the line IJ being the chord of contact.*

In the involution $(E'E,\cdots)$ on AB to the point L corresponds the point H, hence: *The conic (M) touches the line AB in the point H.*

Thus we have three tangents PI, PJ, AB to the conic (M) and their respective points of contact I, J, H. These elements are more than sufficient to generate the conic (M) either by points, or by lines, making use of Pascal's or Brianchon's theorems. For instance, the point of contact T of any tangent t to (M) may be found as follows: Let $X\equiv(t, PI)$, $Y\equiv(t, PJ)$, and $Z\equiv(IY, JX)$, then $T\equiv(t, PZ)$.

If the points I, J are conjugate imaginary we may use in this construction the line AB with its point of contact H and any other tangent which has been determined by means of the conic (0), the corresponding point of contact having been found by either Pascal's or Brianchon's theorem.

Also solved by C. T. BUNNEL, MICHAEL GOLDBERG, W. J. PATTERSON, and MABEL M. YOUNG.

3164 [3160; 1926, 47] Proposed by L. S. Dederick, University of British Columbia.

All values of a continuous variable between 0 and 10 are equally likely. What is the most probable number of values that must be selected at random before obtaining two that differ by more than 9?

Is the result the same if the phrase "most probable" is taken literally as if it has the common meaning as applied to errors, as likely as not to be exceeded.

Solution by the Proposer.

In order that any value shall be available as one member of a pair differing by more than 9, it must be less than 1 or greater than 9. The probability of this is $\frac{1}{5}$. The probability that out of n values exactly m shall be available is $C^n_m(\frac{1}{5})^m(\frac{4}{5})^{n-m}$. The probability that of these m available values exactly k shall be in the lower interval, 0 to 1, is $C_k^m(\frac{1}{2})^m$. If we arrange the available values in the order of magnitude of the *difference* between each one and the lower limit, 0 or 9, of the interval containing it, all orders are equally likely since all positions within the unit interval are equally likely. In this ordering, if no two values differ by more than 9, then all the values from the upper interval will precede all those from the lower. The probability of this is $k!(m-k)!/m!$ The probablity then that at least one pair shall differ by more than 9 is $1-[k!(m-k)!/m!]$ for any particular values of m and k. For an original list of n values the probability that the members of at least one pair shall differ by more than 9 is, therefore,

$$\sum_{m=2}^{n}\ \sum_{k=1}^{m-1} C^n_m\left(\frac{1}{5}\right)^m\left(\frac{4}{5}\right)^{n-m} C_k^m\left(\frac{1}{2}\right)^m\left[1-\frac{k!(m-k)!}{m!}\right]$$

$$= \sum_{m=2}^{n} C^n_m\left(\frac{1}{10}\right)^m\left(\frac{4}{5}\right)^{n-m}[2^m-1-m]$$

$$= \sum_{m=2}^{n} C^n_m\left(\frac{1}{5}\right)^m\left(\frac{4}{5}\right)^{n-m} - \sum_{m=2}^{n} C^n_m\left(\frac{1}{10}\right)^m\left(\frac{4}{5}\right)^{n-m} - \frac{n}{10}\sum_{m=2}^{n}\frac{(n-1)!}{(m-1)!(n-m)!}\left(\frac{1}{10}\right)^{m-1}\left(\frac{4}{5}\right)^{n-m}$$

$$= \left[1-\left(\frac{4}{5}\right)^n-\frac{n}{5}\left(\frac{4}{5}\right)^{n-1}\right]-\left[\left(\frac{9}{10}\right)^n-\left(\frac{4}{5}\right)^n-\frac{n}{10}\left(\frac{4}{5}\right)^{n-1}\right]-\frac{n}{10}\left[\left(\frac{9}{10}\right)^{n-1}-\left(\frac{4}{5}\right)^{n-1}\right]$$

$$= 1-\frac{n+9}{10}\left(\frac{9}{10}\right)^{n-1}.$$

If we call this $f(n)$ and set $f(n)=\frac{1}{2}$, we obtain the most probable value of n in the sense, the value as likely as not to be exceeded. By trial we find $f(16)=.4853$ and $f(17)=.5183$. Hence, the most probable number of values is between 16 and 17. If we select 17 values it is more likely than not that two of them will differ by more than 9; if we select 16 values it is less likely.

The most probable value of n, however, in the literal sense is quite different. Let

$$\phi(n)=f(n)-f(n-1)=\frac{n+8}{10}\left(\frac{9}{10}\right)^{n-2}-\frac{n+9}{10}\left(\frac{9}{10}\right)^{n-1}$$

$$=\left(\frac{9}{10}\right)^{n-2}\left(\frac{n+8}{10}-\frac{9n+81}{100}\right)=\frac{n-1}{100}\left(\frac{9}{10}\right)^{n-2}.$$

This is the probability that the nth value selected differs by more than 9 from one or more of the first $n-1$, but that none of these differs from another by more than 9, i. e., the nth choice is the first successful one. The function $\phi(n)$ continues to increase as long as $\phi(n+1)>\phi(n)$, that is

$$\frac{n}{100}\left(\frac{9}{10}\right)^{n-1}>\frac{n-1}{100}\left(\frac{9}{10}\right)^{n-2}$$

or $n<10$. We have then $\phi(10)=\phi(11)$ and this value is greater than any other value of $\phi(n)$. The probability, therefore, that the first successful choice is the 10th or 11th is greater than that of any preceding or following. In this sense 10 and 11 are the most probable values of n.

Also solved by Michael Goldberg.

3165 [3161; 1926, 47] Proposed by C. K. Robbins, Purdue University.

A non-degenerate conic and its tangents are inverted in $x^2+y^2=1$.

Show that the locus of the centers of the circles, which are the inverses of the tangents to the conic, is a conic.

Solution by W. L. Ayres, University of Pennsylvania.

Given the conic

$$x(a_{11}x + a_{12}y + a_{13}) + y(a_{21}x + a_{22}y + a_{23}) + (a_{31}x + a_{32}y + a_{33}) = 0, \tag{1}$$

where $a_{ij}=a_{ji}$, and where the determinant Δ of the coefficients is not zero, the equation of the tangent at x_1, y_1 is obtained by writing these coördinates for x, y within the parentheses. If in the resulting equation we now replace x, y by $x/(x^2+y^2)$, $y/(x^2+y^2)$, we obtain the equation of the inverse of the tangent, a circle. If $(\bar{x}, \bar{y})$ is the center then

$$\begin{aligned} -2\rho\bar{x} &= a_{11}x_1 + a_{12}y_1 + a_{13}, \\ -2\rho\bar{y} &= a_{21}x_1 + a_{22}y_1 + a_{23}, \\ \rho &= a_{31}x_1 + a_{32}y_1 + a_{33}. \end{aligned} \tag{2}$$

Since x_1, y_1 satisfy (1) we have at once

$$2x_1\bar{x} + 2y_1\bar{y} - 1 = 0. \tag{3}$$

From (2) we have

$$\begin{aligned} \Delta x_1 &= (-2D_{11}\bar{x} - 2D_{21}\bar{y} + D_{31})\rho, \\ \Delta y_1 &= (-2D_{12}\bar{x} - 2D_{22}\bar{y} + D_{32})\rho, \\ \Delta &= (-2D_{13}\bar{x} - 2D_{23}\bar{y} + D_{33})\rho, \end{aligned} \tag{4}$$

where D_{ij} is the cofactor of a_{ij} in Δ. Inserting (4) in (3) we obtain the equation of the desired locus, the conic,

$$2\bar{x}(-2D_{11}\bar{x} - 2D_{21}\bar{y} + D_{31}) + 2\bar{y}(-2D_{12}\bar{x} - 2D_{22}\bar{y} + D_{32}) - (-2D_{13}\bar{x} - 2D_{23}\bar{y} + D_{33}) = 0.$$

Since
$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \Delta a_{33},$$

the locus is of the parabolic type if a_{33} is zero; elliptic, if Δ and a_{33} have the same sign; hyperbolic, if they have opposite signs.

Also solved by Theodore Bennett and Michael Goldberg.

3166 [3162; 1926, 47] Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

Give an integral solution of $z^2=ay^2+x^2$

Solution by William E. Routh, West Allis, Wis.

It is here assumed that a is an integer and that integral values of x, y and z satisfying $z^2=ay^2+x^2$ are desired. From this equation, we have

$$ay^2 = (z + x)(z - x); \; a^2y^2 = a(z + x)(z - x). \tag{1}$$

Since equation (1) must be satisfied by integers, it is evident that $z+x$ and $z-x$ must be integers of the form

$$z + x = mr^2t, \; z - x = ns^2t, \tag{2}$$

where m and n are integers such that $mn=a$, and r, s, and t are arbitrary integers. Then from (1) and (2),

$$y = \pm rst, \tag{3}$$

where the negative sign may be disregarded. Then

$$x = (mr^2 - ns^2)t/2, \quad z = (mr^2 + ns^2)t/2, \tag{4}$$

and the given equation is identically satisfied. Hence the integral solutions of that equation are given by (3) and (4), where t is even, or t is odd and mr^2+ns^2 is even.

Also solved by R. P. Agnew, H. C. Bradley, P. A. Caris, H. E. Trefethen and E. E. Whitford.

3171 [3167; 1926, 104]. Proposed by H. Betz, University of Missouri.

Consider a particle moving in a straight line in the plane of an ellipse, inside the ellipse in such a manner that whenever it strikes the boundary of the ellipse it is "reflected" just as a ray of light would be, striking a mirror. The particle will, therefore, travel indefinitely often back and forth across the ellipse.

Let its path be referred to as its orbit. Now if, initially, the orbit passes through one focus of the ellipse, it will, in accordance with an elementary property of the ellipse, pass through the other focus also, and so on, indefinitely. Show that the orbit will converge, in the limit, to the major axis of the ellipse.

SOLUTION BY THEODORE BENNETT, University of Illinois.

We assume the ellipse in the form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1,$$

the foci of which are $F(-ae,0)$, $F'=(ae,0)$. Let the particle start at $P=(\xi,\eta)$ on the ellipse, pass through F to Q, where it is reflected through F', striking the ellipse again at $P'=(\xi',\eta')$. Let us say that the operation T has moved the particle from P to P'. From geometrical considerations it is evident that the analytic expression of T is a bilinear relation between ξ and ξ' (but *not* between η and η'). We now find this relation. Eliminating y between the equations of the line PF and the ellipse we have a quadratic in x, one root of which is known to be ξ; the other root, the abscissa of Q, is found to be

$$X = -a\frac{\alpha\xi + a}{\xi + a\alpha}, \text{ where } \alpha = \frac{1+e^2}{2e}. \tag{1}$$

Similarly, starting from Q, we find the abscissa of P' to be

$$\xi' = a\frac{\alpha X - a}{X - a\alpha}. \tag{2}$$

Combining these results, we find the analytic expression of T to be

$$\xi' = a\frac{(1+\alpha^2)\xi + 2a\alpha}{2\alpha\xi + a(1+\alpha^2)}. \tag{3}$$

Since the self-corresponding points of T are $\xi=a$ and $-a$, we know that (3) may be written

$$\frac{\xi'-a}{\xi'+a} = k\frac{\xi-a}{\xi+a}, \text{ or } \xi' = a\frac{(1+k)\xi + a(1-k)}{(1-k)\xi + a(1+k)}. \tag{4}$$

By comparing (3) and (4) we find that

$$k = \left(\frac{1-\alpha}{1+\alpha}\right)^2 = \left(\frac{1-e}{1+e}\right)^4.$$

Now $\xi^{(n)}$, the abscissa of the point obtained from P by applying T n times, is obtained from (4) by replacing ξ' by $\xi^{(n)}$ and k by k^n. Since $0<e<1$, we have also $0<k<1$, whence $k^n \to 0$ as $n \to \infty$. Therefore, we see from (4) that $\xi^{(n)} \to a$ as $n \to \infty$, which proves the theorem.

NOTE BY OTTO DUNKEL, Washington University. This may also be proved geometrically by aid of the symmetry and convexity of the ellipse and of the focal property stated in the problem.

Also solved by W. J. PATTERSON, H. S. UHLER, and the PROPOSER.

3173 [3169; 1926, 104]. Prosposed by C. C. Camp, University of Illinois.

Two parallel vertical walls stand upon horizontal ground. A ladder of length a has its foot at the bottom of the first wall and leans against the second. A ladder of length b has its foot at the bottom of the second wall and leans against the first. What must be the distance between the walls so that the ladders will cross at a height h? When is a solution possible?

SOLUTION BY W. J. PATTERSON, Western University, Can. and OTTO DUNKEL, Washington University.

Let x be the distance between the walls, and let the ladders of lengths a and b (both finite) meet their respective walls at the heights y and z. Let t be the distance of the foot of h from the foot of b. Then by similar triangles $t/x=h/z=(y-h)/y$. Hence

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{h}, \qquad h>0. \tag{1}$$

Since $y<a$ and $z<b$, we have as a necessary condition for a real solution

$$\frac{1}{a}+\frac{1}{b}<\frac{1}{h}\cdot \tag{2}$$

It will be shown that (2) is a sufficient condition for a real solution.

If $a=b>2h_0$ then $y=z=2h_0$, and $x=(b^2-4h_0^2)^{1/2}$.

If $a>b$ and if a real solution is possible, then (1) shows that $2h$ is the harmonic mean of y and z, and hence $h<z<2h<y$. In order to determine x we have an additional equation

$$y^2-z^2=a^2-b^2=k^2, \tag{3}$$

and then from (1) and (3) we have

$$f(z)=z^3(z-2h)+k^2(z-h)^2=0. \tag{4}$$

From (4) it is obvious that $f(z)$ is positive if $z\leqq 0$ or if $z\leqq 2h$; moreover $f(h)=-h^4$. Hence there is at least one real root between 0 and h and at least one between h and $2h$. Now $f'(h)=-2h^3, f'(2h)=8h^3+2k^2h$, and since $f''(z)=12z(z-h)+2k^2$, it follows that $f'(z)$ has one and only one root in the interval $h,2h$. Thus $f(z)$ in this interval decreases from a negative value to a minimum and then increases and vanishes only once.

It must now be shown that this root z is less than b. We have after a slight change

$$f(b)=a^2(b-h)^2-h^2b^2, \tag{5}$$

while from (2) it follows that $a(b-h)>hb$. Hence $f(b)>0$, and since $b>h$, it follows that b is greater than this root z. Hence x is uniquely determined by $x=(b^2-z^2)^{1/2}$, and y is then found from (1). This pair of values of y and z satisfy (1) and (4) and hence must also satisfy (3). It now follows that

$$y^2=b^2-x^2+a^2-b^2=a^2-x^2.$$

This concludes the proof that this root and this one alone furnishes a solution.

We shall now consider the remaining roots of (4). If $2k^2-3h^2\geqq 0, f''(z)$ has a double root or imaginary roots; it then follows that $f(z)$ has a pair of imaginary roots. If $2k^2-3h^2<0, f''(z)$ has two real roots between 0 and h and at either one of these points $6f'(z)=(2z-h)f''(z)-4(3h^2-2k^2)z-10k^2h<0$. Since the maximum and minimum values of $f'(z)$ are negative, it must have two imaginary roots, and it then follows that $f(z)$ has a pair of imaginary roots. Thus the only other real root lies between 0 and h. The corresponding value of y for this smaller root is negative by (1). To this smaller root corresponds the case where the ladder a meets its wall below the horizontal, and the prolongations of the two ladders meet outside of the x interval. Here again a condition must be imposed upon a, b, h to obtain a real solution for this situation.

All of these results are easily obtained by an examination of the two hyperbolas in (1) and (3). It may be shown that if z is the greater root

$$2h - \frac{k^2}{8h} < z < 2h - \frac{k^2 h}{8h^2 + k^2}$$

The solution of the case $a=40$, $b=25$, $h=15$, is given in the solution of 2836[1922, 181].

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

On Sunday, March 20th, 1927, will fall the two-hundredth anniversary of the death of Sir Isaac Newton. A meeting to celebrate this Bicentenary will be held on March 18, 19, under the auspices of the Yorkshire Branch of the Mathematical Association, at Grantham, the place associated with Newton's boyhood and with some of his most fundamental discoveries. There will be a scientific meeting with addresses on Newton's work by Sir J. J. Thomas, F. R. S. (Master of Trinity), Sir F. Dyson, F. R. S. (Astronomer Royal), Dr. J. H. Jeans (Sec. R. S.), Professor H. Lamb, F. R. S. After a pilgrimage to Woolsthorpe Manor House (Newton's birth place), there will be a visit to Stoke Rochford, where Mr. Christopher Turnor will speak on "Newton's Countryside."

Professor W. L. Hart has been appointed chairman of the Department of Mathematics in the College of Science, Literature, and the Arts at the University of Minnesota to succeed Professor W. H. Bussey who resigned the chairmanship so that he might devote more time to his duties as Assistant Dean and as Editor-in-Chief of the American Mathematical Monthly.

Dr. L. D. Ames has been appointed professor of mathematics at the University of Southern California.

Assistant Professor H. G. Harp of Wittenberg College is spending this academic year in graduate study at Ohio State University.

Dr. Harold R. Phalen, formerly associate professor of mathematics at the Armour Institute of Technology, has been appointed professor of mathematics at St. Stephen's College, Annandale-on-Hudson, New York.

Miss Una Rudd, instructor in mathematics in Simmons University is on leave this year, doing graduate work in the University of California.

Professor C. E. Horne, formerly Dean of the College of Agriculture at Mayaguez, Porto Rico, has been appointed professor of mathematics at the University of Porto Rico.

At Trinity College, Hartford, Connecticut, Mr. Howard T. Engstrom has been appointed instructor in mathematics. Assistant Professor Frederick J. Burkett is absent on leave for graduate study.

Mr. Alva M. Tuttle has been appointed instructor in mathematics at Heidelberg College, Tiffin, Ohio.

At the Case School of Applied Science, Cleveland, Ohio, Mr. Richard S. Burington has been appointed instructor in mathematics. Professor Max Morris is on leave of absence. Mr. E. M. Justin has returned to the department after a year of study at the University of Chicago.

Miss Esther Pearce, M. A. University of Michigan, has been appointed instructor in mathematics at Georgia Wesleyan College, Macon, Georgia.

Mr. Joseph H. Kusner has been appointed instructor in mathematics at the University of Florida.

At Ohio Wesleyan University, Mr. Sidney A. Rowland is professor of mathematics and head of the department. Earnest Clare Bower, formerly associate astronomer at the U. S. Naval Academy, has been appointed assistant professor of astronomy and mathematics.

At the University of Buffalo, Charles D. Gregory and C. Wallace Munshower have been appointed instructors in mathematics.

Professor Wilfred H. Sherk, of the University of Buffalo, has been elected President of the Association of the Teachers of Mathematics in the Middle States and Maryland.

Miss Lois Dicks (A. B. University of South Carolina), has been appointed instructor in mathematics at Coker College, to succeed Miss Ellen C. Stokes who resigned to accept a similar position in the New York State Teacher's College.

At Colgate University, Professor A. B. Stewart of the department of mathematics will retire from active teaching at the close of the present academic year. Professor A. W. Smith has leave of absence for the second semester of the present year.

At Kansas State Agricultural College, Mr. T. I. Porter, Mr. H. M. Stewart and Miss Irene Eldridge have been appointed instructors in mathematics; Mr. W. C. Janes, and Miss Thirza A. Mossman have been promoted from the rank of instructor to that of assistant professor; and Mr. C. F. Lewis, Mr. W. H. Lyons and Miss Emma Hyde from the rank of assistant professor to that of associate professor. Leave of absence has been granted to Professor A. E. White for study at the University of Iowa.

Mr. Dwight F. Gunder has been appointed instructor in mathematics at the Colorado Agricultural College to succeed Homer W. Craig who has accepted a fellowship at the University of Wisconsin.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 6-7, 1927.
Twelfth Annual Meeting, Nashville, Tenn., December, 1927.
The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 6-7.

INDIANA, De Pauw University, May 6-7.

IOWA, University of Iowa, 29, 30.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, May.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, May.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November.

ROCKY MOUNTAINS, Colorado College, April 22-23.

SOUTHEASTERN, March.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATION: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 2, FEBRUARY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE MAY MEETING OF THE MINNESOTA SECTION

The regular spring meeting of the Minnesota Section was held at St. Olaf College, Northfield, Minnesota, on Saturday, May 22, 1926. Professor M. A. Nordgaard, St. Olaf College, the Chairman of the Section, presided.

The attendance was 45 at the dinner, 60 at the regular session, and included the following 21 members of the Association: W. O. Beal, A. Bogard, R. W. Brink, W. H. Bussey, J. M. Earl, C. H. Gingrich, Gladys Gibbens, W. L. Hart, D. Jackson, R. A. Johnson, D. N. Kingery, W. H. Kirchner, L. W. Moench, W. D. Morgan, M. A. Nordgaard, G. C. Priester, Sister Alice Irene, Ella Thorp, A. L. Underhill, Marion B. White, L. Winkelman.

The following officers were elected for the coming year: Chairman, INEZ RUNDSTROM, Gustavus Adolphus College, St. Peter, Minnesota; Secretary, A. L. UNDERHILL, University of Minnesota; an Executive Committee consisting of the Chairman, the Secretary and M. A. NORDGAARD, A. BOGARD, College of St. Teresa, Winona, Minnesota, and SISTER CLAUDETTE, St. Joseph College, St. Joseph, Minnesota.

A motion was passed expressing the appreciation of the Section for the hospitality of St. Olaf College.

The 1927 meeting will be held May 21 at Gustavus Adolphus College, St. Peter, Minnesota.

The following five papers were read:

1. "The correlation ratio," by Professor DUNHAM JACKSON, University of Minnesota.

2. "On the determination of the last payment in an amortization process," by Professor W. L. HART, University of Minnesota.

3. "An earlier place for the calculus in the curriculum," by Professor M. A. NORDGAARD, St. Olaf College.

4. "Kepler's and Newton's laws," by Professor C. H. GINGRICH, Carleton College.

5. "A mathematical survey course," by Professor R. W. BRINK, University of Minnesota.

Abstracts of papers, numbered as in the above list of titles, follow.

1. It is well known that the correlation ratio, while of primary importance when used with discretion in the analysis of statistical data, is in the last analysis ordinarily incapable of exact definition for a finite set of data, inasmuch as the definition depends essentially on class intervals which are more or less arbitrary, or else on a representative curve which is equally arbitrary. This paper deals with the elementary properties of the determinate correlation ratio corresponding to an idealized continuous frequency distribution.

2. Let $\$A$ be a debt which is to be discharged, principal and interest at the rate i per period included, by payments of $\$R$ at the end of each interest period. Let the equation

$$A = R(a_{\overline{n}|} \text{ at } i) \tag{1}$$

be solved for n by interpolation in ordinary annuity tables, and suppose that the solution is $n=k+t$, where k is an integer and $0<t<1$. Then, in discharging the debt, it is necessary to make k payments of $\$R$, and a smaller final payment at the end of $(k+1)$ periods. Professor Hart showed that the final payment is *exactly* $\$tR$. This fact[1] has escaped mention in current American college texts on the mathematics of investment. This property of the solution $n=k+t$ of equation (1) obtained by interpolation is *not* possessed by the solution which is obtained by solving (1) as an exponential equation in n.

3. Mr. Nordgaard's paper, after short discussion of the growing practice of giving an introductory course in calculus to college freshmen, gave a brief of the reasons for this departure. He also gave a synopsis of the events since 1900 which in France, Germany, and England have led to the inclusion of elementary calculus in the secondary schools and propounded the question whether it would not be feasible for us in the United States to do the same.

4. The three well-known Kepler's laws of planetary motion were stated, and from the mathematical form of these laws and the development of them, the several implications, central force, law of inverse squares, universality, pointing to Newton's law of gravitation, were mentioned.

---

[1] For a proof of it, see E. Foerster, *Politische Arithmetik*, Walter De Gruyter & Co., Berlin, p. 48.

5. The customary sequence of freshman courses in mathematics is well designed to develop the technical skill of students who intend to continue their mathematical studies. But since a large majority of students in mathematics will have no need for this technique in their later work, it would seem to be advisable to offer them, in colleges of liberal arts, a purely cultural course in mathematics which would emphasize the ideas and methods of mathematics, rather than technical drill. This course should be reserved strictly for students who have no intention of continuing in mathematics, and, for this and other reasons, should probably be open only to students who have completed the freshman year. This is in marked contrast to the practice at the University of Montana,[1] where the attempt is made to combine a cultural course with a course required as a preliminary to any other work in mathematics. It is rather in harmony with the earlier suggestions of Professor Lennes.[2] An outline for the course was suggested.

A. L. UNDERHILL, *Secretary*.

SUCCESSIVE GENERALIZATIONS IN THE THEORY OF NUMBERS[3]

By E. T. BELL, California Institute of Technology

1. Contrasted with the laboriously slow evolution[4] of the complex number system now current in analysis, the corresponding development in arithmetic proper has been extraordinarily rapid. Possibly all the scientific progress of the race from prehistoric times to the Nineteenth Century is reflected in the successive extensions and perfections of the usual number system of analysis, culminating in the complex variable and hypercomplex numbers, for each successful attempt to think clearly about nature has left its impress on mathematical analysis and therefore ultimately upon the numbers which analysis subjugates and fertilizes.

The dim beginnings of this long evolution are apparently lost beyond hope of recovery, unless it may be that psychology shall some day find a necessary basis for any self-consistent thinking, and we can only speculate on what induced the first mathematical philosopher to imagine that he knew what he was talking about when he glanced over his offspring and his bananas and

[1] N. J. Lennes, *A new type of freshman course in mathematics*, this Monthly, vol. 33 (1926), pp. 307–315. See also N. J. Lennes, *A Survey Course in Mathematics* (Harper and Bros., 1926).

[2] N. J. Lennes, *Mathematics for culture*, Educational Review, 1914.

[3] A lecture given by invitation of the Mathematical Association of America at the joint meeting with the American Mathematical Society, Columbus, Ohio, September 8, 1926.

[4] Readers of the Monthly in those States of the Union where "evolution" is anathema may substitute for this word "abracadabra," provided they define the latter sensibly.

ejaculated "36". It is conceivable that whatever this nameless but immortal primate thought he was concerned with would bear but little resemblance to the esoteric formulation of 1, 2, 3, · · · achieved by Frege and Russell with their "class of all classes equivalent to a given class"—an abstraction worthy of a Hindu mystic and yet a practical thing.

2. We have alluded to this ethereal definition, which not infrequently provokes the mathematically irreverent to ribald scorn, because it is precisely the concept of class inclusion which Dedekind, in a single flash of penetrating genius, perceived to be the essence of the theory of arithmetical divisibility. The connection, if any, between the Frege-Russell definition and the constant recurrence of the rudiments of the abstract algebra of classes and relations in all theories of algebraic numbers, and in fact in many other theories styled arithmetical by their originators, is not immediately apparent, and it might well repay closer attention.

3. The "numbers" with which much of modern arithmetic is concerned are far removed from the 1, 2, 3, · · · which custom deludes us into thinking we comprehend; yet, if Kronecker's vision of the potential arithmetization of all mathematical analysis be not an illusion, it must be possible to trace the class concept in the theory of numbers from its taproot in 1, 2, 3, · · · to its latest efflorescence in the theories of Prüfer, v. Neumann, and Speiser. This is a task which may be suggested to the mathematically minded philosopher and to the philosophically inclined mathematician, as a respite from their nightmares to circumscribe the ever-shifting quantum theory or sound the bottomless bog of the transfinite.

The unravelling of this tangled skein of pure logic and modernized numbers might well exercise the ablest logician, and it seems strange that mathematical fundamentalists (in the reputable sense of those who busy themselves with scientific foundations) should have neglected arithmetic, whose underlying structure is seemingly identical with that irreducible bed-rock which Boole called "laws of thought," and which today is renovated by, and associated with, the school of Whitehead and Russell, for the transient flux of theories "thrust into the world before their time but half made up."

The foundations of geometry have had their share of attention. Few creative philosophers of this generation would deny that the critical insight gained from a postulational examination of geometry has clarified their outlook on time no less than on space. Is it too much to hope that a like scrutiny of modern arithmetic will also yield its rich reward in a clearer perception of thought itself?

4. We have mentioned the amazing speed with which modern arithmetic reached maturity once the need for an extension of the classic theory of numbers became apparent; the lifetime of a single man might have spanned it all.

Before outlining briefly a few salient characteristics of this explosive growth, we may call attention to a distinction in the abstract natures of the respective generalizations of "number" sought and attained in the domains of analysis and arithmetic.

A true generalization Θ of a theory T would include T as an instance, either by appropriate specialization of the primary elements ("relata") of the theory or by a specific interpretation of the abstract relations between the elements, and T should not similarly include Θ, since otherwise T, Θ would be abstractly identical. The emphasis of the generalization may be either on the relata or on the relations, or it may be distributed over both. Instances of all these possibilities occur in the successive generalizations which finally yielded the field of complex numbers and later the several systems of hypercomplex numbers of modern algebra. Geometry also presents numerous examples.

Incidentally it may be pointed out that generalization and abstraction are radically different processes, with abstraction on the distinctly lower level. In abstraction we exhibit the structure of a theory devoid of everything but sufficient "marks" (both of relata and of relations) to show forth that structure. In generalization we construct a theory which implies the theory generalized but which is not implied by that theory.

There exists no true generalization of the theory of the natural numbers $1, 2, 3, \cdots$. Fragments only of what is known as the classic theory of numbers have been generalized in the above sense, which is usually, by common consent, taken to be substantially the only one worthy the title of generalization. Nor has abstraction, a useful first step toward generalization, been attempted in any systematic manner in arithmetic.

The distinction between the development of the numbers of analysis and those of arithmetic to which we referred a moment ago is this: in the arithmetical generalizations the chief emphasis seems to have been on the relations. Thus, in order to reach a workable theory of divisibility Dedekind was compelled to turn his back on intuition and to imagine a wider relation—that of class inclusion—which is indeed a true generalization of the concept of divisibility. In this he seems to have been followed by subsequent writers, certainly by Kronecker, some of whose theories at first sight bear but little resemblance to his.

There is another type of generalization which, however, is too hazy to be worth pursuing. It may be asked, why draw a distinction between the complex numbers of analysis and those of arithmetic? Are not the latter included in the former? The answer is of course affirmative, but it must be qualified by the obvious remark that arithmetic does not begin until integral elements are created—until atoms and molecules, as it were, each with a recognizable

individuality of its own, emerge from that chaotic smudge which analysis calls the continuum. Attempts to unite the continuous and the discrete in unholy wedlock have proved barren so far as arithmetic is concerned. If such efforts are not contrary to nature they certainly defy history. This of course does not refer to the far reaching applications of analysis to arithmetic—a totally different thing which does not concern us here.

5. In tracing the evolution of the theory of algebraic numbers it is customary to go back at least as far as Euler and Lagrange, both of whom frequently used several of the simpler processes of the modern theory, but neither of whom can be said to have shown any evidence that he suspected the existence of a new universe behind the few curious phenomena which caught his passing attention.

A clearer line of descent can be traced from Gauss. The usual historical reference to Gauss' connection with the theory of algebraic numbers points to his introduction of the complex integers $a+bi$ in his profound study of the law of biquadratic reciprocity.

It is perhaps significant, as has often been observed, that the arithmetical theory of complex numbers is not a necessity for the object which Gauss had in view, but merely a convenience. Can the like be maintained with regard to any existing application of the theory of algebraic numbers and ideals to the theory of the rational integers? And if so, precisely what is gained for rational arithmetic by the introduction of algebraic numbers or ideals beyond a certain compactness of statement?

Without here venturing an opinion on the last questions, we shall point out merely that the germ of the modern theory of ideals is implicit in the classic work of Gauss much earlier than in the memoirs on biquadratic residues. It appears in fact almost at the beginning of the *Disquisitiones Arithmeticae* in the familiar notion of congruence: two rational integers a, b are called congruent with respect to a third rational integer m when their difference is divisible by m, and this is written: $a \equiv b \bmod m$.

Let us examine for a moment this epoch-making definition which originated with Gauss. It yields much more than a suggestive notation pointing the way to innumerable fruitful analogies with the theory of algebraic equations. It does two things, each of them fundamental for the modern theory of numbers.

First, it segregates the infinity of rational integers into a finite number m of classes, two integers being put into the same or different classes according as they are congruent or incongruent modulo m.

This principle of reflecting the properties of an infinite totality of elements in a finite set of suitably chosen representatives permeates all arithmetical theories since Gauss. We have grown so accustomed to its use that we often take the principle for granted and perhaps fail to ascribe due credit to it as a

creative device. Once it has been applied in any given instance we see how inevitable has been the application. The impulse to make use of the obvious is however one of the rare things in mathematics.

An apposite illustration of this reluctance to use what lies close at hand is afforded by the very theories we are about to describe. A Dedekind ideal is an infinite linear set of a particular kind, yet it does not seem to have occurred to anyone until very recently that this infinite totality can be replaced with profit by a representative finite set. That such is apparently the case is evident from the "ideal systems" introduced by Speiser (1926) in his theory of ideals for the Dickson arithmetic of an algebra.

The second epochal conquest achieved by Gauss' definition of congruence is equally elementary, and again it is one of those obvious things which are seen for the first time only by the eye of genius. The statement $a \equiv 0 \mod m$ can be read either "a is divisible by m" or "a is a member of the class of integer multiples of m," and it is the second of these that generalizes to yield the germs of the concepts of divisibility of ideals and of congruence with respect to an ideal modulus.

6. An outstanding difficulty, for which no comprehensive method of attack has been suggested, in the arithmetization of a given theory is the discovery of the correct twist to be given to the familiar relations of common arithmetic in order that the new relata shall yield relations abstractly identical with the particular relations being considered between rational integers. For example, how shall the concept of greatest common divisor be modified so that it may be significant for elements between which order relations (relations of greater or less) are without meaning?

We may suggest that a useful first step is the abstraction of the fundamental relations of common arithmetic, that is, their restatement in terms of "marks," and the comparison of the result with the like abstract formulations of such relations in the theory under investigation as are already well known and interesting. This implies as a preliminary the abstract formulation of the postulate systems of both theories.

In this project categorical systems of postulates or other technically refined sets are beside the point. What is wanted in each case is a workable set of postulates which will reveal at a glance those properties of the elements considered which are of mathematical importance and which it is desirable to abstract and, if possible, generalize. In short, the familiar must not be tortured out of all recognition into the bizarre in order to gain a doubtful logical perfection according to the prevailing whim of the moment among expert postulate makers, otherwise only an impossibly complicated machinery is acquired at the expense of mathematical power. If it be possible by selecting as basic some unusual property of the elements under consideration to reduce the

number of postulates defining them by one or two, the astute postulationalist will doubtless do so. The wily mathematician, on the other hand, with a different object in view, knows better than to blunder on his own scaffold and hang himself with his own rope. It is not here a question of right reasoning or of wrong, but of scientific expediency.

7. We have referred to the arithmetization of a theory, and although the sense in which the term is here used is plain by implication, it may be well to consider it briefly so that there shall be no misunderstanding.

In the first place "arithmetization" is used in what may appear to be a a broader meaning than that of Kronecker but which in the end may be formally equivalent to his.

Kronecker wished among other things to replace mathematical analysis by chains of finite operations upon the positive rational integers alone, and he showed specifically how the calculus of algebraic numbers can be included in such a program. This great project of Kronecker's, the subject of a lifelong dispute with Weierstrass who was of a diametrically opposite scientific temperament, is today of fresher interest than it has been since Kronecker's death, in view of the revolutionary work of Brouwer and his pupils. It begins to appear that Kronecker's prophecy to Weierstrass that, if he (Kronecker) himself should lack the time to push the matter through to a conclusion, his successors would, and see to it that analysis is founded rigorously on a strict arithmetical basis will be fulfilled.

Kronecker's program has never been cordially received, especially by the analysts nor, what is rather strange, by the professed prophets of mathematical rigor. The latter have objected to a thorough overhauling of their wares (including the Cantorian transfinite) on the ground that such a procedure would belong to metaphysics rather than to mathematics, and that rigor as pure logic demands it is a fantasy with which rigor in mathematics need have no traffic. Rigor, in other words, for some who dislike Kronecker's "mathematical anarchy," is on a par with virtue—a vaguely relative term whose intensity depends only upon the taste or distaste of the user. It is little less than sardonic justice that the present distress of the body analytic should have been induced by the execrable metaphysics which analysis swallowed in transfinite doses over Kronecker's protest.

Kronecker's arithmetization would make the positive rational integers central and supreme. The emphasis seems to be on the elements rather than on the relations between them, although the latter also are taken account of in the rejection of irrational and other interminable processes. If the emphasis be shifted from the relata to the relations, we may perceive another aspect of arithmetization, which is as follows.

Suppose it is possible from the given elements of a theory to construct new elements between which there subsist relations abstractly identical with certain relations in the theory of the rational integers. Then, to the extent of these relations, the theory of the new elements, whatever their nature, is abstractly identical with arithmetic. The problem in any case is to construct the new elements or to reconstruct the relations between the original so that an abstract arithmetical structure shall emerge.

The work of Dedekind abounds in striking illustrations of several of the points which we have mentioned. His early expositions are particularly valuable to present day workers because he has spared no pains to show us the subtle trains of reasoning which led him to his inspirations. Truly great, this mighty genius, unlike some others, was above clearing away the scaffolding before permitting the less gifted to view his perfected work. His preliminary bulletins on the theory of ideals are an unsurpassed laboratory in the method of mathematical discovery.

Returning for a moment by way of illustration to the G. C. D., let us observe how it is recast to make it capable of extension to domains in which order relations are meaningless. Evidently the following definition preserves all the properties of the G. C. D. with which we are familiar in common arithmetic, and yet it makes no explicit mention of "greatest" or of any other concept of magnitude: the G. C. D. of the positive rational integers a, b is that common divisor d of a, b which is such that, if δ is any common divisor of a, b then δ divides d. Similarly the L. C. M. of a, b is that common multiple m of a, b which is such that, if μ is any common multiple of a, b, then m is a multiple of μ.

Notice that these definitions, having isolated extensible properties of the G. C. D. and L. C. M., describe them wholly in terms of relations. Uniqueness is included in the definitions merely for brevity. In a given domain uniqueness is either proved to subsist or, if such be not the case, then each of the G. C. D. and L. C. M. is replaced by a class of divisors having the essential properties indicated.

Under these reconstructed definitions we may, for example, evolve an arithmetic either for classes or for relations, in which unique factorization, congruences, etc., of classes or relations are abstractly identical with the like for the positive rational integers. In the arithmetic of classes we may take the G. C. D. of two classes to be a definite one of their logical sum or their logical product, depending upon the definition of division; the L. C. M. is then the other. Division for classes may be defined in either of two ways: "a divides b," where a, b are classes, if (1) a includes b, or (2) a is included in b. The arithmetics developed on these respective definitions are of course distinct in their interpretations; abstractly they are identical. Two arithmetics exist for either classes or relations owing to the Pierce-Schroeder dualism in symbolic logic.

As another illustration, well known and easily accessible in elementary texts, we recall the theory of polynomials modulo a rational prime which is basic in Zolotareff's theory sketched later. In this theory there is unique factorization, etc., precisely as in the arithmetic of the rational integers or of ideals, although the interpretations are distinct. It is interesting in this theory of polynomials to observe the manner in which a large region of infinite rational arithmetic is imaged on a finite mirror.

We shall next state a few definitions and theorems from each of three widely different theories of ideals in order to suggest definite problems in the concluding sections. There is no attempt at completeness in what follows; we have endeavored to select similarly oriented phenomena from three distinct theories (with more space at our disposal we might easily have made it a dozen) which tend toward the same goal. Superficially these three (and there are many more) are violently unlike. It remains to be seen, from a complete abstract formulation, whether the differences go deeper than the surface. My own opinion, based on abstract formulations of several such theories, is that the apparent differences are principally variants of interpretations of one and the same body of pure (abstract) mathematics.

8. Dedekind's theory (of about 1876) belongs to the multiplicative half of arithmetic. Underlying the entire structure is the definition of integer in an enlarged sense. The theory of arithmetical divisibility occupies the central position. Arithmetical divisibility exists in a theory in the strict sense, namely in that of the rational integers, only when there is a law of unique factorization into indecomposable or prime integral elements. This in fact may be taken as the cardinal distinction between algebraic and arithmetical divisibility. It has even been proposed (by Cahen) to define arithmetic as that branch of algebra in which division is only exceptionally possible.

The goal in Dedekind's theory is the restoration of unique factorization by the introduction of properly defined elements (ideals) to replace the original elements (algebraic integers) for which unique factorization into primes fails. To be effective the new divisibility must be in at least one respect formally equivalent to the old—or nothing is gained by the invention of ideals, and the original problem of unique factorization is not solved but is replaced by another which has no significant relation to the original. We recall that two propositions are "formally equivalent" when each implies the other.

The following can be understood as an abstract proposition irrespective of the specific meanings later attached to the technical terms involved: if α, β are algebraic integers, α divides β when and only when the principal ideal (α) divides the principal ideal (β); moreover an ideal is uniquely the product of prime ideals (not necessarily nor in general principal). This is the link connecting factorization of algebraic integers with that of ideals. Evidently on

this basis a formal equivalence between the theories of divisibility of ideals and of algebraic integers, considered as principal ideals, can be established, for a given algebraic integer determines precisely one principal ideal. We shall see later that the desired end can be attained otherwise. First let us explain the terms used above.

9. A root x of an equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

in which the a's are rational numbers is called an algebraic number; if the a's are (rational) integers, x is an algebraic integer. It can be shown without difficulty that the sum, difference, product of two algebraic integers is an algebraic integer.

Irrespective of what interpretations shall be attached to the symbols *, † or to the phrase "* divides †," we shall write "* divides †" as follows: $* \mid \dagger$. Divisibility in the arithmetical sense of algebraic integers $\alpha, \beta, \gamma, \cdots$ is defined as for the rational integers: $\alpha \mid \beta$ when and only when an algebraic integer γ exists such that $\beta = \alpha\gamma$.

An algebraic integer α satisfies precisely one equation irreducible in the rational domain with leading coefficient unity (and, by preceding definitions, the remaining coefficients rational integers). The degree n of this irreducible equation satisfied by α is called the degree of α and of the field $K(\alpha)$ generated by α; the remaining roots of the equation are the conjugates of α. From $K(\alpha)$ is isolated the set $I(\alpha)$ of all algebraic integers in $K(\alpha)$, including those of degree 1, (the rational integers). The product of α and all its conjugates is the norm $N(\alpha)$ of α.

The problem is to study the laws of divisibility in $I(\alpha)$. In $I(1)$, the set of all rational integers, unique factorization into indecomposable integers (primes) is implied by the following theorem: if a prime divides the product of two integers it divides at least one of the factors. It is the breakdown of this fundamental theorem, in general, in $I(\alpha)$ which necessitates the introduction of ideals. A simple $I(\alpha)$ for which ideals are necessary is Dedekind's example $I(i\sqrt{5})$.

A unit is an element of $I(\alpha)$ which divides each element of $I(\alpha)$, the quotient being also in $I(\alpha)$; a decomposable element of $I(\alpha)$ is one which is the product of two elements of $I(\alpha)$ not both units; an indecomposable or prime element of $I(\alpha)$ is one which is not decomposable. Elements of $I(\alpha)$ differing only by unit factors are not distinguished in questions of divisibility. In general an element of $I(\alpha)$ is not uniquely the product of primes.

To restore unique factorization in $I(\alpha)$ Dedekind cast about for some simple construct of elements of $I(\alpha)$ for which the fundamental operations addition and multiplication are well defined. The class proved to be what was wanted, for classes of elements can be added and multiplied even when the individual

elements of the classes cannot. This choice is singularly appropriate in the case of $I(\alpha)$, for not only can sets (classes) of elements of $I(\alpha)$ be added and multiplied, but the like is true (in this instance) within the classes themselves. Even were the latter not the case, as happens in some arithmetical theories, we could still proceed, but the means for doing so would take us too far afield here. Notice that in the case of algebraic numbers multiplication and addition of elements, although abstractly identical with the like for classes of the same elements, are different operations from those for the classes in that their interpretations are distinct in the two instances. Such an abstract identity of relation but difference of interpretation characterizes many situations in general arithmetic; it is ultimately the logical distinction between intension and extension.

The class concept was introduced by Dedekind as follows. A set M of elements of $I(\alpha)$ which is closed under subtraction, and hence under addition and subtraction, is called a module. If M be such that, if β is any element of M, and γ is any element of $I(\alpha)$, then $\beta\gamma$ is an element of M, the module M is called an ideal. Hence an ideal of $I(\alpha)$ is closed under addition and subtraction and multiplication by elements of $I(\alpha)$.

As customary Dedekind ideals of a given $I(\alpha)$ are denoted by small German letters, $\mathfrak{a},\mathfrak{b},\mathfrak{c}, \cdots$.

We saw that the Gaussian concept of congruence is equivalent to a restatement of certain aspects of division in terms of class inclusion. Division in Dedekind's theory becomes class inclusion in the following way. If β, γ are any elements of $I(\alpha)$, and $\mathfrak{a}$ is any ideal of $I(\alpha)$, we say that

$$\beta \equiv \gamma \bmod \mathfrak{a}$$

only when the difference of β, γ is an element of $\mathfrak{a}$. This is otherwise expressed by the locution "$\mathfrak{a}$ divides, or contains, $\beta-\gamma$."

Before stating how unique factorization is restored, let us see how order relations are reintroduced into this theory. It is proved that the elements (algebraic integers) of $I(\alpha)$ fall into a finite number of classes with respect to any given ideal modulus $\mathfrak{a}$, two elements of $I(\alpha)$ being put into the same or different classes according as they are or are not congruent modulo $\mathfrak{a}$; the number $N(\mathfrak{a})$ of distinct classes is called the norm of $\mathfrak{a}$. Theorems in $I(1)$, such as Fermat's or Wilson's whose proofs in $I(1)$ depend ultimately upon order relations or upon enumeration, are in general extended to the ideals of $I(\alpha)$ by replacing the (rational integral) modulus m by a norm as above defined.

Returning now to divisibility, we shall state the method by which unique factorization of elements of $I(\alpha)$ is accomplished. It is shown first that each ideal $\mathfrak{a}$ of $I(\alpha)$ (or, what is the same for our purpose, of $K(\alpha)$) is defined by a finite number, say s, of elements $\alpha_1, \alpha_2, \cdots, \alpha_s$ of $I(\alpha)$, so that, γ being any element of $\mathfrak{a}$,

$$\gamma = \mu_1\alpha_1 + \mu_2\alpha_2 + \cdots + \mu_s\alpha_s,$$

where the μ's are in $I(\alpha)$. Consequently $\mathfrak{a}$ is represented by the symbol $(\alpha_1, \alpha_2, \cdots, \alpha_s)$, and $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_s\alpha_s$ is in $\mathfrak{a}$, where the λ's are any elements of $I(\alpha)$.

Hence, $\mathfrak{a} = (\alpha_1,\alpha_2, \cdots, \alpha_s)$, $\mathfrak{b} = (\beta_1,\beta_2, \cdots, \beta_t)$ being any two ideals of $I(\alpha)$ the ideal represented by the st products $\alpha_i\beta_k (j=1, \cdots, s;\ k=1, \cdots, t)$ is determinate. This ideal is called the product of $\mathfrak{a}$ and $\mathfrak{b}$ and is written $\mathfrak{ab}$. Such is the definition of multiplication of ideals.

Divisibility for ideals is defined as follows in terms of class inclusion: $\mathfrak{a}|\mathfrak{b}$ ('the ideal $\mathfrak{a}$ divides the ideal $\mathfrak{b}$') when and only when $\mathfrak{a}$ contains ($\equiv$ includes) $\mathfrak{b}$, that is, when and only when every element of $\mathfrak{b}$ is in $\mathfrak{a}$.

The definition is justified as more than a mere play on words by the theorem that if $\mathfrak{a} \mid \mathfrak{b}$, then there exists a unique ideal $\mathfrak{c}$, called the quotient of $\mathfrak{b}$ by $\mathfrak{a}$, such that $\mathfrak{b} = \mathfrak{ac}$, precisely as in rational arithmetic.

An ideal defined by the single element β of $I(\alpha)$ is called principal and is written (β); the principal ideal (1) divides every ideal $\mathfrak{a}$ of $I(\alpha)$, and it is the only ideal of $I(\alpha)$ having this property. Hence the unique unit ideal is (1).

Contrary (as usual) to what common sense might predict, an ideal divisor is "greater," meaning "more inclusive" than the dividend. That this is reasonable is seen thus. In $I(1)$ we write (m) for the principal ideal consisting of all rational integral multiples of the rational integer m. Then, for example, $2 \mid 6$ and $(2) \mid (6)$ are true propositions, and (2) is obviously more inclusive than (6).

An ideal $\mathfrak{p}$ is called prime if $\mathfrak{p}$ is not the product of two ideals both different from the unit ideal (1).

The fundamental theorem of Dedekind's theory states that an ideal is uniquely the product of prime ideals.

Since $\beta|\gamma$, where β, γ are any elements of $I(\alpha)$, is formally equivalent to $(\beta) \mid (\gamma)$, the theory has reached its goal and unique factorization is restored in $I(\alpha)$. It would be more accurate however to say that unique factorization in $I(\alpha)$ is "induced" by the like in the set of all ideals of $I(\alpha)$, since the "restoration" is achieved by means of an isomorphism between two relations, one of which concerns the original set of elements for which factorization was sought, while the other does not immediately refer to those elements but to classes of them. The abstract situation presented by this restoration is of very great interest.

When we come to Prüfer's theory we shall frequently require one detail of definition: if $\mathfrak{a}$ is any ideal of $I(\alpha)$, and β an element of $I(\alpha)$, then $\mathfrak{a} \mid \beta$ means $\mathfrak{a} \mid (\beta)$. Conversely, if β is an element of $\mathfrak{a}$, then $\mathfrak{a}$ is said to divide ($\equiv$ contain) β, and when convenient this is written $\mathfrak{a} \mid \beta$. Strictly, of course, $\mathfrak{a} \mid \beta$ is meaningless unless some conventional significance (as above) be assigned to it, since $\mathfrak{a}$ is an ideal and β is not, and we defined division by an ideal only when the dividend also is an ideal.

The part played in the foregoing by the algebra of classes or of relations is conspicuous, and it is the essence of Dedekind's generalization of the concept of divisibility. Two further instances of the importance of the class in this theory may be noticed, as they are simple and require no further definitions for their statement. The G. C. D. of any number of ideals is their logical sum; their L. C. M. is their logical product. The multiplicative properties of ideals based upon these definitions are abstractly identical with the like for the rational integers.

There are many other interesting formal equivalents between the multiplicative theory of numbers, the theory of classes or of relations, and the theory of Dedekind ideals, but these we must pass over with a bare reference to the important concept of a class of ideals and the number—the so-called class number—of ideal classes in a given field. The last suggests interesting developments in the algebra of classes, thus repaying a part of the debt of the theory of algebraic numbers to symbolic logic.

No satisfactory definition of the sum or difference of two Dedekind ideals was devised until very recently, and this new attempt has yet to be scrutinized. Hence, as stated before, a complete generalization of rational arithmetic is a thing of the future. This statement again, however, may require modification with respect to Prüfer's recent theory, but the explicit rounding out of the Dedekind theory has not yet been exhibited.

One point in the foregoing sketch may be noticed particularly. The concept of the product of two ideals may possibly be discarded or replaced by another in the final theory of arithmetic, as forecast by the work of Speiser on Dickson arithmetics, in which the Dedekind module appears to be fundamental. These questions however are still in the melting pot.

10. Dedekind's theory has several advantages over its great predecessor, Kummer's (of about 1857), of which we need here refer only to two. Kummer, the first to define ideal numbers and to restore unique factorization in a field where the fundamental theorem of arithmetic does not hold, considered only fields defined by roots of unity, and not a general algebraic number field as did Dedekind. Despite the fact that Kummer's introduction of ideals into arithmetic was beyond all dispute one of the greatest mathematical advances of the nineteenth century, his work now is commonly said to be chiefly of historical interest, being supplanted by the theories of Dedekind and others, although some of the leading workers on Fermat's last theorem still use Kummer's theory, at least as a point of departure for certain of their researches. The second advantage possessed by theories of the Dedekind type over those of the Kummer species, is that the former explicitly exhibit their ideals while the latter do not.

Abstractly, however, a theory of the Kummer type is of the highest interest. It would take us too far here to justify this remark in detail, but we shall endeavour to indicate where the epistemological interest of such a theory lies. Instances of the abstract situation presented by Kummer's theory are rare in mathematics, although they are the rule rather than the exception in theoretical physics—an observation to which we shall recur in a moment.

Kummer did not define an ideal as a thing in itself, but instead used relations expressed in terms of ideal numbers without concerning himself whether or not such numbers have an independent existence. Thus, if the algebraic integer α has always a certain property A, which in Kummer's theory is the property that α is a root of one or more congruences, α is said to be divisible by a determinate ideal corresponding to the property A.

The abstract situation here is identical with that exemplified in many theories of classical (and now largely discarded) theoretical physics, for instance the hydrodynamical analogies of Kelvin and Maxwell in electrodynamics. Physics never comes to grips with the reality which it believes it is investigating, but invariably describes that hypothetical reality by establishing a correspondence or isomorphism with an ideal construct whose imagined anatomy is supposedly familiar. Kummer's theory is one of the few complete mathematical analogues of this tenuous and somewhat mystical methodology.

Although it is now a historical relic, Kummer's theory has in one respect a singularly modern aspect which is less prominent in Dedekind's and in those of the latter's followers. The emphasis on the relation as the central concept of pure mathematics is a development of the present century. The method of Kummer's theory is purely relational, in contrast to Dedekind's which is largely concerned with the relata (ideals) existing only by implication in another form (ideal numbers) in Kummer's theory. In this respect Kummer's theory is potentially more general than Dedekind's because the ideal elements of Kummer's theory may be any whatever provided they satisfy the relations. There is thus in this respect a higher order of abstraction in Kummer's theory than in Dedekind's.

Before leaving this, we recall that Dedekind himself suggested an interpretation of a part of his theory more general than that now current. The elements of a Dedekind module as defined above belong to $I(\alpha)$, that is, they are algebraic integers. But obviously, as remarked by Dedekind, this restriction is fortuitous so far as modules quâ modules are concerned. The elements $\beta, \gamma, \delta, \cdots$ of a module may be any whatever provided only that they are such that for any pair of them there exists a unique "sum," where "addition" satisfies the commutative law and has a unique inverse and the "sum" is again in the module, so that a module is a set closed under an operation abstractly identical with algebraic addition.

To illustrate Dedekind's remark we may take as the elements of a module either classes or relations, and evidently a similar generalization of Dedekind ideals is feasible. The unique inverse of addition is given by the supplementary class.

11. Intermediate in type between Kummer's theory and Dedekind's is that of Zolotareff (of about 1880), which also has some features in common with the most recent theories of algebraic numbers. Zolotareff's theory, as will be seen from the following sketch of its basic concepts, is abstractly of a structure similar to Kummer's in that its ideal numbers are relational rather than explicit. Again the algebra of classes is fundamental for the arithmetic, but in a less obvious way than in Dedekind's theory.

The algebraic aspects of Zolotareff's theory derive from Galois, and the entire fabric, attractive from several points of view, has an advantage over some other theories of algebraic numbers in that every indicated operation can be performed in a finite number of steps without any tentative process whatever. This advantage it shares with another, better known theory, that of König, which carries to reasonable completion Kronecker's extension of Dedekind's theory. The gain in each case however is perhaps only illusory, as the actual execution of the details—a matter which writers on this branch of mathematics usually disdain—may in practice prove to be prohibitively prolix. In short the arithmetical millenium has not yet arrived either in the rational or in the algebraic theory.

12. Zolotareff starts with an algebraic number field $K(\alpha)$ of degree n, and precisely as in Dedekind's theory, isolates the totality $I(\alpha)$ of integers in $K(\alpha)$. Let $\Pi \equiv \Pi(x)$ be the polynomial in x, algebraically irreducible in $K(1)$, whose vanishing defines $K(\alpha)$, the coefficient of x^n in Π being 1 and the remaining coefficients rational integers. Then, in this theory, the rational prime p is defined to be prime in $K(\alpha)$ if Π is irreducible modulo p. All remaining rational primes are classified as composite in $K(\alpha)$.

It is proved readily that if p is prime in $K(\alpha)$ then p divides the product of two or more integers of $K(\alpha)$ when and only when p divides one of the factors. The theory thus satisfies one of the cardinal requisites of any strict arithmetical theory.

Consider next the rational prime p for which Π is reducible modulo p, and let $V_j (j=1, \cdots, h)$ be the factors of Π modulo p, of the respective degrees $m_j (j=1, \cdots, h)$, so that

$$V_1^{m_1} V_2^{m_2} \cdots V_h^{m_h}$$

is the resolution of Π into powers of distinct (polynomial) factors irreducible modulo p. Then this rational prime p is said to be composite in $K(\alpha)$ and to have the prime ideal factor V_j precisely m_j times $(j=1, \cdots h)$. The factors V_j are said to belong to p.

If now $f(x_0)$ is an element of $I(\alpha)$, we say that $f(x_0)$ is divisible by a factor of p belonging to V if $f(x)$ is divisible modulo p by V. Set

$$\begin{aligned} W &= V_1^{m_1} \quad \cdots V_h^{m_h}, \\ W_1 &= V^m V_2^{m_2} \cdots V_h^{m_h}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ W_h &= V^m V_1^{m_1} \cdots V_{h-1}^{m_{h-1}}, \end{aligned}$$

(where in W_j the factor V_j is omitted), and let λ be a positive rational integer. Then, attending to the typical exponent m, we set $\lambda = km + r (0 \gtreqless r < m)$, and say that $f(x_0)$ has an ideal factor, belonging to V, precisely λ times (or of multiplicity λ), provided

$$f(x)V^{m-r}W^{k+1} \equiv 0[\text{modd } p^{k+1}, \Pi],$$

is true, and

$$f(x)V^{2m-r-1}W^{k+2} \equiv 0[\text{modd } p^{k+2}, \Pi]$$

is false. Similarly for other ideal factors and their multiplicities.

On these definitions, theorems abstractly identical with the fundamental ones of rational arithmetic are proved with comparative ease—certainly after a shorter chain of subsidiary theorems than is required in the usual presentation of Dedekind's theory after Hurwitz. Among these we note only the following.

If $f(x_0)$ is an arbitrary complex integer (element of $I(\alpha)$) which contains the ideal factor of the rational prime p belonging to V exactly λ times, and if $\psi(x_0)$ does not contain this factor, then $f(x_0)\psi(x_0)$ contains this factor precisely λ times.

There is unique factorization of elements of $I(\alpha)$ into prime ideal factors (in Zolotareff's sense).

The G. C. D. and L. C. M. are defined in a manner abstractly identical with that of rational arithmetic. Thus two elements of $I(\alpha)$ are co-prime if they have no common factors; their G. C. D. is the product of their common ideal factors, etc.

Zolotareff's theory refers to any algebraic number field. Its scope therefore is coextensive with that of the Dedekind theory. The special case of Zolotareff's theory in which the field is generated by a root of unity is easily seen to yield the Kummer theory.

We have given enough of this theory to satisfy our immediate purpose here, namely to show that unique factorization can be restored in at least two ways, and that the algebra of symbolic logic underlies the entire structure. Dedekind, Kronecker and others have sketched a third way of restoring unique factorization, that of adjunction to the original field, but this is not very attractive and it has not been followed out completely.

The class concept in Zolotareff's theory again is fundamental, being concealed in the notion of reducibility modulo a rational prime and in the definition of ideal factors.

Before leaving Zolotareff's ideas, we mention an easy extension of his theory which does not seem to occur in the literature: the rational prime p may be replaced by a prime ideal $\mathfrak{p}$, as in the theory of reducibility, unique factorization, etc., modulo p passes over at once to an identical theory modulo $\mathfrak{p}$. The resulting theorems of course belong to a domain different from that of Zolotareff's investigations.

We pass on to another method which has been proposed recently (1925, 1926). This seems to be the most inclusive of all, as will be seen from the summary. The substratum of algebraic logic underlying this theory and the abstract identity of its fundamental constructs with those of the theory of classes or relations is as strikingly apparent as in Dedekind's ideals. It is most clearly seen in v. Neumann's presentation, which we have followed here.

13. Prüfer's theory (of 1925) and its modifications by v. Neumann (1926) aim to be the most comprehensive yet devised for algebraic numbers.

A set of elements closed under multiplication and division is said to form a ray; a set closed under addition, subtraction and multiplication constitutes a ring. Hitherto, as already noted, all attempts to define addition or subtraction for Dedekind ideals have proved unsatisfactory. It is one of the features of Prüfer's theory that the new "integral ideals" which it introduces form a ring. These ideal numbers are an extension of the given algebraic number field, completing it, in a manner similar to that in which Dedekind ideals complete the rays of the field by restoring the laws of rational arithmetic.

That part of the theory which concerns multiplicative properties yields, upon the proper specializations, the theories of Kummer, Dedekind, and Kronecker. Moreover there exists in Prüfer's theory a unique additive decomposition for all ideal numbers, reminiscent of the direct sum in linear algebras, a feature not found in any previous theory. Further, the new theory can be carried over to the theory of algebraic functions of a single variable, but not to the like for more than one variable, an analogy, apparently, with the Dedekind-Weber theory in the same direction. It would be of great interest to see whether this theory is capable of restoring unique factorization in a Dickson arithmetic—the aim of Speiser's entirely different theory.

A new result of a different order states that the theory of relative fields is rendered superfluous. This is indeed good news if it be more than an existence theorem. Briefly the argument whereby this conclusion is reached is as follows. In Dedekind's theory as sketched above and as always presented in its rudiments, we took as the point of departure the rational number field and defined algebraic numbers in relation to it by postulating that the coefficients of the equation defining the field were rational numbers. The immediately suggested generalization of investigating any given algebraic number field in relation to any other, not necessarily the rational, leads soon to serious difficulties. Prüfer's

theory exorcises these difficulties by not raising them. That is, his proofs are cast ab initio in a mould which is independent of the properties of the rational numbers quâ rational numbers, and in particular the unique decomposition of a rational integer into rational primes is nowhere assumed in the proofs of the fundamental theorems. The theory of the rational integers is therefore included, not presupposed as customary, in the new theory.

The rational unique factorization theorem is here replaced by another, due to Dedekind, and this substitute theorem does not imply, nor is it implied by, the rational theorem. It is this: if $\alpha_j (j=1, \cdots k)$ belong to an algebraic number field, then in the same field there exist numbers $\beta_j (j=1, \cdots, k)$ such that each product $\alpha_i\beta_j (i, j=1, \cdots, k)$ is an integer and $\sum \alpha_i\beta_i = 1$ $(i=1, \cdots, k)$. Having avoided by this means the peculiarities of the rational field in his proofs, Prüfer attains a theory which includes that of the rational integers as a special case, and moreover it is clear that, no particular field being given precedence over any other in the demonstrations, the theory of relative fields is no longer necessary for complete generality.

Prüfer's theory has been recast by v. Neumann, who also has removed certain restrictions (concerning "infinite" numbers and divisors of zero) of the original theory. To give some idea of this latest development of arithmetic we shall follow v. Neumann's exposition so far as may be done with the concepts of Dedekind's theory already sketched.

14. In Prüfer's theory are defined certain "ideal numbers" which accomplish unique factorization, as in the Dedekind theory, and which, unlike the ideals of that theory, form a ring. Ideals (Dedekind) and ideal numbers (Prüfer) are distinct things; we shall use these terms in full, so that "ideal" is not an abbreviation for "ideal number." To each Dedekind ideal corresponds (in a technical sense) in Prüfer's theory a class of ideal numbers, and there exist ideal numbers corresponding to no Dedekind ideal. Prüfer's theory is therefore a true generalization of Dedekind's.

An ideal number to which corresponds no Dedekind ideal is called infinite; the rest, except zero and its divisors, are called finite. Finite ideal numbers, as in the Dedekind theory, have a unique factorization theorem. Prüfer introduces his ideal numbers as 'ideal solutions' of infinite systems of congruences; v. Neumann reaches results similar to Prüfer's otherwise and he obtains a multiplicative decomposition (unique factorization) for all ideal numbers, including the finite, the infinite, zero and the divisors of zero. The finite ideal numbers justify their name thus: a finite ideal number differs from the other varieties only in that it is the product of a finite number of factors. When an infinity of factors are required in the decomposition the ideal number is called infinite; if at least one prime factor is repeated an infinite number of times the ideal number is a divisor of zero; if all prime factors enter, each an infinity of times, the ideal number is zero.

One stumbling block in the terminology must be avoided. "Real number" is used always in contradistinction to "ideal number." Thus in this theory a real number is in general complex. The real numbers of Prüfer and v. Neumann are the elements (algebraic numbers) of any given algebraic number field of degree n. The paradox then is one of nomenclature, not of fact. Dedekind's theory is presupposed, at least in v. Neumann's presentation.

We proceed to the explanation of v. Neumann's excessively abstract ideal numbers. The following definitions and theorems are as beautiful an example of the arithmetical formulation of a theory as any in the literature. They can be grasped on their own merits independently of their intrinsic significance in relation to algebraic numbers.

15. Rational numbers are denoted by $m, n, \cdots$; real numbers by $\alpha, \beta, \cdots, \xi, \eta, \cdots$; Dedekind prime ideals by $\mathfrak{p}, \mathfrak{q}, \cdots$; sequences of real numbers by $R, S, \cdots$; ideal numbers by $\mathfrak{A}, \mathfrak{B}, \cdots$. The previous notation $* \mid \dagger$ for "$*$ divides $\dagger$" is retained; "divides" can in each case be read "includes" or "contains" in the sense of class inclusion as in mathematical logic. The case of $\alpha \mid \beta$, an apparent exception, can be referred to $(\alpha) \mid (\beta)$, (α), (β) principal ideals.

Let α be a real number, m a rational integer, $\mathfrak{p}$ a prime ideal. Then α is said to have the factor $\mathfrak{p}^m$ when there exists a real (algebraic in the given field), integer ξ such that $\mathfrak{p}^m \mid \alpha\xi$.

A sequence $R \equiv [\alpha_1, \alpha_2, \cdots]$ of real numbers is called fundamental when for every $\mathfrak{p}$, m almost all ($\equiv$ all with a finite number of exceptions) differences $\alpha_r - \alpha_{r+1}$ have the factor $\mathfrak{p}^m$.

Any two sequences $R \equiv [\alpha_1, \alpha_2, \cdots]$, $S \equiv [\beta_1, \beta_2, \cdots]$ of real numbers are called equivalent, $R \sim S$, when for each $\mathfrak{p}$, m, almost always, $\mathfrak{p}^m \mid \alpha_r - \beta_r$ (viz., with a finite number of exceptions the differences $\alpha_r - \beta_r (r = 1, 2, \cdots)$ are elements of the mth power of the prime ideal $\mathfrak{p}$).

Hence equivalence of sequences of real numbers is abstractly identical with equality, being reflexive, symmetric, and transitive. Moreover the set of all sequences of real numbers falls into mutually exclusive classes such that all elements (sequences) of a given class are equivalent and no two elements of different classes are equivalent. Further, if $R \sim S$, then both or neither of R, S are fundamental. Hence all or none of the sequences in a particular class are fundamental. Thus classes of sequences fall into two classes, fundamental and non-fundamental; each sequence in a fundamental class is fundamental; no sequence in a non-fundamental class is fundamental.

A fundamental class is now called an ideal number.

If R is an element of the ideal number $\mathfrak{A}$, then $\mathfrak{A}$, R are said to appertain to one another, and this is written $\mathfrak{A} \sim R$. Each fundamental R appertains to precisely one ideal number; to each ideal number appertain an infinity of R's.

The relation $\mathfrak{A}\sim R$ has the abstract properties of equality; thus $\mathfrak{A}\sim R$, $R\sim S$ imply $\mathfrak{A}\sim S$; $\mathfrak{A}\sim R$, $\mathfrak{A}\sim S$ imply $R\sim S$; $\mathfrak{A}\sim R$, $\mathfrak{B}\sim R$ imply $\mathfrak{A}\sim\mathfrak{B}$.

Fundamental sequences $R\equiv[\alpha_1, \alpha_2, \cdots]$, $S\equiv[\beta_1, \beta_2, \cdots]$, $\cdots$, form a ring in which addition $R+S$, subtraction $R-S$, multiplication RS are the operations yielding respectively the sequences (easily shown to be fundamental)

$$[\alpha_1+\beta_1, \alpha_2+\beta_2, \cdots], \qquad [\alpha_1-\beta_1, \alpha_2-\beta_2, \cdots], \qquad [\alpha_1\beta_1, \alpha_2\beta_2, \cdots].$$

Hence $R'\sim R''$, $S'\sim S''$ together imply

$$R' \pm S' \sim R'' \pm S'', \qquad R'S' \sim R''S''.$$

Let $\mathfrak{A}\sim R$, $\mathfrak{B}\sim S$ ($\mathfrak{A}$, $\mathfrak{B}$ ideal numbers). Then the ideal numbers appertaining to $R\pm S$, RS respectively depend only upon $\mathfrak{A}$, $\mathfrak{B}$ and not upon R, S.

Hence we define the sum $\mathfrak{A}+\mathfrak{B}$, the difference $\mathfrak{A}-\mathfrak{B}$, the product $\mathfrak{A}\mathfrak{B}$ of the ideal numbers $\mathfrak{A}$, $\mathfrak{B}$ as the ideal numbers appertaining respectively to $R+S$, $R-S$, RS.

Under this definition ideal numbers form a ring, since obviously addition, subtraction and multiplication of ideal numbers as just defined obey the commutative, associative and distributive laws and since, moreover, subtraction as defined is the unique inverse of addition.

An integral ideal number is one to which appertains at least one fundamental sequence all of whose elements are algebraic integers. Hence, immediately, if $\mathfrak{A}$, $\mathfrak{B}$ are integral ideal numbers so also are $\mathfrak{A}\pm\mathfrak{B}$, $\mathfrak{A}\mathfrak{B}$.

Each sequence of the type $[\alpha, \alpha, \cdots]$ is fundamental; the ideal number appertaining to $[\alpha, \alpha, \cdots]$ is written[1] α'.

Hence we have

$$\alpha' + \beta' = (\alpha+\beta)', \ \alpha' - \beta' = (\alpha-\beta)', \ \alpha'\beta' = (\alpha\beta)'$$

where $(\alpha+\beta)'$ is the ideal number appertaining to $\alpha+\beta$, and similarly for the others; also

$$\begin{aligned} \mathfrak{A} + 0' = \mathfrak{A} - 0' &= \mathfrak{A}, \qquad & \mathfrak{A} - \mathfrak{A} &= 0', \\ \mathfrak{A}1' &= \mathfrak{A}, & \mathfrak{A}0' &= 0'. \end{aligned}$$

Division of ideal numbers is defined as the inverse, in the arithmetical sense of multiplication. Thus, if there exists an integral ideal number $\mathfrak{G}$ such that $\mathfrak{B}\sim\mathfrak{A}\mathfrak{G}$, then by definition $\mathfrak{A}\mid\mathfrak{B}$, and this division has the abstract properties of arithmetical division, so that in particular

$$\begin{aligned} &\mathfrak{A}\mid\mathfrak{B}.\ \mathfrak{B}\mid\mathfrak{C} :\supset: \mathfrak{A}\mid\mathfrak{C}; \\ &\mathfrak{A}\mid\mathfrak{B}.\ \mathfrak{A}\mid\mathfrak{C} :\supset: \mathfrak{A}\mid\mathfrak{B}\pm\mathfrak{C}; \\ &\mathfrak{A}\mid 0', \qquad\qquad \mathfrak{A}\mid\mathfrak{A}, \end{aligned}$$

in which $\supset$ is the customary symbol for "implies," and, for example, the first states that if $\mathfrak{A}$ divides $\mathfrak{B}$ and $\mathfrak{B}$ divides $\mathfrak{C}$, then $\mathfrak{A}$ divides $\mathfrak{C}$.

---

[1] I have here changed v. Neumann's bars to accents to suit machine printing.

The connecting link with common division is the following easily seen theorem,

$$\alpha \mid \beta : \equiv : \alpha' \mid \beta',$$

that is, each of these division relations implies the other.

Recalling the definition of the relation "α has the factor $\mathfrak{p}^m$" which was given near the beginning of this section, we can now close this chain of definitions and theorems as follows. The ideal number $\mathfrak{A}$ is said to have the factor $\mathfrak{p}^m$ when, for each fundamental sequence $R \equiv [\alpha_1, \alpha_2, \cdots]$ appertaining to $\mathfrak{A}$, almost all α_r have the factor $\mathfrak{p}^m$. Hence if $\mathfrak{A}$ has the factor $\mathfrak{p}^m$ it also has the factor $\mathfrak{p}^n$, $n \leqq m$; if also $\mathfrak{A} \mid \mathfrak{B}$, then $\mathfrak{B}$ has the factor $\mathfrak{p}^m$, and then each of $\mathfrak{A} \pm \mathfrak{B}$ has the factor $\mathfrak{p}^m$; while if $\mathfrak{A}$, $\mathfrak{B}$ have the respective factors $\mathfrak{p}^m$, $\mathfrak{p}^n$, then $\mathfrak{A}\mathfrak{B}$ has the factor $\mathfrak{p}^{m+n}$. Finally, either both or neither of α', α have the factor $\mathfrak{p}^m$; the equations $\alpha' = \beta'$ and $\alpha = \beta$ are formally equivalent, and in all operations which have been defined for both real and ideal numbers (addition, subtraction, multiplication, divisibility, having a factor $\mathfrak{p}^m$), the real numbers α and the ideal numbers α' play precisely the same parts. Hence the theories of real and of ideal numbers are abstractly identical up to the point considered, and the structure of the ideal theory is abstractly identical with that of rational arithmetic.

For the ideal numbers above described, v. Neumann obtains the results summarized in § 14. He also finds much more, and his further theory, like that of Prüfer, has close affiliations with Hensel's theory of p-adic numbers. It also, in its definitions concerning limits, is allied to Kurschak's work. Thus the scope of the Prüfer–v. Neumann theory is wide indeed. Its further development will be eagerly awaited. At the present time a detailed application to some number field of degree greater than 2 would be welcome, as only by concrete application can the power of a theory of algebraic numbers in obtaining new results be appraised.

16. Looking back over the three specimen theories of algebraic numbers selected to illustrate some of the features of such theories, and having in mind numerous other theories whose aims are the same or closely similar to those of the three chosen, we see the need of several definite projects, all of which are well within reach of our existing technique. Of these things to be done we need mention only a few; others will occur to anyone who tries to find his own way.

First, as a general enquiry, it may be asked why have some theories of algebraic numbers survived and others perished? What are the defects, if any, which have embalmed certain theories, for example those of Zolotareff and Sochocki, as beautiful things but dead, in the short notices of the *Fortschritte*, while others still masquerade in at least the semblance of life? This enquiry however may be only of antiquarian interest.

More vitally, it would not be a waste of time to exhibit each of these theories, dead, living and in process of being born, in purely abstract form, so that we may see at a glance precisely what the theory is as an object, not of art, but of pure mathematics. Until this be done it seems unlikely that we shall be able to say in what significant way one theory differs from another.

Having abstracted the several theories we may classify them and isolate the part common to all as abstract arithmetic. Rational arithmetic falls naturally into two main divisions, additive and multiplicative. It is perhaps too much to expect that any considerable part of both divisions shall pass over into the abstraction common to several theories.

With the relational aspects of the theories before us, we may modify the abstract structure of the postulates and develop the theories of the resulting systems. This has been done, for example, in Dickson's theory of the arithmetic of a rational algebra. In other examples some curious possibilities emerge. Thus it appears that we may have sets of integral elements for which division is not unique but which posses a unique multiplicative decomposition into indecomposable elements of the set.

Incidentally the postulational treatment of the whole of existing arithmetic is a desideratum. The like discussion of projective and other kinds of geometry yielded unexpectedly interesting results; the same may prove to be the case for arithmetic. Sets of postulates, say for the G. C. D., the L. C. M., arithmetical divisibility (as opposed to algebraic), integral elements, sufficiently broad to include the known instances, do not seem to have been attempted.

How far is it possible to go in arithmetic without using order relations? Again, exactly what is arithmetic and what part of any given theory (including the theories of geometry) is abstractly arithmetical in structure? Finally there is the problem of generalizing each definite abstraction achieved and producing interpretations of the generalizations.

If one guess may be permitted, it is this: a thoroughgoing examination of the pure mathematics underlying the numerous theories of algebraic number fields of the past half century will show that no one of these theories has gone much if at all deeper than the profound work of Richard Dedekind.

17. In conclusion we recall the famous aphorism of Kronecker: "God made the integers; all the rest is the work of man." History does not record whether Kronecker intended the second part of this remarkable theorem as a boast on behalf of man's superior ingenuity or a lament over his prolific perversity. It may be taken either way. Looking soberly over the welter of arithmetic theories that have jostled their way into existence after Gauss' initial improvement over $1, 2, 3, \cdots$, a pessimistic critic might mistake the theory of numbers for a branch of physics.

NOTE BY THE EDITOR: In a later issue of this Monthly there will be published a short list of readings in connection with the topics mentioned in Professor Bell's paper.

THE VALUE OF MATHEMATICAL MODELS AND FIGURES[1]

By ARNOLD EMCH, University of Illinois

1. The reason for the choice of the title for this paper is not to put up a defense for the use of models and figures in mathematical instruction. The value of the heuristic element, especially in elementary mathematical teaching, has been and is presumably recognized by the majority of teachers of mathematics. But even in more advanced fields and certain domains of mathematical research the establishment of the constructive features of a mathematical theory may sometimes be greatly aided and illuminated by appropriate graphs, diagrams and models.

Not infrequently it happens that, after the actual construction of a figure or of a model, a close examination of the finished product reveals or suggests the existence of new properties of the form investigated which were not anticipated before the construction.

Another important factor in the construction of figures and models lies in its strengthening of the geometrical imagination and of mathematical intuition in general.

The use of physical images of certain mathematical forms and relations, moreover, is an incomparable aid in fortifying mathematical memory. The acquisition of a concrete picture of a certain theory or of certain forms is a powerful means of mathematical orientation. As a matter of fact, mathematical symbols and formulas must be classed in this category of concrete pictures of mathematical relations. Geometrical figures in the plane and in space (models) serve in a general way the same purpose.

The ability to visualize mathematical truth, gained by training in constructive methods, together with a keen logical reasoning faculty and a strong intuition make a very powerful mathematical combination. These abilities are not always found combined among the mathematicians.

Steiner is a conspicuous example of a visualizing and intuitional genius lacking the critical power of analysis even as it existed at his time. Much of his early work, n its initial stage, was accompanied by marvelously drawn figures. Later, after long practice, when his mental visualizing power had become very strong, he did not find it necessary to make use of figures any more. This personal custom of Steiner was unfortunately carried to an extreme by some of his disciples and successors, like R. Sturm, who in his magnificent "Lehre von den geometrischen Verwandtschaften" has not drawn a single figure.

Klein, among the modern mathematicians of note, may be mentioned as one who combines both intuitional and critological power.

---

[1] This is an abstract of a paper, with screen projections, presented to the Mathematical Association of America at Columbus, Ohio, September 7, 1926.

It seems to me that the rejection of either the standpoint of utmost rigor or that of intuitional methods from mathematics in general must lead in the end to one-sidedness or sterility. A happy combination of both seems to be the most fruitful. No historic instance will illustrate this better than that of Riemannian function theory and Riemannian geometry, which have opened up entirely new mathematical vistas, which the Berlin school under the leadership of Weierstrass was unable to foresee.

It must be understood, of course, that in recommending the use of figures and models, these must be merely considered as useful auxiliaries, not as ends

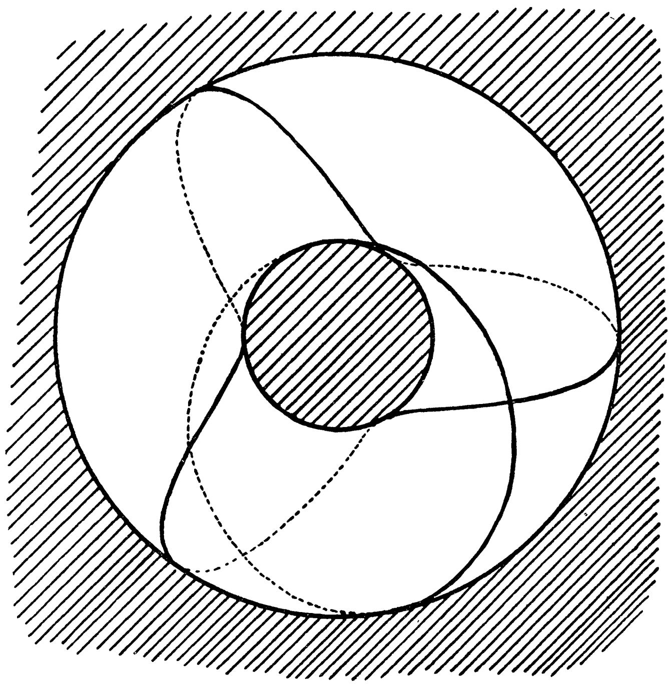


FIG. 1

in themselves, although they may often prove to be of great value in bridging over to applied mathematics.

I shall now speak specifically of some examples which may help to illustrate and clarify some of the ideas expressed above, in order to point out the possibilities of mathematical visualization for mathematical instruction by such means.

2. The plane sections of a torus form a particularly interesting class of bicircular quartics, i. e. quartics with the bicircular points at infinity as double points. Their self inverse (anallagmatic) properties are well known. On account of their extremely simple construction and representation they may

easily be visualized and classified as to their genus. A generic plane cuts the torus in a bicircular quartic of genus 1; a simple tangent plane, in a rational quartic, a Bernoulli lemniscate for example, with the point of tangency as the double point. When the intersecting plane is doubly tangent, the quartic, having now four double points, two real and two imaginary, must necessarily degenerate into two conics which, passing through the circular points, consist of two circles intersecting at the two points of tangency. This is beautifully illustrated by a model which shows the additional property that those circles are loxodromics, i. e., lines which intersect the parallels, or meridians, of the torus at a constant angle. This opens the question of loxodromics on a torus in general, which was answered by the author in connection with an application of elliptic functions. When the radius of the core-circle is R, the radius of the rotating, generating circle is r, and a closed loxodromic L_1 winds p_1 times around the axis of the torus and q_1 times around the core-circle of the torus, then the orthogonal loxodromic closes also after p_2 and q_2 similar windings when satisfying the relation $(p_1/q_1)(p_2/q_2)=(R^2-r^2)/r^2$ with R^2/r^2 rational.

Figure 1 illustrates the case $p_1=q_1=1$ of a loxodromic circle $(R=2r)$. For the orthogonal loxodromic, $p_2=1$ and $q_2=3$ in this case. The projection is a sextic.[1] In case of closure the loxodromics are algebraic curves.

As another example of effective visualization may be mentioned the space-sextic curve of genus 4, which is obtained as the intersection of a quadric and cubic surface. Projecting the sextic from a generic point upon a generic plane a plane sextic with 6 double points is obtained. The 6 apparent double points of the space sextic are beautifully seen in the model—Figure 2. Moreover the possibilities of real tri-tangent planes of the space sextic are easily seen from the same model. It is not difficult to construct the space sextic of genus four with the maximum number of ovals, namely five,[2] from which may be enumerated the 120 tri-tangent planes.

3. Group theory is a field in which visualization by means of constructive geometry has not been used extensively. And yet modern geometric research in many of its ramifications is closely associated with certain groups. It is true that the regular triangle, the square, the regular polygons in general, and the regular polyhedrons offer beautiful geometric illustrations of certain important finite groups; likewise the well known regular divisions of the complex plane, by circular arcs, which are associated with the groups of automorphic functions and the movements of non-euclidean geometry.

---

[1] The result contained in the preceding formula, referred to by E. Pompiani as Emch's theorem, was generalized in a memoir by Pompiani on *Rappresentazione grafica delle ciclidi di Dupin e delle loro lossodromiche*, Reale Istituto Lombardo di Scienze e Lettere (3) vol. 21 (1915), pp. 205–242. The author's own investigations on loxodromics were published in this Monthly, vol. 6 (1899), pp. 136–139 and vol. 9 (1902), pp. 277–280.

[2] Hilbert, Mathematische Annalen, vol. 38, 1891, p. 122.

As a new example I mention the model[1] for the symmetric group of order 24 on four letters. If the four letters represent four real numbers, then their 24 permutations represent the substitutions of the symmetric group G_{24} of order 24 and at the same time 24 points in projective 3-space. These 24 points lie on a quadric and by sixes on 16 conics, whose planes by fours pass through four lines of a plane quadrilateral. They lie by twos on 72 lines which pass by twelves through 6 vertices of said quadrilateral, thus belonging to 6 involutions on the

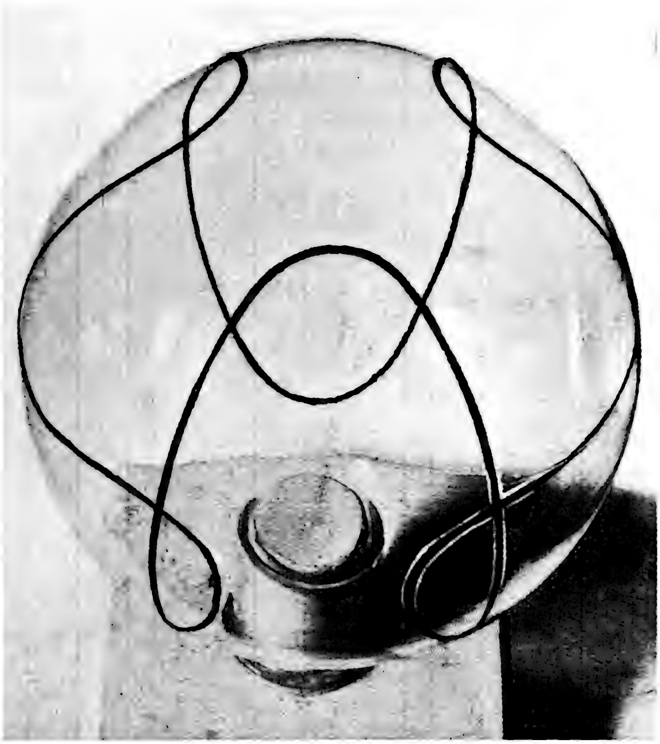


FIG. 2. On the models the lines run smoothly and evenly. Refraction and reflexion of light mar the effect on the photographs and are the cause of irregularities.

quadric. The model reveals the fact, which may be verified, that in certain groups, their joins meet in points outside of the points mentioned before. This would hardly be anticipated without the model.[1]

4. I hope that these examples from various mathematical realms, whose number might be increased indefinitely, will be impressive enough to show that mathematical visualization even in more advanced mathematical instruction and certain lines of research may often serve as a useful auxiliary to strengthen mathematical intuition and initiative.

---

[1] *Some geometric applications of symmetric substitution groups.* American Journal of Mathematics, vol. 45 (1923), pp. 192–207.

ON THREE INTERESTING TERMS RELATING TO AREA

By SOLOMON GANDZ, Rabbi Isaac Elchanan Theological Seminary, New York.

In the early texts on geometry there are three terms relating to area which have unusual etymological interest. They show, as is not infrequently the case in scientific literature, an interrelation among ancient peoples that is significant in the history of primitive culture. Since these terms do not seem to have heretofore been discussed from the standpoint of comparative philology, and since they have great mathematical interest, it is proposed to set forth the problem briefly as it appears in the light of present knowledge.

The first of these terms is *meshîḥah,* as it appears in its earliest form, at which time it seems to have been common to all Semitic languages. The earliest Arabic geometry, so far as known is the so-called *Bab-al-Misâḥah,* as it appears in al-Khowârizmî's[1] algebra[2] (c. 830). The word *Misâḥah* has three meanings: (1) mensuration,—the act and process of measuring: (2) area or superficial content,—the result obtained by measuring; and (3) geometry, —the art of measuring, including the rules for finding areas and volumes; all three of these meanings being recognized by al-Khowârizmî himself.

In the title *Bâb al-Misâḥah* the word means geometry, this particular chapter of his algebra furnishing rules for ascertaining areas and volumes. In the first sentence of the chapter,[3] however, it is used to mean area. The sentence, which defines the latter term, should be translated: "Know that *one in one* certainly means area and its meaning is one cubit in[4] one cubit." The term is used later[5] with the same meaning, but on the same page[6] it is used to denote mensuration in the sense of the process of measuring. The sentence should be translated as follows: "And truly the process of measuring this is that you shall measure its surface[7] and ascertain its area. And this, multiplied by the depth, gives the volume."[8] Rosen, in his translation, failed to distinguish between these two meanings, and hence a confusion naturally resulted. He gave "mensuration" as the title and then translated the first sentence as follows: "Know that the meaning of the expression 'one by one' is mensuration,"—which has no sense.

---

[1] Mohammed ibn Musa al-Khowârizmî.

[2] Rosen ed., London, 1831; Arabic text, pp. 50–64; English text, pp. 70–85.

[3] Arabic text, p. 50, lines 11–12.

[4] This use of "in" to denote multiplication is the origin of our use of "into" in the expression "$a+b$ into $a-b$." Compare Smith, *History of Mathematics*, II, 102. See also the note at the end of this paper.

[5] Arabic text, p. 53, lines 10, 18.

[6] Lines 7, 8.

[7] That is, of the base.

[8] Rosen (p. 74) translates this: "You must calculate it by ascertaining at first the area of its basis. This, multiplied by the height, gives the bulk of the body." This gives correctly the sense of the passage, but does not give the precise meaning of the terms *misâḥah* and *saṭch.*

This first sentence, with the definition of area, was misunderstood not only by Rosen but also by Aristide Marre in his French translation[1] and by Ruska in the German.[2] It is not due to al-Khowârizmî but was evidently taken from a much older work,—the Hebrew geometry called the *Mishnath ha-Middoth.*[3] In this, however, the usual term for area is *Meshîḥah*, this being also used for volume.[4] As a term for area it appears in the form *meshîḥah* in the *Liber Embadorum* of Abraham Savasorda (c. 1100), which bears the Hebrew title *Ḥibbûr ha-Meshîḥah weha-Thishboreth* (*The book of the area and the superficial content*).[5] In the Talmudic and Syriac literature the word *meshîḥah* is also well-known, although there it has the meaning of a surveyor's rope or measuring cord. In each of these languages *mashôḥa* means a surveyor or a geometer.[6] The root *meshaḥ*, *mashaḥ*, or also *mataḥ* means "to extend," "to stretch," "to measure," and appears in the Aramaic and Syriac translations of the Bible for the Hebrew *madad.* It should not be assumed, however, that *mashaḥ* is a later form, since there is reason to believe that the biblical word *madad* is the later Hebrew word for the much older common Semitic root *mashaḥ*, a form that was in common use in Assyrian literature.[7] The word *meshîḥah* as it appears in the *Mishnath ha-Middoth* is undoubtedly used as a well-known ancient scientific term for which the definition is given. The author of the work seems proud and anxious to write in the classic Hebrew style, and twice he tries to introduce the biblical word *medîdah* instead of *meshîḥah.*[8] This he has to give up, however; the old habit is victorious and the ancient term *meshîḥah* holds place throughout the rest of the book as the only word for area. It seems clear, therefore, that we may safely assume the existence of an ancient and well-known Semitic term for area namely *meshîḥtu*, *meshîḥah*, or *misâḥah.* The earth measurers (geometers) were called *mashîḥanu* or *mashôḥa*, in Arabic *massâḥ*; they were the measurers of area, the ones who handled the measuring cords (*meshîḥah*).

---

[1] *Annali di Matematica pura ed applicata*, VII, 271.

[2] *Zur ältesten Arabischen Algebra*, p. 103, note.

[3] The date is very uncertain, as stated by Professor Smith, *History*, I, 174; but the writer is confident that it belongs early in the Christian era. The definition appears in I, 6.

[4] See I, 2, 3, 4, 5, 6, 7, etc., for the area, and II, 6, 7, 9 for the volume (*meshîḥath ha-gûf*, or *ha-'ammûd*).

[5] The word *Thishboreth* is discussed later.

[6] See the Talmudic dictionaries of Levy, Jastrow and Kohut, and the *Thesaurus Syriacus* (II, 2235 seq.) of Payne Smith.

[7] See the Assyrian dictionaries of Delitzsch (pp. 430, 431) and Muss -Arnold (pp. 600, 601), which relate *mashaḥu* (to measure, to measure a field) *meshiḥu* (measure in general, also measure of a quantity of grain or dates), *meshiḥtu* (measure, extent of ground, field), and *mashiḥanu* (land surveyor). In *meshiḥtu* we probably have the old Assyrian word for area.

[8] See the *Mishnath ha-Middoth*, I, 1, 6, where *medîdah* is properly translated "area" (not understood by Schapira). The passage I, 1 should be translated: "There are four ways to grasp (ascertain) the area"; and (I, 6), "and the *medîdah* (area) contains four times the square unit."

We also find traces of the measuring cord in early Egypt, used in connection with length and with the square unit employed in land measure. The *khet*, meaning the reel of cord, was 100 cubits long, and its square was the unit of area, the Egyptian *setat*, in Greek the *ároura* (ἄρουρα).[1] The Greek writer Democritus (c. 400 B.C.) refers to the *harpedonaptae* (ἁρπεδονάπται), rope stretchers,[2] possibly analogous to the Hindu *śulvasūtra* and to the Semitic *mashôḥa*. The Akkadeans, the ancient predecessors of the Assyrians, also used *ashlu* (cord) and *śubban* (half cord) as terms for measure.[3]

We therefore find an ancient term adopted by all Semitic writers; this term is analogous to Greek, Egyptian, and Sanskrit terms, all relating to surveying, and the parent in idea though not etymologically of the Greek word which gave us the term "geometry."

The second of the terms under consideration is the Greek word *chroia* (χροιά, χρῶμα; chroia, chroma), meaning "skin" and "color." It was used by the Pythagoreans,[4] but the date of its first appearance is quite uncertain. It is derived from the word *chrio* (χρίω, χράω, χραίνω, χρώξω), meaning to "rub," "anoint," "paint," "smear," and in general "to slightly touch the surface of a body." Now it is worth while to notice the relation of *chroia*, which became the Greek term for "surface," to the Semitic *meshîḥah*, already mentioned. Besides the meanings stated in the first part of this article it also means, like the Greek word, to "rub," "anoint," or "smear" the skin. Hence we have the Hebrew *mashîaḥ*, *Messiah*, and the Greek *Christos*, the anointed one. The Assyrian language recognized only the geometric meaning (to measure); the biblical Hebrew, only the second one (to anoint); while the Syriac, the Aramaic, the post-biblical Hebrew, and the Arabic used both meanings. From the first root of *mashaḥ* (to measure) there developed in the Semitic languages the term for area, while in the Greek language this came from *chroia*, corresponding to the second root of *mashaḥ* (to anoint). This suggests a possible connection between the two roots,—to touch the surface lightly either with the hand or with a measuring cord. On the other hand the Greeks may have received the word for area from the Semitic as they are known to have received many others.

This connection of geometric terms in different languages is also seen in the case of the late Greek expression τὸ ἐμβαδόν (*embadȯn*, *embadum*, area, as in the *Liber embadorum*). It comes from the word ἐμβαίνω (*embaino*, ἐμβαδίζω, *embadizo*, to step upon, walk apace, and then to measure with the feet

[1] T. E. Peet, *Rhind Mathematical Papyrus*, London, 1923, pp. 24, 25; F. L. Griffith, *Proceedings of the Society of Biblical Archaeology*, XVI, 236, seq.

[2] See T. L. Heath, *Greek Mathematics*, I, 121; Smith, *History*, I, 81; II, 288; Peet, *Rhind Papyrus*, p. 32. The Greek word is from ἁρπεδόνη, rope or cord, and ἁπτώ, to seize or handle.

[3] *Revue d' Assyriologie*, XVII, 133. Compare also the English "chain" as unit of measure.

[4] Smith, *History*, II, 276.

or by pacing).[1] Now the Arabic *masaḥa* (Semitic *mashaḥ*) means not only to measure and to survey a field, but also to walk apace, as in the Greek. The verse in Isaiah XL, 12, "Who hath measured the waters in the hollow of His hand and meted out heaven with the span," should, according to Rashi (c. 1100), read "Who hath measured the waters with the hollow of His sole or with His steps."[2]

We therefore have an interesting connection relating to the geometric *superficies*, surface, measure, area, skin, rope stretchers, *śulvasutra* (?), *harpedonaptae*, *chroia*, and *meshîḥah.*

The third of the curious names for area is *taksîr* or *thishboreth*, a word meaning "to break into pieces" and hence "fractions." After using *misâḥah* for area, as already stated, al-Khowârizmî changed to the word *taksîr* and used this throughout the chapter, with only a few exceptions. These exceptions were where he wrote *misâḥah* because he was in the immediate context using *taksîr* for volume.[3]

The question now arises as to the origin and meaning of *taksîr*. Thus far we have found area represented only by terms relating to surface, appearance, color, extent, measure, and the like,[4] it being quite clear why they were chosen. It is, however, not at first sight apparent why *taksîr* was chosen,—a word meaning fractions, or to break in pieces. Naturally we turn to Hindu literature, the source of so many ideas of al-Khowârizmî, and here we find a similar term, *cuṭṭaca* or *cuṭṭa*, meaning a pulverizer or a grinding, pulverizing multiplier.[5] This word, however, is used both for algebraic computations in general and for a special algebraic process,—namely to find a quantity such that when a given number is multiplied by it and the product is added to another given number, the sum is exactly divisible by a given

---

[1] The same idea occurs in the Latin *podismus*, Greek ὁ ποδισμός measuring by the feet, area. (Tropfke, IV, 2nd ed., pp. 37, 126–127.)

[2] Compare also Rashî on Numbers XXII, 42. Rashî's opinion receives support in the Chaldean use of the word *sha'ala* for hollow of the sole, step. See the dictionaries of Levy and of Jastrow.

[3] See p. 53, lines 10, 18. The name *taksîr* is used regularly for area on pp. 51 (lines 7, 9, and at the bottom), 52 (lines 3, 4, 15), 53 (lines 3, 7), 54 (lines 8–10), 57 (lines 14, 16), 55 (lines 8–14), 56 (lines 3–6, 12), etc. For volume the same word is used on pp. 53 (5, 8, 10), 62 (line 11), and 63 (line 6).

This use of *taksîr* is also given in the dictionaries of Lane (p. 2612[c]) and Dozy (II, 465). The latter quotes many geographical writers who use the term. It often occurs in the Arabic writings of the Hebrew philosopher Moses ben Maimon (c. 1200); see Friedlaender, *Lexicon to Maimonides*, under *kasara*; in addition to this see the manuscript commentary of Maimonides on the *Mishna Erubin*, I, 5, and II, 5, now in the Jewish Theological Seminary, New York City.

The new term *taksîr* did not, however, entirely crowd out the old *misâḥah*, and so we find Omar Khayyam (Khayyâmî, c. 1100) and Behâ Eddîn (c. 1600) using only the term *misâḥah* and *saṭch* (roof, surface). See Omar Khayyam, ed. Woepke, pp. 9 (lines 10, 12, 18), 27 (line 3), 29 (line 5), and the chapter on geometry in Beḥâ Eddîn's *Kholâsat al-Ḥisâb.*

[4] Smith, *History*, II, 276; Tropfke, IV, 2nd ed., pp. 37, 126–127.

[5] H. T. Colebrooke, *Algebra* · · · · *from the Sanscrit*, London, 1817, pp. 112, 113 seq., and 325 seq.

divisor.[1] It therefore appears that the term does not assist us with respect to the word *taksîr*.

Turning then to the Hindu terms for area, we find in the writings of Bhāskara (c. 1150) and Brahmagupta (c. 628) the words *phala*, *gañita*, *cshêtra-phala*, and *sama-côshta-miti*, which the commentators interpret to mean the "measure of like compartments, or number of equal squares of the same denomination (as cubit, fathom, finger, etc.) in which the dimensions of the side is given: i. e., the area or superficial content." Another explanation reads, "The area of a figure is represented by little square compartments formed by as many lines as are the numbers of the upright and side."[2] The related term *g'hana-phala* is used for the solid content, apparently meaning a number of equal cubes of the same denomination.[3] This suggests the idea that the term *taksîr* had the origin in the idea of breaking the whole area into equal square parts or units, thus obtaining an exact measure of superficial content.

This idea is corroborated by the medieval Hebrew use of this very term. The Hebrew name of the *Liber Embadorum* of Savasorda is *Ḥibbûr ha-Meshîḥah weha-Thishboreth*, *Meshiḥah* meaning "area," as already stated, and *Thishboreth* (breaking) being both a translation and a formal imitation of the Arabic *taksîr*, and very likely being first used in this work[4] (c. 1100). Although both *Meshiḥah* and *Thishboreth* are used, in the text itself the former gives place to the latter, a procedure already followed by al-Khowârizmî, as we have seen. Since that time *Thishboreth* and *Shibbûr* have been the current Hebrew terms for both area and geometry,[5] but none of this assists us in our understanding

[1] *Ibid.*, p. 113, note 1.

The best explanation of the term *cuṭṭaca* seems to be that given by Al-Bîrûnî, (c. 1030) in his *India*, ed Sachan, Arabic text, p. 74, lines 8-10, Engl. text, I, p. 155: "19. On *Kuṭṭaka*, i. e. the pounding of a thing. The pounding of oil-producing substances is here compared with the *most minute and detailed research*. This chapter treats of algebra and related subjects." The same association of ideas is to be found in the Hebrew form *diqdûq*. It means "pulverising," or "pounding of oil-producing substances," and is used as a term for the science of grammar with its minute research. *Diqdûq haberîm*, "the pounding of the fellow-scholars" is used for the dielectic discussions on Talmudic law.

[2] *Ibid.*, p. 69, note 6; p. 296, note 2.

[3] *Ibid.*, p. 88, note 2.

[4] Abraham bar Chiia Savaṣorda (c. 1070–c. 1136). As is well known the name Savasorda came from the Arabic *Sâḥib al-Shorta*, "Chief of the Guards." It seems first to have been used by Plato of Tivoli (c. 1120), see Smith, *History*, I, 206. Professor Guttman, introduction to the *Ḥibbûr ha-Meshihah*, p. XIII, note 3; and Professor Wolfsohn, *Jubilee Volume of the Hebrew Union College*, Cincinnati, p. 301, are both mistaken in their derivation of *Thishboreth* from the Arabic *shibr*. This would contradict all rules of Semitic philology, as well as Savasorda's own statement (*Ḥibbŭr ha-Meshihah*, p. 23) that it is a translation from the Arabic *taksîr*.

[5] See the Hebrew astronomy of Savasorda *Ṣûrath ha- Areṣ*, chapter IX; the Hebrew translation of the commentary of Maimonides on the *Misna Erubin*, I, 5; II, 5; IV, 2, in the first edition, Naples, 1492, the later editions erroneously giving *shi'ûr* instead of *shibbûr*. Compare further the astronomy of Isaac Israeli, *Jesod 'Olam*, I, 2, 3, pp. 2b, 3b (2d. ed., Berlin); the astronomy of David Gans, *Neḥmad we-Na'îm*, p. 17; § 31; Luzzato, *Kerem Ḥemed*, II, 70, 76; Steinschneider, *Gesammelte Schriften*, p. 399.

of the key word *taksîr*. We have, however, certain earlier Hebrew works which use the old Aramaic form *thebaryâthâ* (fractions) for area. For example Rabbi Ḥananel of Kairuan (c. 990-1050), in his commentary on *Erubin*, 14b, says: "10 cubits upon 10 cubits are 100 cubits in fractions" (*bi thebaryâthâ*), and the same form appears in his commentary, *Migdal Ḥananel*,[1] in an old Genizah fragment,[2] and in Ibn Ezra's commentary on Hosea, XIV, 6, where he says that "fractions" in geometry are analogous to "roots" in arithmetic.[3] Here then we have the proof that the "fractions" are merely the "square units," called *jadhr* (root, which also corroborates the writer's opinion[4] that the *jadhr* was originally a name for the square unit and not for an abstract root or for a side of one dimension. A similar explanation of the word *shebarîm* (fractions) is also given by Samuel ibn Tibbon in his lexicon of technical terms to Maimonides's *Guide to the Perplexed*.[5]

The clue to the meaning of the term is thus apparent. The ancient mathematicians made a great effort to bring home the idea that square measure is entirely different from linear and that therefore the square unit is not a line but a "fraction," a part of the whole area, and a magnitude of two dimensions. The same reasoning was also responsible for the introduction of the *jadhr*.[6] It is all a part of the age-long efforts to make the immature mind realize that the doubling of the side of a square does not double the area, but quadruples it. The *Mishnath ha-Middoth* (I, 7), for example, uses a figure precisely like the one found in our modern textbooks, and the *Liber Embadorum* of Savasorda does the same.[7] So the Genizah fragment above mentioned says that to find the area, *ḥalqînah bi-Thebaryâthâ* ("divide it into fractions"); Maimonides[8] (c. 1200) says: "*wa-huwa mauḍi'un fî misâḥatihi 'inda at-taksîrî* ("and this is a place whose area is found by breaking it into fractional parts"); and a Hindu commentator[9] remarks: "The area is represented by little square compartments formed by as many lines as are the numbers of the upright and side." The result of this breaking of an area into small fractions is to have a figure resembling the plaiting or twisting of basket work,

[1] Edited by Berliner, p. 38. Berliner confesses (p. XXV) to have written *thishboreth* for *thebaryâthâ*. This is purely a corruption of the text.

[2] The exact date is unknown. It was edited by Professor Louis Ginzberg, *Geonica*, II, 36.

[3] See also his commentary on Numbers, XXXIV, II. Ibn Ezra uses the Hebrew *shebarîm* and the Aramaic *thebaryâthâ* for fractions, and the Hebrew *shorashîm* for roots. *Shoresh* was first used by Ibn Ezra for the Arabic *jadhr*. See this Monthly, vol. 33, p. 262.

[4] This Monthly, vol. 33, p. 263.

[5] Under *thishboreth*. See also Iḥwân as-Safâ, ed. Bombay, p. 37, quoted by Ruska, *loc. cit.*, p. 69, note 1.

[6] See this Monthly, vol. 33, p. 263.

[7] P. 28, 2; Hebrew text p. 23.

[8] Commentary on the *Mishna Erubin*, I, 5. There are three MS. copies of it in the Jewish Theological Seminary, New York City.

[9] Colebrooke, p. 297.

and so we find *Mishnath ha-Middoth* saying (II, 5): *Meṣaref ha-orekh bethokh ha-roḥab* ("He who has to measure square figures has to plait, or twist length into breadth"). This is the original meaning of *ṣaref* or *ṣoref*, "to plait" or "melt", and later "to multiply."[1]

From the facts here adduced we may fix the approximate dates of introduction of the terms mentioned. The earliest known Hebrew geometry set forth the method of calculating an area by dividing it into "fractions," but did not know the terms *taksîr*, *thebaryâthâ*, or *thishboreth*. Mohammed ibn Musa (c. 800), Rabbi Ḥananel (c. 1000) and the Genizah fragment (c. 900-1000) introduced the terms without any explanation, as if they were well-known at that time. We can trace them to the Hindu terms *phala* and *cshêtra-phala* as given in the works of Brahmagupta (c. 628). Whether this suffices to prove the Hindu origin of the idea must be left to the determination of Sanskrit scholars, and it is hoped that this study will lead them to undertake this definite and important piece of work.[2]

ON THE LEAST MULTIPLE OF AN INTEGER EXPRESSIBLE AS A DEFINITE QUADRATIC FORM

By H. S. VANDIVER, University of Texas

In his *Report on the Theory of Numbers*, H. J. S. Smith[3] discussed methods for the solution of the quadratic congruence $x^2 \equiv D (\mathrm{mod} P)$ where D and P are integers, D prime to P. After discussing a method due to Gauss,[4] he describes a simplification of it as follows:–

"By a known property of quadratic forms, whenever the congruence $r^2+D\equiv 0 \ (\mathrm{mod} P)$ is resoluble, the equation $mP=x^2+Dy^2$ is resoluble for some value of $m<2(D/3)^{1/2}$, if $D>0$. By assigning therefore to m all values in succession which are inferior to that limit and which satisfy the condition

$$\left(\frac{m}{P}\right)=\left(\frac{P}{D}\right)$$

(these are Legendre's quadratic residue symbols) and then obtaining by Gauss' method all prime representations of the resulting products by the form x^2+Dy^2, we shall have

$$r \equiv \pm x'/y', \quad r \equiv \pm x''/y'', \cdots (\mathrm{mod}\ P)$$

[1] The terms used in multiplication have been mentioned by Professor Smith, *History*, II, 114–116, and 102 seq. The subject is very extensive in relation to the Semitic languages and deserves careful study. To *saref* = "plait" compare *sarîf* = "roof of twisted reeds."

[2] The writer wishes to express his appreciation of the assistance of Professor David Eugene Smith in the preparation of his manuscript.

[3] British Association Report, 1860, Art. 68, p. 120; Collected Mathematical Papers, vol. 1, p.148.

[4] *Disquisitiones Arithmeticae* (1801), Articles 327, 328.

$x', y'; x'', y''$; etc. denoting the different pairs of values of x and y in the equation $mP=x^2+Dy^2$."

In the first place it is possible to show that this method does not always yield all the roots of the congruence when P is composite. For examine the congruence $u^2+11\equiv 0(\text{mod}\,45)$. It has four incongruent roots ± 13, ± 22. Following Smith's method we find $1+11\times 2^2=45$ and $6^2+11\cdot 3^2=3\cdot 45$. The first relation gives ± 22 as solutions of the proposed congruence, but clearly the second relation, in itself, does not give any roots. Smith's use of the notation $r\equiv \pm x'/y'$, and use of the term prime representation indicate that he meant that x' and y' could always be taken prime to each other, which is contrary to the situation in the above example. Also, in quoting Smith's[1] results the statement "x and y prime to each other" was included by the writer.

The "known property of quadratic forms" referred to by Smith, namely that whenever the congruence $r^2+D\equiv 0(\text{mod}\,P)$ is resoluble the equation $mP=x^2+Dy^2$ is resoluble for some value of $m\leqq 2(D/3)^{1/2}$, was proved by Oltramare,[2] for the case P a prime, and by Aubry,[3] for the general case the latter using an unnecessarily complicated method.

In another paper on factoring integers it will be necessary for me to make use of a slightly more specific result which includes Smith's statement.

THEOREM: *If $r^2+a\equiv 0(\text{mod}\ n)$, where a is positive and prime to $n>1$, there exists at least one set (x, y) such that*

$$x^2+ay^2=kn,\quad k\leqq 2(a/3)^{1/2} \text{ and } ry\equiv \pm x(\text{mod}\ n)\ ,$$

x and y being positive integers or zero.

The argument we shall employ depends on well known methods in the theory of definite quadratic forms.[4] If $0<r<n$ and $r^2+a\equiv 0(\text{mod}\ n)$, a positive, consider the form, $F=(n, r, (r^2+a)/n)$. The form F is equivalent to a certain reduced form, say (A, B, C), where $C\geqq A\geqq 2\,|\,B\,|$. From this we have

$$AC\geqq A^2 \text{ and } B^2\leqq \tfrac{1}{4}A^2 \text{ whence, } AC-B^2=a\geqq 3A^2/4 \text{ or } A\leqq 2/(a/3)^{1/2}.$$

If the substitution

$$T\colon \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\quad \alpha\delta-\beta\gamma=1 \quad \text{converts } (A, B, C) \text{ into } \left(n,\ r,\ \frac{r^2+a}{n}\right),$$

then $$A\alpha^2+2B\alpha\gamma+C\gamma^2=n \quad\text{and}\quad (A\alpha+B\gamma)^2+a\gamma^2=An,$$

---

[1] Vandiver, Bulletin of the American Mathematical Society, vol. 30, (1924) p. 548, last line.

[2] Journal für die reine und angewandte Mathematik, vol. 49 (1855), pp. 142–160.

[3] Association française pour l'avancement des sciences, vol. 40 (1911), pp. 55–60.

[4] Dirichlet-Dedekind *Vorlesungen über Zahlentheorie*, 4th edition (Braunschweig, 1894), pp. 154–158.

and putting $x=A\alpha+B\gamma$, $y=\gamma$, we have

$$x^2+ay^2=kn \qquad \text{with} \qquad k\leqq 2(a/3)^{1/2}.$$

To complete the proof of the theorem it is necessary to show that $ry\equiv \pm x(\text{mod } n)$. Since T converts (A, B, C) into

$$\left(n, r, \frac{r^2+a}{n}\right),$$

it follows that

$$A\alpha^2+2B\alpha\gamma+c\gamma^2=n$$
$$A\alpha\beta+B(\alpha\delta+B\gamma)+c\gamma\delta=r$$
$$A\beta^2+2B\beta\delta+c\delta^2=(r^2+a)/n \text{ with } \alpha\delta-\beta\gamma=1.$$

Using the first and last of these relations it is easy to verify that

$$(A\alpha+B\gamma)\equiv-(A\alpha\beta+B(\alpha\delta+\beta\gamma)+c\gamma\delta)\gamma, \text{ (modulo } n).$$

Hence $x\equiv\pm ry(\text{mod } n)$.

Recalling Smith's apparent idea that x and y are prime to each other, we note that, if $A\alpha+B\gamma$ has a factor p in common with γ, then $A\alpha\equiv 0(\text{mod } p)$; that is $A\alpha$ has a factor in common with γ, a situation which is not unusual in the relation $A\alpha^2+2B\alpha\gamma+C\gamma^2=n$.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

A PARADOX RESULTING FROM INTEGRATION BY PARTS

By J. L. WALSH, Harvard University.

We start with the usual formula for integration by parts

$$\int udv=uv-\int vdu,$$

and wish to evaluate the integral $\int dx/x$. We set $u=1/x$, $dv=dx$, deriving of course $du=-dx/x^2$, $v=x$. The above formula for integration by parts becomes

$$\int\frac{dx}{x}=1+\int\frac{dx}{x}.$$

Cancellation of the two integrals then yields the familiar equation $0=1$.

QUESTION

56. Proposed by J. E. Trevor, CORNELL UNIVERSITY.[1]

[1] The proposer needs an answer to this question for use in some work on the conditions of stability of certain thermodynamic equilibria.

Let $F(x_1, x_2, x_3, x_4)$ be a homogeneous analytic function of the independent real variables x_1, x_2, x_3, x_4, of degree one, wherefore

$$t \cdot F(x_1, x_2, x_3, x_4) = F(tx_1, tx_2, tx_3, tx_4)\ ; \tag{1}$$

and let quantities X, y_1, y_2, y_3 be defined by the equations

$$X = x_3 + x_4, \quad y_1 = x_1/X, \quad y_2 = x_2/X, \quad y_3 = x_3/X.$$

When we put $t=1/X$, the equation (1) becomes $F/X=F(y_1, y_2, y_3, 1-y_3)$, which shall be written

$$F = X \cdot f(y_1, y_2, y_3). \tag{2}$$

Let the four x_i be given independent increments δx_i, whereupon the three y_j obtain increments δy_j, which are equal to zero when the conditions

$$\frac{\delta x_1}{x_1} = \frac{\delta x_2}{x_2} = \frac{\delta x_3}{x_3} = \frac{\delta x_4}{x_4}$$

are satisfied. The corresponding increments of the functions F and f shall be denoted by δF and δf, that is

$$F(x_1 + \delta x_1, \cdots, x_4 + \delta x_4) - F(x_1, x_2, x_3, x_4) = \delta F$$
$$f(y_1 + \delta y\ , \cdots, y_3 + \delta y_3) - f(y_1, y_2, y_3) \qquad = \delta f,$$

and Taylor's expansion shall be written in the notation

$$\delta F = \delta^1 F + \delta^2 F + \cdots, \qquad \delta f = \delta^1 f + \delta^2 f + \cdots, \tag{3}$$

where $\delta^n F$, or $\delta^n f$, is the sum of terms of order n in the expansion.

In a note in this MONTHLY, vol. 26(1919), p. 445, it is shown that $\delta^1 F$ is connected with $\delta^1 f$, and that $\delta^2 F$ is connected with $\delta^2 f$, by the relations

$$\frac{\delta^1 F}{X} = \frac{\delta X}{X} f + \left(1 + \frac{\delta X}{X}\right)\delta^1 f, \qquad \frac{\delta^2 F}{X} = \left(1 + \frac{\delta X}{X}\right)^2 \delta^2 f. \tag{4}$$

Now since, by (2), F is equal to the function Xf of X and f, and X, f are independent, it follows that

$$\delta F = f\delta X + X\delta f + \delta X \delta f. \tag{5}$$

Elimination of δF, $\delta^1 F$, and $\delta^2 F$ between (4), (5), and the first of equations (3) yields

$$\left(1 + \frac{\delta X}{X}\right)\delta f = \left(1 + \frac{\delta X}{X}\right)\delta^1 f + \left(1 + \frac{\delta X}{X}\right)^2 \delta^2 f + \text{terms in } \delta^3 f, \text{ etc.}$$

It is desired to know whether it is generally true that

$$\frac{\delta^n F}{X} = \left(1 + \frac{\delta X}{X}\right)^n \delta^n f, \quad n > 1.$$

RECENT PUBLICATIONS

Edited by W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Elementarmathematik vom höherem Standpunkte aus. I. By FELIX KLEIN. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen. Band XIV. Dritte Auflage. Berlin, Julius Springer, 1924. viii+321 pages, 125 figures.

Before making its appearance as the fourteenth number of the Courant series, this work of the late Felix Klein appeared in two mimeographed editions, the first in 1908, the second in 1911. The present printed form is the original text with the incorporation of the changes in the second edition and the inclusion of additional footnotes not found in the first or second editions. These latter are enclosed in brackets.

As the title indicates, the book deals with elementary mathematics from the point of view of higher mathematics. It is in the form of a series of lectures, primarily intended for teachers of mathematics in secondary schools and for mature students. Klein and Schimmack had published a report (April, 1907) on the mathematics courses in German secondary schools and their organization. The present work of which the first volume is under review is by way of amplification and expansion of this report.

Klein represented in Germany a movement with respect to the teaching of mathematics such as did Perry in England and the National Committee on Mathematical Requirements, J. W. Young, chairman, in the United States. It was prompted by a desire to vitalize the teaching of the subject by giving the teacher a breadth of view with regard to the interrelationships of the various parts of mathematics that might enable him to inspire in his students a real interest and enthusiasm for mathematical study. Klein's purpose was to give the teacher a panorama or survey of the field from the heights which would both deepen his insight and provide wider vistas. As the most prominent German mathematician of his generation, and equally one of the greatest and most versatile, Klein was well qualified to succeed in just such an undertaking.

The book has a decidedly conversational flavor, quite in the style of a heart-to-heart talk and hence is very readable; it exhibits everywhere the accuracy and authority of one who gained world renown. Homely illustrations abound. The graphical side is stressed, as is attested by the presence of 125 figures in the book itself and emphasis placed on pictorial representation. Much interesting historical matter is included to aid in the presentation of the systematic development of mathematics. From Euclid to Poincaré, from axiomatic foundations to the most practical multiplying machine, the reader

tours through the time-space domain of mathematics. Here is mathematical pedagogy of the highest type, shorn of all pseudo-scientific speculation. Here is mathematical history by one who knows not alone names and dates, but significance and interpretation.

The present volume (I) deals with arithmetic, algebra and analysis. Geometry is reserved for volume[1] II and does not enter within the precincts of this review. Part I begins with the natural numbers and includes the extension of the number concept, some elementary number theory, complex numbers together with quaternions. Part II treats mainly of the solution of algebraic equations, with a maximum attention to graphical methods. Solvability is also considered. Part III in analysis treats the logarithmic, exponential and trigonometric functions, differentiation, integration and Taylor's expansion. Two appendices to this part deal with the transcendence of e and π, and an introduction to point sets.

The text itself was prepared by E. Hellinger which ensures its accuracy. Two short chapters are added by Seyfarth giving the developments in mathematical courses in secondary schools in Germany since 1911 and a list of recent mathematical works of interest to teachers. Throughout the book a bibliography is to be found of the classic mathematical treatises. The typographical errors are very few and unimportant.

As an inspiration for the best kind of mathematical teaching this book is recommended to all teachers of mathematics of high school and elementary college grade.

H. J. ETTLINGER.

Éléments de la Théorie des Probabilités. By ÉMILE BOREL. Third Edition. Paris, J. Hermann, 1924. vii+226 pages.

This book deals with discontinuous probabilities, continuous probabilities, and the probabilities of causes, four or five chapters being devoted to each subject. The above mentioned parts appear in this edition as books I, II, and III. The first edition, which appeared in 1909, contained these three books, and the only essential change in the third edition is the inclusion of 39 pages containing four notes. The first note is on radioactivity, probability and determinism; the second, on the employment of the differential method for the comparison of statistics; the third, on the expression of probabilities in certain games of chance in terms of integral equations; and the last, on games of chance as affected by the ability of the players.

To go a little more into detail, the first book deals with *a priori* methods of probability and the theory is illustrated by the usual exercises in coin tossing.

[1] Volume II has already appeared. ED.

The definition of mathematical expectation is given and total and compound probability are illustrated by drawings from urns. The first and second approximations of the formula of Stirling are developed and the normal probability function is defined and a number of its relations are developed. The law of great numbers receives considerable attention in this book, in fact, the book leads up to and concludes with the development of this subject.

The second book is occupied with geometrical probability which is illustrated by numerous examples well known in the classical theory of this subject. The author also deals with arbitrary functions and makes an application to the distribution of the minor planets and another one to the kinetic theory of gases. The book concludes with a chapter on errors of observation and the method of least squares and the various notions arising in connection with this theory.

The third book introduces Bayes' theorem and it is illustrated by various problems. The first approach to modern statistics as developed through empirical processes appears in this book. The notions of empirical probability are illustrated by means of urn schemata and by statistics relating to masculine and feminine births, births of twins, the number of births in a family, and mortality statistics. The book concludes with a brief discussion of continuous probabilities as applied to the determination of causes. The author considers in this connection the distribution of the stars upon the celestial sphere, the values of atomic weights and some biometric applications.

One leaves this book with the general feeling that the author has explained the simple ideas of *a priori* and empirical probability and illustrated them up to a certain point, but falls short of bringing the student to a realization of how much is behind—or perhaps one should say ahead—of these elementary matters which have been set forth. The more extensive applications to large subjects are not sufficiently developed and the reader is left in a state of doubt as to what it is all about. All this of course merely emphasizes the fact that the subject has so many ramifications and applications that if one attempts to circumscribe them all in a relatively small book the general effect must be that of sketchy outline lacking tremendously in illuminating and guiding details. The point I wish to make is well illustrated by the proposed new[1] publication of four volumes and seventeen sub-titles,—in reality seventeen books, of which four have already appeared—dealing either with theory or application. Each one of these seventeen books is about as bulky as the one under review and is devoted to a topic which perhaps in the latter is covered in three or four pages or omitted altogether.

---

[1] *Traité du Calcul des Probabilités et de ses Applications.* Par Émile Borel, avec collaborateurs.

I might conclude by saying that the author affords the student a fair start in the elements of the theory of probability and perhaps did as well as anyone could do. The fact remains, however, that any book on this subject which attempts to cover so much ground by way of suggestion and an occasional application—where often the real significance of the application is missed—is almost bound to fail if its object is to leave the reader in a lucid state of mind concerning the fundamentals of the theory of probability and its very great power in varied fields of application.

J. W. GLOVER.

Curve Sghembe Speciali Algebriche e Trascendenti. Volume two. Curve trascendenti. By GINO LORIA. Bologna, N. Zanichelli, 1925. 255 pages. Price L 50.

There are several extensive works on the properties of particular plane curves, among them this author's excellent Special Plane Curves.[1] No correspondingly complete treatment of particular space curves was available before the appearance of this work. We welcome it accordingly as filling a real need.

Although this second volume[2] is entitled Transcendental Curves, its first chapter deals with algebraic curves of arbitrary order. This is followed by a chapter on spherical curves, after which comes a discussion of curves defined by their intrinsic equations. The last chapter is devoted to curves lying on given surfaces. The book is crammed with varied, useful, and well selected information on a wide variety of topics. The chief theorems are proved in full for each type of curves. Numerous (but not exhaustive) references in footnotes serve to guide the reader who wishes to pursue any subject further. The incidental discussion of special surfaces is considerable and adds to the value of the book. The author's wide reading and thorough mastery of his subject is evident on every page.

Minor errors are rather frequent. The introduction of new topics and the statement of restrictive hypotheses are sometimes inadequately displayed.

An index of names for the entire work appears at the end of this volume. The absence of a subject index seems unpardonable in a work of this kind but its place is filled in part by a good table of contents.

C. H. SISAM.

Great Circle Sailing. By L. M. BERKELEY. New York, White Book and Supply Co., 1925. vi+45+25 pages. Price $1.50.

The writer proposes in this pamphlet to show that the advantage of following a great circle track is greater than is commonly supposed, and to encourage

[1] *Spezielle algebraische und transcendente ebene Kurven*, two volumes, Fritz Schütte, 1910–1911.

[2] A review of the first volume will be found in the Bulletin of the American Mathematcal Society, vol. 31, p. 557.

the use of great circle sailing by offering to navigators a method of computing a course that is simpler than any in common use, and a method of correcting the course from time to time, that is not commonly used. He specifically offers a gain of five or six hours on each trans-Atlantic trip as a reward for following a certain set of formulas.

Undoubtedly, increase in the expense of rapid ocean transit and a rising valuation of passengers' time make it more and more important to choose the quickest course, but it is not clear that the practice of the managers of steamship lines is more than normally belated in following the changing conditions. The great circle tracks that would make the greatest gain for the swift liners are precisely those that are most blocked by land and hindered by difficult northern waters; some cannot be sailed, and in others, the loss through lessened speed, battered ships and distressed passengers would offset the greater length of the more southerly passage. These liners do follow the northern lanes in the favorable season. In our Navy, certainly the great circle track is well understood, and is followed, when desirable, with no dearth of convenient methods of navigating. For tramp steamers, the problem is not so acute; short runs between ports mean more cargoes and less fuel to be carried, and there are no fixed schedules that will insure the time gained on a run from being swallowed up in time lost in port.

The author's later argument against these views is not sound, but the promises of his introduction should tempt a navigator to read on, if he is unused to great circle sailing and wishes to improve his practice, or if he is more experienced and is attracted by the promise of an easier method of setting his course. The navigator who does read the text will find himself confronted at the start by an unfamiliar notation and by tables (three figure Gaussians) unfamiliar in form and use. In his usual practice, he already has the choice of the Azimuth Tables, from which he can read his course in a minute, as closely as he can use it, or the Hydrographic Office Azimuth Chart, or Captain Weir's, from either of which he can read his course directly; or, if he prefers to compute, he can choose his favorite from the many methods of solving the Altitude-Azimuth sight. This sight, if he is up-to-date, he uses from eight to twelve times a day in the Marc St. Hilaire method of determining a line of position, and its problem is identical with that of the great circle course and distance. It is difficult to feel that he will think the promise of greater simplicity has been kept, for the intensive study and practice of the fundamental sight has evolved so many formulas that hardly any taste can be left ungratified. Neither is he likely to value the method of occasionally correcting his course, for he ordinarily resets his course five or six times a day, when he works out his routine sights, and that is often enough. No calculation of future settings, unchecked by later observations, is likely to improve on the seamanship of an experienced navigator.

There remains the distinctly polemical part of the pamphlet. In this, the reader finds citations from five modern authorities, all belittling the advantage of the great circle track. He is told that the numerical values quoted, if put to the practical test of solution by the correct method, will be found to be so inaccurate as to disprove the contention of the authorities. He may then turn to the last chapter for an exposition of correct comparison of loxodromic and great circle distances; he is destined to be disappointed again, but he will find grounds for guessing why the authorities have been unanimous in disagreement with the author.

It seems probable that the five authorities were content to make their comparisons by means of the first approximation to the figure of the earth—the terrestial sphere. Through an error of judgment, the author fails to do this, apparently not realizing that storms, prevailing winds, ocean currents and the unavoidable inaccuracies of the steersman are of greater consequence than the ellipticity of the earth's meridians. Worse than this, he makes an improper comparison, and makes it incorrectly. His initial procedure is correct; he sets up the differential triangle for the second approximation, the terrestial spheroid, with the element of arc of the meridian as one leg, and finds a correct differential expression for the slope of the loxodrome. (His deduction would have been simpler if he had followed the ordinary practice of introducing the eccentric angle.) He then obtains the second approximation to the loxodromic course. His integration is correct, but furnishes an awkward formula for computation; use of the Binomial Theorem would have replaced his elaborate correction term by the expression $e^2(\sin L_2 - \sin L_1)$, which would have been adequate, especially as he has used a two-figure value of the eccentricity. At this point he makes a blunder; with the improved course, he returns to the sphere for his element of distance, discarding the element of arc of the elliptic meridian. It is difficult to say what unit of length he should use in giving the distance that results from this hybrid approximation; he uses the nautical mile; if he had completed his second approximation consistently, his use of the earth's equatorial radius would have given his result in minutes of equatorial arc; converting these to nautical miles would have offset in great measure the difference between his first and second approximations. The radius of the terrestial sphere is of course so chosen as to secure this effect. A more fundamental error is his comparison of first approximations of great circle distances with what at least purport to be second approximations of the corresponding loxodromes. If he is to do anything with second approximations, he should find the distance and courses of the track on the spheroid having at each point the latitude and longitude of a corresponding point located on the great circle of the terrestial sphere. When this is done, the courses and distances in the two examples that he presents are these:

		Initial Course	Final Course	Distance (naut. mls.)	Advantage of G.C. (naut. mls.)
Ex. 1. Sphere	Great Circle	41° 33!8	48° 35!6	5364.5	9.0
	Loxodrome	38°	35!5	5373.5	
Spheroid	Great Circle	41° 42!5	48° 42!4	5360.9	0.2
	Loxodrome	38°	45!3	5361.1	
Hybrid	Great Circle	— —	— —	—	—
	Loxodrome	38°	45!3	5386	
Ex. 2. Sphere	Great Circle	50° 11!3	78° 13!0	2721.1	89.5
	Loxodrome	74°	6!6	2810.6	
Spheroid	Great Circle	50° 18!0	78° 14!7	2728.1	89.9
	Loxodrome	74°	9!5	2818.0	
Hybrid	Great Circle	— —	— —	2721	98
	Loxodrome	74°	9!6	2819	

In the first example, the loxodrome crosses the equator along in the middle part of the run, and there is consequently little difference between the great circle and loxodrome distances; the author does not compare these distances. The second great circle, from Nantucket Shoals to Liverpool, is a characteristic example of a track in moderately high latitudes (it reaches 54.3 N.), showing a considerable difference between the distance run on it and on the loxodrome. The comparison would be more valuable if the direct loxodrome from Sandy Hook were compared with this great circle distance plus the run to Nantucket Shoals. But this great circle is characteristic in another way; it has been started far enough out to clear Long Island, as the author says, and it consequently clears New England (the direct track from New York would go overland until it reached a point well up on the Labrador coast), but after skirting the coast of Nova Scotia, it cuts across land north of Cape Canso, later crossing Newfoundland (from about 47°4 N, 56°2 W to 48°6 N, 53°0 W), and still later, of course, Ireland.

The two examples indicate that the use of the terrestial sphere causes little error in the comparison of the two kinds of distance; the author makes about sixteen times as great an error in the second example by flirting with the spheroid. It is fair to say that in his disagreement with the five authorities, he was probably wrong in his criticism of their numerical results.

Advantageous great circles are actually rare, especially in those waters where saving of distance would be most highly appreciated. The northern continents, the Antarctic Ocean and the easterly bulge of South America spoil a great many attractive circles, leaving mainly some from Australia, some around the Cape of Good Hope, and some across the Atlantic over which the shortening of the distance is not great. On one occasion a navigator carefully

laid out a great circle track on a long Pacific run, but found when he finished the run, that on account of corrections he had been obliged to make for ocean currents, he had followed a loxodrome almost exactly.

The concluding paragraphs of the pamphlet give estimates of the saving in cost made by following the track across Newfoundland, but the zest has gone from it.

PAUL CAPRON.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Transactions of the American Mathematical Society, volume 28, No. 3, July 1926: "Double binary forms with the closure property" by A. B. Coble, 357–383; "On a class of polynomials in the theory of Bessel's functions" by J. H. McDonald, 384–390; "On the zeros of the function β (z) associated with the gamma function" by T. H. Gronwall, 391–419; "Osculating curves and surfaces" by P. Franklin, 400–416; "On Laplace's integral equations" by J. D. Tamarkin, 417–425; "On Volterra's integro-functional equation" by J. D. Tamarkin, 426–431; "On the discriminant of ternary forms and a certain class of surfaces" by A. Emch, 432–434; "On the convergence of certain methods of closest approximation" by E. Carlson, 435–447; "Functions of plurisegments" by A. J. Maria, 448–471; "Sets of postulates for the logic of propositions" by B. A. Bernstein, 472–478; "Multiple-sheeted spaces and manifolds of states of motion" by H. Hotelling, 479–490; "A theory of a general net on a surface" by V. G. Grove, 491–501; "Cubic curves and desmic surfaces" by R. M. Mathews, 502–522; "Expansion problems in connection with homogeneous linear q-difference equations" by M. G. Carman, 523–535; "Some properties of limited continua, irreducible between two points" by W. A. Wilson, 536–553; "Applications of the theory of relative cyclic fields to both cases of Fermat's last theorem" by H. S. Vandiver, 554–560.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3236. Proposed by H. S. Uhler, Yale University.

Find the equation of a circle (if it exists) such that the sum of the squares of the (radial) distances from the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$, $(4, 3)$ to the circumference shall be a minimum.

3237. Proposed by Nathan Altshiller-Court, University of Oklahoma.

A variable chord of an ellipse subtends a right angle at the center of the curve. Find the envelope of the chord.

3238. Proposed by Malcolm Foster, Williams College.

Determine the necessary and sufficient condition that the loci of the centers of first and second curvature of a given curve have orthogonal tangents at corresponding points.

3239. Proposed by Alex S. Wiener, Cornell University.

Solve the following simultaneous equations for u, v, w, x, y, and z:

$$(u - a_1)(x - b_1) - (v - b_1)(w - a_1) = 0,$$
$$(u - a_2)(z - b_2) - (v - b_2)(y - a_2) = 0,$$
$$(w - a_3)(z - b_3) - (x - b_3)(y - a_3) = 0, \text{ and}$$
$$u^2 + v^2 = w^2 + x^2 = y^2 + z^2 = r^2.$$

3240. Proposed by Lü Ling, Tsing Hua University, Peking.

On a given base, construct a triangle such that its vertex may be on a given straight line, and the difference between its base angles may be equal to a given angle.

3241. Proposed by F. A. Berger, Washington University.

With a center on the circumference of a given circle an arc of a second circle is drawn which bisects the area of the first circle. Find the ratio of the radii of the two circles.

3242. Proposed by R. S. Underwood, Auburn, Alabama.

A man finds that a pile of cocoanuts is exactly divisible by n after giving an extra nut to a monkey. He takes away $1/n$th of the remaining nuts and leaves the rest. A second man repeats the process with the rest giving one nut to the monkey and taking away exactly $1/n$th of the rest. This is continued and the nth man leaves at the end a pile of nuts which is exactly divisible by n. How many nuts were there at the beginning and how many at the end?

3243. Proposed by J. L. Riley, Ouachita College.

Show that the number of points whose coördinates are integers (exclusive of the origin) contained within the circle $x^2+y^2=m$ is four times the number contained within the area bounded by the line $x=0$ and the hyperbolas

$$y(4x+1)=m \text{ and } y(4x+3)=m. \quad \text{(Eisenstein, Crelle, XXVII, p. 248.)}$$

SOLUTIONS

2831 [1920, 227; 1925, 481]. Proposed by B. J. Brown, Kansas City, Mo.

Given that the series $\sum_0^\infty a_n y^n$ is absolutely convergent when $|y|<1$, prove that the series $\sum_0^\infty a_n(2x\cos\theta-x^2)^n$ may be arranged in powers of x provided that $|x|<2/5$.

Prove that if x lies between 1 and 2 the series $\sum_0^\infty(2x-x^2)^n$ is convergent and consists only of positive terms, but that the series obtained by arranging it in powers of x diverges.

Solution by Otto Dunkel, Washington University.

In the first part of the problem where x may be real or complex, if $|x|\leqq\rho$, then

$$|y| = |2x\cos\theta - x^2| \leqq 2\rho + \rho^2. \tag{1}$$

The function $\rho^2+2\rho$ increases from 0, when $\rho=0$, to 1, when $\rho=2^{1/2}-1>2/5$. Hence the series

$$\sum a_n(2x\cos\theta - x^2)^n \tag{2}$$

is absolutely convergent if $|x|<2^{1/2}-1$, and this is surely true if $|x|\leqq 2/5$. It will now be shown that the series may be rearranged as a power series in x. We know that, if x is small enough, say $|x|<\delta$, we may

rearrange (2) as a power series in x. Now (2) is an analytic function of an analytic function, if $|x|<2^{1/2}-1$ and it may therefore be expanded as a power series in x within this circle. This expansion must coincide with the one within the δ circle, and hence this latter expansion can be extended, if $\delta<2^{1/2}-1$, to the larger circle. This completes the proof of the first part.

In the second part of the problem x is real. The given series is convergent if $|2x-x^2|<1$. The function $2x-x^2$ decreases from 1 at $x=1$ to -1 at $2^{1/2}+1$. Hence the given series is convergent if $1<x<2^{1/2}+1$. If we restrict x to the shorter interval $1<x<2$, then $2x-x^2$ is positive. This series cannot be written as a power series in x if $|x|>1$. For, if the power series in x were convergent for $x=\bar{x}$ where $|\bar{x}|=\rho>1$, it would be absolutely convergent, x real or complex, in the circle $|x|<\rho$. In this case $1/[1-(2x-x^2)]=1/(1-x)^2$ could be written either as a power series in x or in $(2x-x^2)$ in this circle. It would then follow that $1/(1-x)^2$ is an analytic function in a circle with a radius larger than 1: but this is impossible since there is a singular point at $x=1$. Hence the power series in x cannot converge if $|x|>1$.

3160 [3156; 1926, 46]. Proposed by R. E. Gaines, University of Richmond.

A *given* ellipse always touches a fixed straight line at a given point. Find the locus of its center. Find the locus of the focus of a parabola under similar conditions.

SOLUTION BY ROSCOE WOODS, State University of Iowa.

The locus sought under the conditions of the above problem is known under the name of glissette, that is, a curve traced by a point that is carried by a curve two of whose points glide on a fixed curve. In this case the fixed curve is a straight line and the two points are coincident. One should consult W. H. Besant's *Roulettes and Glissettes*. (This book is not accessible to the writer.) The method used to obtain these loci is general for this type of problem.

Consider the ellipse $x^2/a^2+y^2/b^2=1$. Take any point P ($a\cos\theta$, $b\sin\theta$) on the curve. The equations of the tangent and normal at this point are

$$bx\cos\theta+ay\sin\theta-ab=0,$$

and similarly

$$ax\sin\theta-by\cos\theta-a^2e^2\sin\theta\cos\theta=0,$$

where $a^2e^2=a^2-b^2$.

We may now easily calculate the distances from the origin to these lines. Call the distances Y, X, respectively. Then we find that $X=\pm(a^2e^2\sin\theta\cos\theta)/(b^2+a^2e^2\sin^2\theta)^{1/2}$, $Y=\pm ab/(b^2+a^2e^2\sin^2\theta)^{1/2}$. Consider now the tangent and the normal as fixed lines. Then (X, Y) are the coordinates of the center of the ellipse referred to this pair of lines as axes. If θ is eliminated from these equations, the equation of the center of the ellipse will be given. This elimination gives $X^2Y^2=(a^2-Y^2)(Y^2-b^2)$.

This quartic consists of two loops symmetrical to the axes and origin. The Y-intercepts are $\pm a$, $\pm b$. For values of Y such that $|a|>Y>|b|$, X is real. The loop obtained depends upon which side of the Y-axis the ellipse is placed.

Consider now the parabola, $y^2=4px$ with the focus $(p, 0)$. In precisely the same way as before we can obtain the locus of the focus of the parabola that is always tangent to the y-axis at the origin. Consider the point $(pt^2, 2pt)$. The equations of the tangent and normal at this point are

$$ty-x-pt^2=0,$$

$$y+tx-2pt-pt^3=0.$$

Calculating now the distances from the focus $(p, 0)$ to these lines, we have $Y=\pm p(1+t^2)^{1/2}$, $X=\pm pt(1+t^2)^{1/2}$. The elimination of t from the equations gives the locus of the focus to be $p^2X^2=Y^2(Y^2-p^2)$.

This quartic cuts the Y-axis at the points $(0, \pm p)$ and is symmetrical to the axes and origin. X is real for all values of Y such $Y>|p|$. This curve has the origin as a conjugate point.

Also solved by E. F. ALLEN, THEODORE BENNETT, E. M. BERRY, RUFUS CRANE, ALEXANDER DILLINGHAM, and WILLIAM ROTH.

3161 [3157; 1926, 47]. Proposed by Dr. Bernardo Ig. Baidaff, Buenos Aires, Argentina.

Calculate

$$\underset{p}{(}x\,\underset{p-1}{(}x\cdots\underset{2}{(}x\,\underset{1}{(}xe^x\underset{1}{)'}\underset{2}{)'}\cdots\underset{p-1}{)'}\underset{p}{)'},$$

$$\underset{p}{(}x\,\underset{p-1}{(}x^2\cdots\underset{3}{(}x^{p-2}\underset{2}{(}x^{p-1}\underset{1}{(}x^p\underset{1}{)'}\underset{2}{)''}\underset{3}{)'''}\cdots\underset{p-1}{)^{p-1}}\underset{p}{)^{(p)}}$$

where the accents $'$, $''$, $\cdots$, mean the first, second, $\cdots$, derivatives, and where the parentheses are paired as indicated by the numbers under them.

SOLUTION BY THEODORE BENNETT, University of Illinois.

Let $\phi_0=e^x$, $\phi_1=(xe^x)'$, $\phi_2=(x(xe^x)')'$, $\cdots$, so that ϕ_i is the result obtained by the i-th differentiation. Then we have

$$\phi_{i+1}=(x\phi_i)'. \tag{1}$$

It is evident that ϕ_i is the product of e^x and a polynomial in x of degree i; moreover, the coefficient of x^i and the constant term in this polynomial are both equal to unity. Therefore, let us write

$$\phi_i=e^x\sum_{j=0}^{i}{}_iu_jx^j, \tag{2}$$

where we know that ${}_iu_0={}_iu_i=1$. Using this expression for ϕ_i (and the corresponding one for ϕ_{i+1}), equation (1) becomes

$$e^x\sum_{j=0}^{i+1}{}_{i+1}u_jx^j=\frac{d}{dx}\left[e^x\sum_{j=0}^{i}{}_iu_jx^{j+1}\right]=e^x\sum_{j=0}^{i+1}\left[(j+1){}_iu_j+{}_iu_{j-1}\right]x^j.$$

Therefore, ${}_iu_j$ is such a function of i and j that

$${}_{i+1}u_j=(j+1){}_iu_j+{}_iu_{j-1},\quad j=0,1,2,3,\cdots,i+1, \tag{3}$$

where we know in advance that ${}_iu_0={}_iu_i=1$, and ${}_iu_j=0$ if $j>i$ or $j<0$.

If we write the first $j+1$ equations of this set (obtained by setting $j=0,1,2,\cdots,j$) and from them eliminate all the u's with right hand subscripts less than j, we arrive at a linear difference equation of order j with constant coefficients, i.e. they involve j but not i. We solve this equation by a method analogous to that of solving a linear differential equation with constant coefficients. We shall not give the details of this work (which would be a bit tedious) but shall state the result and verify that it is correct. This result may be written

$${}_iu_j=\frac{1}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}(j-r+1)^i. \tag{4}$$

We propose to show by mathematical induction that (2) gives the correct value of ϕ_i provided ${}_iu_j$ has the value given by (4). If $i=1$ we have from (4) ${}_1u_0=1$, ${}_1u_1=1$, which is correct, since by direct differentiation $\phi_1=(xe^x)'=e^x(1+x)$. To complete the proof by induction we have only to show that ${}_iu_j$ as given by (4) satisfies (3). Substituting in (3) we have

$$\begin{aligned}\frac{1}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}(j-r+1)^{i+1}&=(j+1)\frac{1}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}(j-r+1)^i\\&\quad+\frac{1}{(j-1)!}\sum_{t=0}^{j-1}(-1)^t\binom{j-1}{t}(j-t)^i.\end{aligned} \tag{5}$$

The terms of the first two sums for which $r=0$ balance each other; omitting these two terms and rewriting the third sum in the form

$$\frac{1}{(j-1)!}\sum_{r=1}^{j}(-1)^{r-1}\binom{j-1}{r-1}(j-r+1)$$

we see that (5) is an identity, since

$$\frac{1}{j!}\binom{j}{r}(j-r+1)\equiv\frac{1}{j!}(j+1)\binom{j}{r}-\frac{1}{(j-1)!}\binom{j-1}{r-1}.$$

Hence we see that ϕ_i is given correctly by (2) and (4). The expression written in the statement of the problem is ϕ_p.

From the numbers ${}_iu_j$ we may form a triangular array which is similar to Pascal's triangle, and in fact the easiest way to compute ϕ_i for small values of i is to form this array, using equation (3). It is interesting to note that in the Pascal triangle the equation corresponding to (3) is ${}_{i+1}u_j={}_iu_j+{}_iu_{j-1}$.

To evaluate the second expression given in the problem we let

$$\phi_0 = 1, \quad \phi_1 = (x^p)', \quad \phi_2 = (x^{p-1}(x^p)')'', \cdots$$

so that

(6) $$\phi_{i+1} = (x^{p-i}\phi_i)^{(i+1)}.$$

It is fairly evident that ϕ_i is a constant multiplied by the $i(p-i)$th power of x; accordingly we assume

(7) $$\phi_i = C_i x^{i(p-i)}.$$

Substituting this value of ϕ_i in (6) we have

$$C_{i+1}x^{(i+1)(p-i-1)} = (C_i x^{(i+1)(p-i)})^{(i+1)} = C_i \frac{[(i+1)(p-i)]!}{[(i+1)(p-i)-(i+1)]!}x^{(i+1)(p-i-1)}.$$

Hence we wish to find C_i so that

(8) $$C_{i+1} = \frac{[(i+1)(p-i)]!}{[(i+1)(p-i-1)]!}C_i.$$

Then evidently

(9) $$C_i = \prod_{j=1}^{i}\frac{[j(p-j+1)]!}{[j(p-j)]!}.$$

Then from (7) we see that the desired quantity, namely ϕ_p, is given by (9) for $i=p$.

3168 [3164; 1926, 104]. Proposed by R. H. Sciobereti, University of California.

Study qualitatively the variations of the function

$$y = 2xe^{(1/x)} + 4x^3 - 15x^2 + 18x,$$

(the numerical values of the extrema are not required).

Find a polynomial $F(x)$ such that the difference $y-F(x)$ approaches zero when x becomes infinite.

SOLUTION BY THE PROPOSER.

The function y is defined and continuous for any value of x, except zero. Since the polynomial $4x^3-15x^2+18x$ vanishes only for $x=0$, it follows that y is negative for $x<0$ and positive for $x>0$. At $x=0$, the function being discontinuous, let us find limit y when x approaches zero; then

$$\lim_{x\to 0} y = \lim_{x\to 0} 2xe^{1/x} = \lim_{x\to 0} 2e^{1/x};$$

hence the limit of y is either $+\infty$ or 0, according as x approaches 0 through positive or negative values. Let us now determine the extrema of the function by finding the roots of the derivative

$$\tfrac{1}{2}y' = (x-1)(e^{1/x}\cdot x^{-1} + 6x - 9).$$

For $x<0$, the derivative is always positive; for $x=0$, it is discontinuous, while for $x>0$ the discussion is somewhat more complicated and the roots of the expression $u=e^{1/x}\cdot x^{-1}+6x-9$ must first be found. Denoting by ϵ a positive infinitesimal, we shall have $u(\epsilon)=+\infty$, $u(1)=e-3<0$, $u(\frac{3}{2})=\frac{2}{3}e^{2/3}>0$; therefore the function $u(x)$ which is continuous for $x>0$ vanishes an odd number of times in each of the intervals $(0,1)$ and $(1,\frac{3}{2})$. The first two derivatives of u are $u'=-x^{-2}\cdot e^{1/x}(x^{-1}+1)+6$ and $u''=x^{-4}\cdot e^{1/x}(2x+4+x^{-1})$. We readily see that u' vanishes only once between 0 and 1, since $u'(\epsilon)=-\infty$, $u'(1)=-2e+6>0$ and since u' is an increasing function of x, the second derivative u'' being always positive for $x>0$. Consequently the function $u(x)$ vanishes but once for $x=x_1$ in the interval (0,1) (Rolle's theorem). In the same manner it can also be shown that $u(x)$ vanishes only once for $x=x_2$ in the interval $(1,\frac{3}{2})$. It follows that the derivative y' vanishes and changes sign at the points $x=x_1$, $x=1$ and $x=x_2$; hence

the function y has a relative maximum at $x=1$ and two relative minima at $x=x_1$, $x=x_2$. The other special values of the first derivative and the sign of the second derivative $\frac{1}{2}y''=x^{-3}\cdot e^{1/x}+12x-15$ are easily determined and will be given in the following table:

x	$-\infty$		$-\epsilon, 0, +\epsilon$		x_1		1		x_2	$3/2$		$+\infty$
y''	$-\infty$	$-$	$-30 \mid +\infty$	$+$		$+0-$	$e-3$	$-0+$			$+$	$+\infty$
y'	$+\infty$	$+$	$18 \mid -\infty$	$-$	0	$+$	0	$-$	0	$+$		$+\infty$
y	$-\infty$	$-$	$-\eta \mid +\infty$	$+$	min.	$+$	max.	$+$	min.	$+$		$+\infty$

The polynomial $F(x)$ such that $\lim_{x\to\infty}[y-F(x)]=0$ is evidently of the form $4x^3-15x^3+2ax+2b$, and the coefficients a and b will now be determined by the condition

$$\lim_{x\to\pm\infty} 2\left[x(e^{1/x}+9-a)-b\right]=0, \text{ or } \lim_{x\to\pm\infty}\frac{e^{1/x}+9-a}{1/x}=b,$$

which shows that the quantity $(e^{1/x}+9-a)$ must be an infinitesimal of the same order as $1/x$ (b being of course finite and $\neq 0$); hence $\lim_{x\to\pm\infty}(e^{1/x}+9-a)=0$, which gives $a=10$ and therefore $\lim_{x\to\pm\infty}(e^{1/x}-1)\div 1/x=1=b$, so that $F(x)=4x^3-15x^2+20x+2$. This shows that the curve representing the function $y=2x\,e^{1/x}+4x^3-15x^2+18x$ is asymptotic to the cubic $y=4x^3-15x^2+20x+2$, the curve being above the cubic for $x=+\infty$ and below for $x=-\infty$.

3170 [3166; 1926, 104]. Proposed by A. A. Bennett, Lehigh University.

Given an isosceles triangle ABC, in which $AC=BC$, and a circle with center at C. Find a point, P, on the circle such that the tangent to the circle at P bisects the angle APB.

I. Solution by Velma Maness, Oklahoma University.

Let the circumcircle of ABC cut the given circle in the points P and P_1, and let CD be a diameter of the circumcircle. Then, since CD is perpendicular to AB, arc AD = arc BD, and hence angle APD = angle BPD. Moreover, PD is a tangent to the given circle at P. Hence P and P_1 are the desired points. It is here assumed that the radius of the given circle does not exceed the length of the equal sides, otherwise no solution is possible for internal bisectors. Two solutions for external bisectors are always possible by taking the points of intersection of the given circle with CD.

II, III, Solutions by Nathan Altshiller-Court, University of Oklahoma.

We shall generalize the problem by assuming that the given triangle ABC is scalene, i.e. the perpendicular to the line AB from the center C of the given circle (C) does not necessarily pass through the midpoint of the segment AB. Furthermore, we shall consider that a point P of (C) satisfies the conditions of the problem, if either the internal or the external bisector of the angle APB is tangent to the circle (C).

Let P be a required point on the given circle (C), and PM, PM' the bisectors of the two angles formed by the lines PA, PB. The traces M, M' of these bisectors on the line AB are harmonically separated by the points A, B. If PM is the tangent to (C) at P, then PM' passes through the center C of (C). Moreover, the polar m of M passes through the pole S of the line AB with respect to (C) and meets AB in the conjugate N of M in the involution of conjugate points on AB with respect to (C). The point P is thus the intersection of the polar m of M with the line joining C to M'.

If M is made to vary on AB we have

$$C(M'\cdots)\overset{=}{\overline{\wedge}}(M'\cdots)\overset{=}{\overline{\wedge}}(M\cdots)\overset{=}{\overline{\wedge}}(N\cdots)\overset{=}{\overline{\wedge}}S(N\cdots);$$

hence, the point $P\equiv(CM', SN)$ describes, in general, a conic (H) passing through the points C and S. Two other points of (H) are given by the intersections of the lines CA and CB with the circle having CS as a diameter. The tangent to (H) at C passes through the midpoint of the segment AB.

The polar $m \equiv SN$ of M with respect to (C) is perpendicular to MC, hence CM' will be parallel to SN, if CM is perpendicular to CM'. Thus (H) will have a real point at infinity, if a segment MM' can be found harmonic to the segment AB and subtending a right angle at C. Such a segment is determined on AB by the bisectors of the angles formed by the lines CA, CB. When M coincides with either end of this segment, we will obtain a point at infinity of (H). Thus the locus of P is an equilateral hyperbola whose asymptotes are parallel to the bisectors of the angles formed by the lines CA, CB.

The hyperbola (H) meets the circle (C), in general, in four points, hence the problem has four solutions. Since (H) passes through the center C of the circle (C), two of these solutions are always real, the other two may be real or conjugate imaginary.

Now if we suppose, as it is done in the proposed problem, that the points A, B are symmetric with respect to the line CS, the hyperbola (H) will degenerate into two straight lines, one of which is the perpendicular bisector CS of the segment AB, and the other a line parallel to AB. It is readily seen that this parallel line is the common chord of the circle (C) and the circumcircle of the triangle ABC.

The reader may consider the case when the points A, B, C are collinear.

If one of the bisectors PM, PM', say, PM of the angles formed by the lines PA, PB is tangent to the given circle (C), then the other bisector PM' passes through the center C of (C).

The bisectors PM, PM' are the tangents to the two conics passing through P and having their foci at the points A, B. Thus a point P satisfying the conditions of the problem is the point of contact of a tangent drawn from the given point C to a conic of the confocal pencil of conics having for their foci the points A, B. Now it is known that the locus of the points of contact of the tangents drawn from a fixed point to the conics of a homofocal system with foci at A, B is a circular cubic (F) passing through A, B and having a double point at C, the tangents to (F) at C being the bisectors of the angle ACB. The four finite points of intersection of (F) with the circle (C) constitute the solution of the proposed problem. Two of these points, namely the points of intersection of the circle (C) with the two infinite branches of (F) are always real. The other two points of intersection lie on the node of (F), and may be real or imaginary conjugate, depending upon the relative size and position of the circle and the segment AB.

Also solved by J. P. ALBERT, E. F. ALLEN, H. C. BRADLEY, THEODORE BENNETT, A. G. CLARK, P. A. CARIS, ALICE A. GRANT, MICHAEL GOLDBERG, F. H. MILLER, W. J. PATTERSON, R. S. SHAW, H. S. UHLER, ROSCOE WOODS, and the PROPOSER.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor W. D. CAIRNS, the secretary-treasurer of the Mathematical Association, is spending the year in California, on leave of absence from Oberlin College. He will be at Berkeley until early in May and then at the University of California at Los Angeles until early in August. The Association business is being carried on by the assistant secretary, Professor C. H. Yeaton, at Oberlin.

The Third Carus Monograph, on "Mathematical Statistics," by Professor H. L. RIETZ, is in type but the press work and binding will delaythe publication till sometime in March.

The "Rhind Mathematical Papyrus" is also delayed on account of final work on the hieroglyphic plates by the Egyptologist. All orders are being filed and delivery is anticipated not later than April ist.

Dr. HELEN BARTON has been appointed professor of mathematics and head of the department at Alabama College, which is a State College for women at Montevallo, Alabama.

Mr. N. C. GRIMES has been appointed professor of mathematics and head of the department at Grove City College, Grove City, Pennsylvania, to succeed Professor ARTHUR RAMSAY who resigned at the end of the last college year.

At Wheaton College, Wheaton, Illinois, Mr. C. R. HILLIARD has been appointed professor of mathematics. He was formerly registrar and professor of mathematics at the College of Ozarks. Professor H. O. TAYLOR (Ph.D., Cornell) is teaching part time during the second semester of this college year.

Mr. L. G. BUTLER has been appointed professor of mathematics at Albany College, Albany, Oregon.

Dr. D. R. DAVIS has been appointed assistant professor of mathematics at the University of Oregon.

Mr. P. S. DWYER has been appointed assistant professor of mathematics at Antioch College.

Dr. W. W. ELLIOTT has been appointed assistant professor of mathematics at Duke University.

Associate Professor M. G. GABA, of the University of Nebraska, has been promoted to a full professorship of mathematics.

Dr. MAYME I. LOGSDON, of the University of Chicago, has been promoted to an assistant professorship of mathematics.

Assistant Professor F. S. NOWLAN, of the University of Manitoba, has been appointed professor of mathematics at the University of British Columbia

The following appointments to instructorships in mathematics are announced: Columbia University, Dr. Edgar DEHN; Lehigh University, Mr. George RIDDLE; Yale University, Mr. T. W. MOORE.

Professor E. BLASCHKE, of the Vienna Technical School, died October 30, 1926, at the age of seventy.

Professor ALEXANDER TSCHUPROW, formerly of the Technical Institute at Petrograd, died April 19, 1926, at the age of fifty-two. He was known for his work in mathematical statistics.

Professor LOUIS SIFF, of Louisville University, died December 25, 1926, at the age of fifty-seven.

Professor J. H. TUDOR, of Pennsylvania State College, died June 7, 1926, at the age of sixty-nine. Professor TUDOR was a member of this Society.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 6-7, 1927.

Twelfth Annual Meeting, Nashville, Tenn., December, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 6-7.

INDIANA, De Pauw University, May 6-7.

IOWA, University of Iowa, April 29-30.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, May.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, Neb., May 14, 1927.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November.

ROCKY MOUNTAINS, Colorado College, April 22-23.

SOUTHEASTERN, March.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATION: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 3, MARCH

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE ELEVENTH ANNUAL MEETING OF THE ASSOCIATION

The eleventh annual meeting of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Thursday and Friday, December 30–31, 1926, in conjunction with the annual meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. Two hundred and sixty-four were present at the various sessions, including the following one hundred and eighty-seven members of the Association:

C. R. Adams, Brown University.
V. W. Adkisson, University of Pennsylvania.
R. C. Archibald, Brown University.
C. S. Atchison, Washington and Jefferson College.
W. L. Ayres, University of Pennsylvania.

Clara L. Bacon, Goucher College.
Ida Barney, New Haven, Conn.
A. A. Bennett, Lehigh University.
William James Berry, U.S. Bureau of Standards.
O. F. H. Bert, Washington and Jefferson College.
E. G. Bill, Dartmouth College.
G. D. Birkhoff, Harvard University.
G. A. Bliss, University of Chicago.
Henry Blumberg, Ohio State University.
J. W. Bradshaw, University of Michigan.
B. H. Brown, Dartmouth College.
H. S. Brown, Hamilton College.
Margaret Buchanan, West Virginia University.
R. W. Burgess, Western Electric Co., New York.

Minnie W. Caldwell, Chowan College.
W. B. Campbell, Cornell University.
P. A. Caris, University of Pennsylvania.
Mildred E. Carlen, Brown University.
I. S. Carroll, Syracuse University.
J. A. Clarke, West Philadelphia High School for Boys.
J. W. Clawson, Ursinus College.
A. B. Coble, University of Illinois.
Abraham Cohen, Johns Hopkins University.
J. B. Coleman, University of South Carolina.
J. W. Colliton, High School, Trenton, N. J.
Julia T. Colpitts, Iowa State College.
Rufus Crane, Ohio Wesleyan University.
E. S. Crawley, University of Pennsylvania.

C. H. Currier, Brown University.
D. R. Curtiss, Northwestern University.

H. M. Dadourian, Trinity College.
Marian E. Daniells, Iowa State College.
J. E. Davis, Drexel Institute.
F. F. Decker, Syracuse University.
L. S. Dederick, St. Stephen's College.
Alexander Dillingham, U. S. Naval Academy.
C. E. Dimick, U. S. Coast Guard Academy.
Annie W. Doughty, Dana Hall School.

T. W. Edmondson, New York University.
J. A. Eiesland, West Virginia University.
L. P. Eisenhart, Princeton University.
J. D. Eshleman, University of Pennsylvania.
H. B. Evans, University of Pennsylvania.
H. S. Everett, Bucknell University.

B. F. Finkel, Drury College.
W. B. Fite, Columbia University.
T. M. Focke, Case School of Applied Science.
F. A. Foraker, University of Pittsburgh.
W. B. Ford, University of Michigan.
C. H. Forsyth, Dartmouth College.
Tomlinson Fort, Hunter College.
Philip Franklin, Massachusetts Institute of Technology.
T. C. Fry, Western Electric Co., New York.

C. A. Garabedian, University of Cincinnati.
R. W. Gardner, Eastern Nazarene College.
W. V. N. Garretson, Rutgers University.
Leslie J. Gaylord, Agnes Scott College.
H. M. Gehman, Yale University.
D. C. Gillespie, Cornell University.
C. H. Gingrich, Carleton College.
O. E. Glenn, University of Pennsylvania.
Michael Goldberg, Bureau of Ordnance, Washington, D. C.
T. E. Gravatt, Pennsylvania State College.
C. C. Grove, Brooklyn Polytechnic Institute.

W. M. Hamilton, U. S. Nautical Almanac Office.
Harris Hancock, University of Cincinnati.
Isabel Harris, Westhampton College.
Mary G. Haseman, University of Illinois.
Olive C. Hazlett, University of Illinois.
E. R. Hedrick, University of California, Southern Branch.
Archibald Henderson, University of North Carolina.
Robert Henderson, Equitable Life Assurance Society, New York.

C. L. Herron, Hillsdale College.
Goldie P. Horton, University of Texas.
H. M. Hosford, Southern Methodist University.
Jewell C. Hughes, University of Arkansas.
E. V. Huntington, Harvard University.
W. A. Hurwitz, Cornell University.

Louis Ingold, University of Missouri.
M. H. Ingraham, Brown University.

Dunham Jackson, University of Minnesota.
Myra I. Johnson, Blackstone College.
R. A. Johnson, Hunter College.
L. S. Johnston, Pennsylvania State College.

O. D. Kellogg, Harvard University.
M. S. Knebelman, Princeton University.

W. D. Lambert, U. S. Coast and Geodetic Survey.
Solomon Lefschetz, Princeton University.
Harry Levy, National Research Fellow.
Florence P. Lewis, Goucher College.
S. B. Littauer, Boys' High School, Brooklyn, N. Y.
L. L. Locke, Brooklyn Training School for Teachers.
Mayme I. Logsdon, University of Chicago.
W. F. Long, Franklin and Marshall College.
H. M. Lufkin, University of Pennsylvania.

Emilie N. Martin, Mount Holyoke College.
Margaret E. Mauch, Randolph-Macon Woman's College.
A. E. Meder, Rutgers University.
W. L. Miser, Vanderbilt University.
H. H. Mitchell, University of Pennsylvania.
E. C. Molina, American Tel. and Tel. Co., New York.
R. L. Moore, University of Texas.
Richard Morris, Rutgers University.
H. M. Morse, Harvard University.
F. D. Murnaghan, Johns Hopkins University.
F. H. Murray, Dalhousie University.

J. J. Nassau, Case School of Applied Science.

G. D. Olds, Amherst College.
F. W. Owens, Pennsylvania State College.
Helen B. Owens, State College, Pa.

B. C. Patterson, Washington and Jefferson College.
E. K. Paxton, Washington and Lee University.

E. W. Pehrson, University of Utah.
E. C. Phillips, Georgetown University.
J. H. Pitman, Swarthmore College.
R. G. Putnam, New York University.

Virginia Ragsdale, North Carolina College for Women.
A. G. Rau, Moravian College.
G. E. Raynor, Wesleyan University.
C. J. Rees, University of Delaware.
W. D. Reeve, Teachers College, Columbia University.
L. W. Reid, Haverford College.
C. N. Reynolds, West Virginia University.
J. B. Reynolds, Lehigh University.
J. N. Rice, Catholic University.
R. G. D. Richardson, Brown University.
H. L. Rietz, University of Iowa.
J. F. Ritt, Columbia University.
Fannie H. Robinson, East Orange, N. J.
E. D. Roe, Jr., Syracuse University.
W. H. Roever, Washington University.
R. E. Root, U. S. Naval Academy.
J. E. Rowe, College of William and Mary.
S. A. Rowland, Ohio Wesleyan University.

F. H. Safford, University of Pennsylvania.
Joseph Seidlin, Alfred University.
Wayne Sensenig, High School, Philadelphia, Pa.
W. A. Shewhart, Bell Telephone Laboratories.
J. Shibli, Pennsylvania State College.
V. Z. Shippy, Central High School, Philadelphia, Pa.
L. L. Silverman, Dartmouth College.
W. G. Simon, Western Reserve University.
Lao G. Simons, Hunter College.
H. E. Slaught, University of Chicago.
L. L. Smail, Lehigh University.
A. W. Smith, Colgate University.
W. M. Smith, Lafayette College.
Virgil Snyder, Cornell University.
Ellen C. Stokes, N. Y. State College for Teachers.
K. D. Swartzel, University of Pittsburgh.
Elijah Swift, University of Vermont.

J. D. Tamarkin, Dartmouth College.
Ruth Thompson, New Jersey College for Women.
A. A. Titsworth, Rutgers University.
W. A. Titsworth, Alfred University.
Marian M. Torrey, Goucher College.
Bird M. Turner, West Virginia University.
H. W. Tyler, Massachusetts Institute of Technology.

H. S. Uhler, Yale University.

H. S. Vandiver, University of Texas.
Oswald Veblen, Princeton University.
Rozana Vivian, Wellesley College.

Helen M. Walker, Teachers College, Columbia University.
J. L. Walsh, Harvard University.
J. H. M. Wedderburn, Princeton University.
F. M. Weida, Lehigh University.
V. H. Wells, Williams College.
J. M. West, Pennsylvania State College.
Anna Pell Wheeler, Bryn Mawr College.
A. Marie Whelan, Dominican College.
Willis Whited, Pennsylvania State Department of Highways.
C. E. Wilder, Dartmouth College.
F. B. Williams, Clark University.
F. G. Williams, Pennsylvania State College.
K. P. Williams, Indiana University.
A. H. Wilson, Haverford College.
Elizabeth W. Wilson, Ann Arbor, Mich.
Ruth G. Wood, Smith College.
Emily K. Wyant, University of Missouri.

C. H. Yeaton, Oberlin College.
J. W. Young, Dartmouth College.
Mabel M. Young, Wellesley College.

This record attendance was very gratifying to your officers. It is, of course, inevitable that a large percentage of those present should come from the general vicinity of the meeting place. In this case about eighty-six percent were from points east of Ohio. However, it is intended to distribute the meeting places over the country as much as possible and it is especially hoped that the next annual meeting in Nashville, Tennessee, will have a record attendance for the middle west. The Philadelphia record can be beaten if all members within

five hundred miles of Nashville begin now to lay plans to be present at that meeting.

The sessions of the American Association for the Advancement of Science began on Monday evening with the retiring address of the president, Doctor Michael I. Pupin, Professor of electro-mechanics in Columbia University, who spoke on "Fifty years' progress in electrical communication." This address appeared in *Science* for December 31, 1926. As has become customary in connection with the meetings of the American Association, numerous lectures of a semi-technical character and of general public interest were given during the week. Among them were:

(1) A symposium on "Research, especially in colleges and professional schools," arranged by Dr. Maynard M. Metcalf, of Johns Hopkins University, and participated in by Dr. John C. Merriam, president of the Carnegie Institution of Washington, Dr. Florence R. Sabin, Rockefeller Institute for Medical Research, Professor Walter W. Cook, Yale Law School, Professor H. B. Goodrich of Connecticut-Wesleyan University, and Dr. Metcalf himself, the two latter speakers referring especially to research in colleges.

(2) An address under the auspices of the society of Sigma Xi by Mr. Herbert Hoover, Secretary of Commerce, Chairman of a Board of Trustees acting under the National Academy of Sciences to collect a large national fund for the support of research in pure science. Mr. Hoover spoke on "The Nation and Science" and gave a most inspiring address.

(3) A symposium on "The role of science in education" arranged by Dr. Otis W. Caldwell, of Lincoln School, New York City.

(4) An address by Professor W. F. G. Swann, of Yale University, on "The new quantum dynamics," delivered before the American Physical Society.

(5) A symposium arranged by the Engineering Section on "Contributions of pure science to the advancement of engineering and industry." The sciences represented were: Astronomy, Biology, Chemistry, Economics, Geology, Mathematics, Medical Science, Physics, Psychology. The speaker for mathematics was Professor G. A. Bliss, University of Chicago.

Full accounts of the American Association meetings and many of the addresses will appear in the columns of *Science*.

On the program of the History of Science Society were three papers relating to mathematics; namely, "Colonial mathematics" by Lao G. Simons, Hunter College, New York, "Mathematical instruments of the colonists" by E. W. Schreiber, Proviso Township High School, Maywood, Illinois, and "Methods of teaching the history of mathematics" by Professor David Eugene Smith, Columbia University.

Professor Dunham Jackson, University of Minnesota, was chosen vice-president of Section A of the American Association and Professor R. C. Archibald, Brown University, was continued as secretary of the section.

The fourth Josiah Willard Gibbs Lecture, presented under the auspices of the American Mathematical Society, was accorded a place on Tuesday afternoon as one of the public addresses. The speaker was Doctor H. B. Williams, Dalton Professor of Physiology in the College of Physicians and Surgeons of Columbia University. He spoke on "Mathematics and the biological sciences." He paid a high compliment to Gibbs and gave an illuminating insight into the important service which mathematics can render to the biological sciences. There was a good attendance but such an address deserved a hearing many times greater.

At the joint dinner of mathematicians held in the dining room of the Aldine Hotel, Professor E. V. Huntington presided and the speaking, aside from words of welcome by Professor Crawley, was limited to four persons, thus leaving ample time for informal social converse which lasted till late in the evening. Professor H. E. Slaught spoke on the great progress made during the past five years in properly financing the activities of mathematics as a whole in this country, and how this has been accomplished without burdensome increase in dues, the combined dues of the Society and the Association (ten dollars per year) being, in fact, considerably lower than in many other scientific organizations. Professor Anna Pell Wheeler spoke appropriately on behalf of the ladies. Professor G.D. Birkhoff spoke on the experiences of his recent sojourn in Europe and told some ways in which American mathematics may well profit by the example of European mathematics. President G. D. Olds spoke entertainingly of his mathematical reminiscences and, though absorbed in administrative duties, still wished to be considered as one of us, declaring that his greatest pleasure was in the companionship of his mathematical friends.

The American Mathematical Society held its thirty-third annual meeting on Tuesday and Wednesday and engaged in a joint session with the Mathematical Association of America and Section A of the American Association on Thursday morning. In the separate sessions of the Society there were sixty-seven papers read and at the joint session there were three addresses as follows:

1. "A mathematical critique of some physical theories," by Professor G. D. Birkhoff, retiring president of the American Mathematical Society.

2. "The weight field of force of the earth," by Professor W. H. Roever, of Washington University, retiring vice-president of Section A of the American Association.

3. "The duty of exposition, with special reference to the Cauchy-Heaviside theorem," by Professor F. D. Murnaghan, of Johns Hopkins University, representing the Mathematical Association.

Professor Birkhoff's address was adjudged by the committee of the American Association to be the most outstanding among all the papers presented at these meetings and he was awarded the prize of one thousand dollars. This is the second time since the establishment of this prize that the award has been made to a mathematician.

Professor Murnaghan brought out strongly the need of intelligent and critical exposition, as well as of sound and scientific research. He illustrated this contention with keen humor and an admirable exhibition of high class exposition, showing, as one example, how Heaviside and others were puzzled and baffled over matters which Cauchy had fully set forth many years before. Professor Murnaghan will prepare this paper for the MONTHLY and thus give all our readers the benefit of enjoying it first hand.

An abstract of Professor Roever's paper was given in the report of Section A in *Science* for January 28.

At the close of the joint session the following resolution was adopted by the combined body of mathematicians:

That the American Mathematical Society, the Mathematical Association of America, and Section A of the American Association for the Advancement of Science, in joint meeting assembled, do hereby express their grateful appreciation of the gracious hospitality extended them by the University of Pennsylvania, the American Philosophical Society, the Academy of Natural Sciences, the Drexel Institute and the Franklin Institute, and do hereby tender their sincere thanks to the various local committees, sub-committees and other local representatives, who by their efficient and thoughtful arrangements have contributed greatly not only to the intellectual profit but also to the physical comfort of these meetings.

SEPARATE SESSIONS OF THE ASSOCIATION

At the two separate sessions of the Association the following papers were presented:

1. "Cauchy's integral theorem" by Professor D. R. CURTISS, Northwestern University.

2. "On various conceptions of correlation" by Professor F. M. WEIDA, Lehigh University.

3. "Here and there in Europe" by Professor R. C. ARCHIBALD, Brown University.

4. "The use of the trinometer in engineering and in the teaching of mathematics," (exhibiting an instrument) by Professor J. E. ROWE, College of William and Mary.

5. "Two geodesists of the eighteenth century" by Mr. W. D. LAMBERT, U. S. Coast and Geodetic Survey.

6. "Modern methods and results of stellar parallax investigations" by Professor J. H. PITMAN, Swarthmore College. Illustrated by lantern slides.

7. "The maintenance of mathematical interest" by Professor E. R. HEDRICK, University of California, Southern Branch.

8. "The apportionment of representatives in congress" by Professor E. V. HUNTINGTON, Harvard University.

Abstracts of these papers follow:

1. Professor Curtiss' paper gave an historical account of various demon strations of Cauchy's integral theorem. This theorem states that if a function of a complex variable is single-valued and analytic throughout a connected region T, then the integral of that function taken around the whole boundary is zero. In one form of this theorem the function is assumed to be analytic throughout the closed region T; in another form this assumption applies only to open T, and the function is supposed continuous in closed T. After a survey of the history of integration of complex functions before Cauchy, a brief account of Cauchy's famous paper of 1825 was given. Its proof of the integral theorem, in the current style of the calculus of variations, was traced down to more modern forms, particularly that of Jordan. The use of Green's lemma was shown to characterize another set of proofs of which Cauchy seems to have given the first, in 1846. A third class, containing Goursat's proof, reduces the problem to the consideration of rectangles or triangles, and does not assume that the derivative of the integrand is continuous.

2. One of the most important topics which can be discussed by statistical methods is the theory of the measure of the statistical relations between quantitative or qualitative modalities of two characters. If one character changes, does the change seem to have no influence on the changes of the other character?

The theory of correlation was first developed on definite assumptions as to the form of the distribution of the frequency, the so-called normal distribution being assumed. Bravais introduced the product-sum, but not a single symbol for a coefficient of correlation. Galton developed the practical method, determining his coefficient—Galton's function—graphically. Edgeworth developed the theoretical side further, and Pearson introduced the product-sum formula.

In the problem of correlation, we wish to find some measure of agreement or disagreement between two sets of paired characters. Under this common term correlation, two very different meanings have frequently been considered, connection and concordance. Gini appears to be the first author to systematically settle the theory in that he established a clear distinction between the different meanings of connection and concordance; he introduced a new principle of dissimilarity between two statistical distributions and put it as the

basis of the theory of statistical relations; he established a suitable measure for each of the different aspects according to which the quantitative or qualitative modalities of two characters may be presented. Gini's researches deal with the quantitative or qualitative modalities of two characters, which may be distributed according to rectilinear or disconnected series. Recently Pietra has completed the subject in that he deals with the dissimilarity, connection, and concordance between cyclical series.

The theory of correlation that has been much applied in recent years to statistical data has been developed largely as an extension of error theory. The theory of correlation may also be developed as an integral part of the theory of a-priori probability. Rietz used certain Urn Schemata as a basis for the development of the theory of correlation. Tschuprow has made correlation theory an integral part of the theory of probability and has approached the whole subject from an a-priori point of view.

It is the purpose of the present paper to give an exposition of (1) various aspects of correlation theory developed largely as an extension of error theory; (2) the theory of connection as developed by Gini; (3) the theory of concordance as developed by Gini; and (4) the theory of correlation developed by Tschuprow as an integral part of the theory of probability and approached from an a-priori point of view.

3. Professor Archibald reported on some of the more interesting information gleaned by him during a tour of five months in ten countries of Europe during 1926. He referred particularly: to mathematics, mathematicians, and the Gennadius Library in Athens; to the Guggenheim and International Education Board Fellowships; to Bieberbach's projected Mathematical Dictionary, and the possible publication of Valentin's great mathematical Bibliography from the invention of printing down to 1900; to mathematicians at Prague, the Sokol Society, and to the extraordinary Czecho-Slovakian Mathematical Society with 1400 members, a splendid library, its own house and printing press which does a flourishing business; to Wieleitner, Pringsheim, Caratheodory, Sir Thomas Heath, Klein, Eneström, Heiberg and Mittag-Leffler. He related also some Hilbert stories and told of his contacts with Egyptologists and Babylonian scholars in preparing his Bibliography of Egyptian Mathematics which, among other things deals with a description of some fifty documents dating from 3500 B. C. to 1000 A. D. This Bibliography is soon to be published in Chancellor Chace's work on the Rhind papyrus.

4. Professor Rowe gave a description of the trinometer and of different types of the instrument, and then presented a practical demonstration of the solution of triangles by means of it. He then set forth the rôle of the instrument in practical engineering and in the teaching of elementary mathematics with special emphasis upon the use of the trinometer as a safe and sane method of

shortening courses in elementary mathematics and in making it possible for the student who finishes his formal training with the secondary school to have a more comprehensive, a more thorough, and a more useful knowledge of mathematics.

5. The two geodesists discussed in Mr. Lambert's paper were Charles Mason and Jeremiah Dixon who, between 1763 and 1768, laid out the celebrated line bearing their names to separate the territories of Lord Baltimore from those of the Penn heirs. Since Delaware was a part of the Penn territory, the boundary between Maryland and Delaware was also a part of the line. Mason and Dixon were often described in the documents of the times as mathematicians, but their mathematics seems to have been entirely of a practical nature, as was most of the English mathematics of that time. The few biographical data known about these men were given, also some account of the long chain of circumstances leading up to their employment. The level nature of the country along the Maryland-Delaware line and the fact that this line runs nearly north and south suggested that it might be advantageous to determine the length of a degree of latitude in this region. This was done under the auspices of the Royal Society of London. Lengths were determined not by triangulation but directly. Instead of a chain, the usual surveying instrument of the time, so-called "levels" were used. These were wooden frames having a measuring rod in the top. They were laid end to end in alignment along the "visto" cut through the forest for the demarcation of the boundary. The report of this scientific part of their work is given by them in the Philosophical Transactions for 1768, where their work is discussed in connection with other measures of a degree by Mason's friend and patron, Rev. Nevil Maskelyne, the Astronomer Royal. This was the first geodetic operation in America. From this it appears that, even then, some thought had been given to the effect of irregularities in deflecting the plumb line, but no systematic study was made until nearly a century later. Lantern slides showing some of the boundary monuments and otherwise illustrating the paper were shown.

6. Professor Pitman discussed the various methods of determining the distances of the stars and nebulae. (1) The trigonometric method or triangulation by photography is used to determine the constants in the other methods. This method is practicable only for the nearer stars. (2) The spectroscopic method is a correlation of the ratio of the intensities of certain pairs of lines in the spectrum with the absolute magnitudes. This method can be applied to most stars whose spectra can be photographed in a reasonable length of time. (3) The period luminosity law derived from the study of Cepheid variables is a correlation of the period of variation and the absolute magnitudes of stars. By means of this relationship we can reach further into space than by other methods.

In 1910 there were 500 determinations of the distances of 365 stars and we knew nothing of the distances of nebulae and clusters. Since then the number

of stars investigated has reached 5000 and the number of determinations is more than 11,000. The familiar Russell diagram shows giant stars of all spectral classes with practically equal absolute magnitudes and a dwarf branch starting with the A and F stars in which the magnitude rapidly decreases as the stars become more red. A study of the masses of the binary systems, whether visual binaries, spectroscopic binaries or eclipsing variables shows that the very bright stars are also the most massive. In other words, the Russell diagram, which arranges the stars with respect to brightness and spectral class or color, is also the arrangement of stars according to mass and color.

7. In this paper, Professor Hedrick discusses at some length the desirability of maintaining the interest of teachers of mathematics in their subject. It is emphasized that this is absolutely essential for proper instruction in colleges. Among means to this end, research and publication of original papers are the most widely recognized. In this paper an attempt is made to extend the means for maintaining the interest of the individual teacher, both by increasing the field of recognized activities, and by increasing the range of recognized activities in those fields. Fields mentioned are applications to engineering, physics, and other sciences. Activities mentioned are lecturing, book publication, work for organizations, and editorial work. It is urged that all these are necessary in the creation of a greater American school of mathematics.

8. Interest in the problem of the apportionment of representatives in Congress has been revived by the recent hearings held before the House Committee on the Census. Professor Huntington's paper continued his study of the mathematical aspects of this problem (begun in the Quarterly Publication of the American Statistical Association for September, 1921), and presented a new and exhaustive examination of the thirty-two different ways in which the exact equation of proportionality can be written. Only five methods of apportionment appear to be possible, among which the "Method of Equal Proportions" occupies the central position. The paper contained some interesting new examples, showing that certain plausible tests prove to be unworkable in practice, because they fail to lead to a determinate result.

Reported for the Secretary by H. E. Slaught.

Meetings of the Board of Trustees

Ten members of the Board of Trustees were present at the various sessions.

The following fifty-seven persons and seven institutions were elected to membership on applications duly certified:

To Individual Membership

V. W. Adkisson, A. B. (Drury). Instr., Univ. of Pennsylvania, Philadelphia, Pa.

Margaret Amig, A.M. (Wellesley). Teacher, St. Catherine's School, Richmond, Va.

Edith I. Atkin, A.M. (Columbia). Asst. Prof., Ill. State Normal Univ., Normal, Ill.

C. A. Balof, M.S. (Iowa). Instr., Howard College, Birmingham, Ala.

L. S. BARNES, A.B. (Occidental). Asst., Lehigh Univ., Bethlehem, Pa.

A. H. BEILER, B.S. in E.E. (Cooper Union). New York, N. Y.

CLIFFORD BELL, Ph.D. (California). Instr., Univ. of Calif., So. Br., Los Angeles, Calif.

PAUL BOEDER, A.M. (Pennsylvania). Instr., Univ. of Del., Newark, Del.

R. W. BOLTON, E.E. (Illinois). Glendale, Calif.

LILIAN H. BROWN, A.M. (Illinois). Teacher, LaSalle-Peru Twp. High School and Junior College, LaSalle, Ill.

R. S. BURINGTON, A.M. (Ohio State). Instr., Case School of Appl. Sc., Cleveland, Ohio.

ELIZABETH CARLSON, Ph.D. (Minnesota). Instr., Univ. of Minnesota.

R. W. E. CARUTHERS, B.S. in E.E. (Miss. A. and M.). General Electric Co., Schenectady, N. Y.

SISTER MARIA CORONA, M.S. (Notre Dame). Teacher, Mt. St. Joseph Coll., Mount St. Joseph, Ohio.

E. A. CUMMINGS, A.M. (Chicago). Teacher, North High School, Denver, Colo.

JEANNETTE G. DABOLL, A.B. (Mount Holyoke). Teacher, Northfield Seminary, East Northfield, Mass.

ANNIE W. DOUGHTY, A.M. (Radcliffe). Teacher, Dana Hall School, Wellesley, Mass.

FRANCES G. FENTON, A.B. (Denver Univ.). Teacher, Acctg. and Bkpg., North High School, Denver, Colo.

EDWARD FLEISHER, M.S. (New York Univ.). Asst. Prof., Coll. of the City of N. Y., Brooklyn Branch, Brooklyn, N. Y.

C. R. GALLOWAY, B.M.E. (Kentucky). Prof., Idaho Tech. Inst., Pocatello, Idaho.

J. A. GARDINER, A.B. (Penn. State). Instr., Howard High School, Wilmington, Del.

C. A. GILLEY, A.B. (Texas). Austin, Texas.

MICHAEL GOLDBERG, B.S. in E.E. (Pennsylvania). Jr. Ord. Engr., Bureau of Ordnance, Navy Dept., Washington, D. C.

R. W. HART, A.M. (Illinois). Asso. Prof., Kansas State Teachers Coll., Pittsburg, Kans.

P. R. HILL, M.S. (Georgia). Instr., Univ. of Ga. Athens, Ga.

ANNIE B. HORTON, A.M. (Texas). Head of Dept., High School, Edinburg, Texas.

H. M. HOSFORD, Ph.D. (Illinois). Asso. Prof., Southern Methodist Univ., Dallas, Texas.

F. H. ISHAM, A.M. (Teachers Coll., Columbia). Teacher, Potsdam State Normal School, Potsdam, N. Y.

E. H. JOHNSON, A.B. (DePauw). Grad. Asst., Lehigh Univ., Bethlehem, Pa.

F. W. KOKOMOOR, Ph.D. (Michigan). Instr., Univ. of Mich., Ann Arbor, Mich.

K. W. LAMSON, Ph.D. (Chicago). Asst. Prof., Lehigh Univ., Bethlehem, Pa.

A. F. LEARY, Ed.M. (Boston College). Jr. Master, English High School, Boston, Mass.

HARRY LEVY, Ph.D. (Princeton). National Research Fellow, Maplewood, Mass.

ENRIQUE LINARES, Jr., B.S. in E.E. (Univ. of Santa Clara). Actg. Chief Engr., Central Board of Roads, Panama City, Republic of Panama.

H. L. LUTZ, A.M. (Columbia). Instr., Cooper Union, New York, N. Y.

F. S. MACGREGOR, A.B. (Harvard). Mgr., Textbook Dept., Harper & Bros., New York, N. Y.

BROTHER MAJELLA, A.M. (Catholic Univ.). Head of Sc. Dept., St. John's Prep. School, Danvers, Mass.

MARGARET E. MAUCH, M.S. (Chicago). Instr., Randolph-Macon Woman's Coll., Lynchburg, Va.

H. H. MITCHELL, Ph.D. (Princeton). Prof., Univ. of Pennsylvania, Philadelphia, Pa.

DAVID MOSKOVITZ, M.S. (Carnegie Inst. of Tech.). Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.

MARIE M. NESS, A.M. (Minnesota). Research Associate, Univ. of Minn., Minneapolis, Minn.

GRETA NEUBAUER, A.B. (Wyoming). Instr., Univ. of Wyo., Laramie, Wyo.

C. H. NORDSTROM, B.S. (Worcester Poly. Inst.). Grad. Asst., Lehigh Univ., Bethlehem, Pa.

J. A. NYSWANDER, Ph.D. (Chicago). Asst. Prof. of Math., University of Michigan, Ann Arbor, Mich.

C. T. OERGEL, Diploma (Drexel Evening School). Stud., Penn. State Coll., State College, Pa.

J. H. PITMAN, A.M. (Swarthmore). Asst. Prof., Math. and Astronomy, Swarthmore Coll., Swarthmore, Pa.

FLORENCE ROTHERMEL, A.M. (Pennsylvania). Teacher, West Philadelphia High School, Philadelphia, Pa.

R. E. SCAMMON, Ph.D. (Harvard). Professor of Anatomy, Univ. of Minn., Minneapolis, Minn.

ELLEN C. STOKES, A.M. (Brown). Instr., N. Y. State Coll. for Teachers, Albany, N. Y.

SARAH E. TRACY, B.L. (Swarthmore). Teacher, John Burroughs School, Clayton, Mo.
YETTA VELINSKY, A.B. (Centenary). Instr., Centenary Coll., Shreveport, La.
MIRIAM WERNER, Asst. Prof., Hunter College, New York, N. Y.
PAUL WERNICKE, Ph.D. (Göttingen). Patent Examiner, U. S. Patent Office, Washington, D. C.
HEINRICH WIELEITNER, Ph.D. (Munich). Director, Neues Realgymnasium, Munich, Germany.
FRANCES E. WOLEVER, A.B. (Rockford). Grad. Student, Univ. of Ill., Urbana, Ill.
C. L. WRAY, A.B. (York College). Teacher, North High School, Denver, Colo.
EMILY K. WYANT, A.M. (Missouri). Instr., Univ. of Missouri, Columbia, Mo.

To Institutional Membership

ST. IGNATIUS COLLEGE, San Francisco, Calif.
UNIVERSITY OF MONTREAL, Montreal, Canada.
LOUISIANA POLYTECHNIC INSTITUTE, Ruston, La.
THE COLLEGE OF THE SACRED HEART, Manhattanville, New York, N. Y.
UNIVERSITY OF AKRON, Akron, Ohio.
HIRAM COLLEGE, Hiram, Ohio.
BRYN MAWR COLLEGE, Bryn Mawr, Pa.

The following were appointed associate editors of the MONTHLY for the year 1927:

H. E. BUCHANAN	H. J. ETTLINGER	C. N. MILLS
ELIZABETH CARLSON	H. S. EVERETT	F. D. MURNAGHAN
W. B. CARVER	B. F. FINKEL	H. L. OLSON
OTTO DUNKEL	H. W. KUHN	D. E. SMITH

President Jackson presented for record in these minutes the following expression of appreciation of the able work of Professor W. B. Ford as Editor-in-Chief of the MONTHLY:

In reluctantly accepting the resignation of Professor Ford as Editor-in-Chief of the MONTHLY, the Trustees desire to place on record their grateful appreciation of his devoted service in maintaining the place of the MONTHLY among the important mathematical journals of the world, in strengthening its scientific content, and in increasing the range and significance of its service to the membership of the Association.

The Trustees unanimously approved and directed that the Secretary-Treasurer transmit this expression of their sentiments to Professor Ford.

Reporting on the plan for expository lectures to be given at certain meetings of the Association or of its sections, President Jackson announced the appointment of Professors Graustein, Ingraham, MacDuffee, and Slaught as members of the committee. It was voted that President Jackson be requested to act as chairman of this committee.

Professor Young announced that the Houghton Mifflin Company had offered to reprint Part I and certain supplementary material of the *Report of the National Committee on Mathematical Requirements* in the Riverside Series of Mathematical Monographs. This monograph will contain about 200 pages and will probably sell for $1.20 less the regular discount to teachers.

Voted that the officers of the Association be authorized to sign papers giving the Houghton Mifflin Company the right to handle this volume without royalty to the Association.

Professor Slaught reported on the activities of the Louisiana-Mississippi Section in cooperation with the National Council of Teachers of Mathematics (a secondary school organization) in creating interest in mathematics in those states and in promoting better understanding between high school and college teachers.

In response to a letter from Professor A. A. Bennett, it was voted that the President should appoint a committee of three to draw up a list of suggested assignments in collateral reading in mathematics for freshmen and sophomore students in American colleges. President Jackson appointed this committee as follows: A. A. Bennett, chairman, R. C. Archibald, and J. A. Nyswander.

Having consulted Professor Fort, President Jackson reported that it seemed best to let the work of the committee on standard departments of mathematics lapse and to release the committee. This action was approved.

The Trustees voted to approve the organization of a PHILADELPHIA SECTION of the Association, subject to the submission of suitable by-laws, a petition to that effect having been sent by a meeting of thirteen members of the Association. This section is intended to serve more than one hundred members living in the eastern part of Pennsylvania, in southern New Jersey and in Delaware, meetings usually being held in Philadelphia. This is the seventeenth section of the Association.

The invitation from President Thomas to hold the summer meeting in 1930 at Rutgers University and the invitation from President Davis and Professor Fort to hold the annual meeting in December 1928 at Hunter College were received and the secretary was directed to acknowledge the same with the thanks of the Association and to say that the decision will be made at a later date.

Professor Wedderburn's request that the Association be represented on the Editorial Staff of the Annals of Mathematics by two associate editors was favorably considered. The Trustees authorized President Ford to appoint a committee of three, including himself, with power to select and nominate two associate editors of the Annals of Mathematics. President Ford appointed Professors Cairns and Slaught as the other members of this committee. It was understood that the Annals volume will be still further enlarged and it was felt that our subvention to the Annals is now inadequate. The Trustees, therefore, voted to increase the annual subvention to $300.

In connection with the publication by the Association of the Rhind Papyrus, it was voted that Professor Archibald be empowered to arrange with Chancellor Chace to accept a cash gift for the Association, it being understood that the

Association will then pay the bill for the printing. This work is to be issued in the name of the Association.

The Trustees discussed the important question of having official delegates from the national body attend the meetings of the sections. One plan proposed was to have the authors of the monographs, as they are issued, give one or more lectures at the section meetings. The merits of the plan were obvious but the difficulty is in financing such a project. It was felt that the sections might well be expected to contribute at least a part of such expenses, but it was recognized that a fund for such purposes was much needed, and that the sections would profit greatly by expository lectures not only on the Monograph topics but in many other lines. The whole matter was laid over for later consideration, it being felt that the Association is not now justified in entering upon an extensive program of this kind.

Professor Archibald suggested the desirability of spending some money each year for binding back volumes of the exchanges and other paper-covered books in the Association library. Action was postponed till the next meeting.

The financial statement for the year 1926, in the absence of Secretary Cairns, was presented by Professor Slaught. The report was approved, subject to the examination and approval of an auditing committee consisting of Professors W. G. Simon of Cleveland, Ohio, and F. E. Carr of Oberlin, Ohio. This committee is to meet in Oberlin and report at a later date. It was voted that Professor Simon's expenses be paid to Oberlin.

It was voted that Professors R. C. Archibald and Florian Cajori constitute a committee to cooperate with the History of Science Society in arranging for an exhibit and program in connection with the celebration in commemoration of the bicentenary of the death of Newton.

It was the sense of the Trustees that the publication of the great mathematical bibliography of Georg Valentin, covering the period from the beginning of printing to 1900, would be of great value to mathematicians throughout the world and that such publication in America with English sub-headings would be very desirable if funds can be provided.

It was voted to hold the annual meeting in December, 1928 at New York City, in affiliation with the American Association for the Advancement of Science.

The publication of a new register early in the autumn of 1927 was authorized.

President Ford was authorized to appoint committees on program and arrangements for the coming summer meeting in Madison, Wisconsin, and for the annual meeting in Nashville, Tennessee.

These committees will be announced later.

Annual Business Meeting of the Association

Professor Slaught announced that fifty-seven individuals and seven institutions had been elected to membership at the meeting of the Board of Trustees. He reported also the death of the following members:

G. N. Armstrong, Professor of applied mathematics, Ohio Wesleyan University (January 8, 1926).

W. S. Barlow, Instructor in mathematics, Detroit Junior College (August 11, 1925).

S. M. Barton, Professor of mathematics, The University of the South (January 5, 1926).

Sarah Beall, Mathematician, U. S. Coast and Geodetic Survey (February 11, 1926).

R. D. Bohannan, Professor of mathematics, Ohio State University (June 20, 1926).

G. W. Brindle, Principal of High School, Surrency, Georgia (July 1926).

B. E. Carter, Assistant professor of mathematics, Colby College (June 10, 1926).

S. R. Cruse, Assistant professor of mathematics, University of Arizona (February 8, 1926).

Walter Denston, Assistant professor of mathematics, Kenyon College (April 8, 1926).

Albert Heinz, Head of department of mathematics, Tsing Hua College, Peking, China (June 11, 1926).

H. A. Howe, Dean of College of Liberal Arts and professor of astronomy, University of Denver (November 2, 1926).

E. W. Martin, Professor of mathematics, Western State College, Gunnison, Colorado (September 8, 1926).

Gertrude W. Mendenhall, State Normal College, Greensboro, North Carolina (April 15, 1926).

G. H. Scott, Dean and professor of mathematics and physics, Illinois College (September 12, 1926).

S. G. Stein, Muscatine, Iowa (December 3, 1926).

C. A. Waldo, Professor emeritus of mathematics, Washington University (October 1, 1926).

L. D. Wills, Teaching fellow in mathematics, University of Oregon (April 9, 1926).

The election of officers for the year 1927 was conducted by mail and in person at this meeting, the tellers, Dr. R. W. Burgess and Professor C. N. Reynolds, reporting the result of the balloting as follows:

For President: W. B. Ford, 324 votes; J. W. Young, 206 votes.

For Vice-Presidents: C. F. Gummer, 236 votes; W. A. Hurwitz, 245 votes; A. J. Kempner, 284 votes; Clara E. Smith, 255 votes.

For additional members of the Board of Trustees, to serve until January 1930: E. T. Bell, 267 votes; A. A. Bennett, 253 votes; R. D. Carmichael, 333 votes; A. B. Chace, 224 votes; H. J. Ettlinger, 164 votes; E. R. Hedrick, 352 votes; Dunham Jackson, 352 votes; H. W. Kuhn, 158 votes.

The following were accordingly declared elected:

President: W. B. Ford, University of Michigan.

Vice-Presidents: A. J. Kempner, University of Colorado; Clara E. Smith, Wellesley College.

Additional members of the Board of Trustees: E. T. Bell, California Institute of Technology; R. D. Carmichael, University of Illinois; E. R. Hedrick, University of California, Southern Branch; Dunham Jackson, University of Minnesota.

In accordance with the amended by-laws, the term of Professor Ford as President will be for two years. The officers appointed by the Board, namely,

the Secretary-Treasurer, the Librarian, the Assistant Librarian, the Assistant Secretary, the Editor-in-Chief, the Manager and the third member of the Committee on Official Journal, all hold over as for 1926, and until their successors shall be appointed. Professor W. H. Bussey, University of Minnesota, was elected Editor-in-Chief at the Columbus meeting, but his term of office began in January, 1927.

C. H. YEATON, *Assistant Secretary.*

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 16, 1926.

RECEIPTS

Balance Dec. 17, 1925		$7,299.78
1925 indiv. dues	281.85	
1925 instit. dues	37.50	
1926 indiv. dues	5,377.19	
1926 instit. dues	690.70	
1926 subscriptions	725.97	
Initiation fees	252.00	
Advertising	764.80	
Sale copies of MONTHLY	133.76	
Sale First Carus Mon.	130.75	
Sale Second Carus Mon.	905.75	
For *Annals* subscriptions	1.50	
Recd. for expense La.-Miss. Section	18.00	
Life memb. payment	33.44	
From matured bond	500.00	
Int. Oberlin Savgs. Bk.	130.57	
Int. Peoples Bkg. Co.	110.79	
Int. Treasury Note	10.94	
Int. Liberty Bonds	53.12	
Int. Hardy Fund	120.00	
		10,278.63
Total 1926 receipts		$17,578.41

EXPENDITURES

Publisher's bills (Nov. '25-Oct. '26)		$4,993.64
President's office		15.00
Manager's office		30.76
Editor-in-Chief's office		533.37
Register		439.26
Committee on Membership		19.02
Secretary-Treasurer's office:		
Postage	$336.30	
Bond	5.00	
Insurance back issues MONTHLY	12.80	
Safety deposit	4.00	
Office supplies	36.86	
Express, tel., etc.	33.70	
Clerical work	716.80	
Printing	331.81	
Library expense	46.50	
Pd. copies MONTHLY	21.45	
Kansas City meetg.	108.16	
Columbus meeting	37.75	
Refd. subscriptions	9.00	
Forwarded for subscribers Monograph	2.00	
		1,702.13
Annals subvention		150.00
Pd. *Annals* subscriptions		4.50
Pd. to sections from initiation fees		133.46
Cost of new bond		515.07
Pd. Chauvenet Prize		100.00
Pd. int. Hardy Fund to B. F. Finkel		120.00
Honorarium Carus Monograph		300.00
Transferred to Monograph Fund		600.00
Transferred to Gen. End. Fund		1,000.00
Expense acct. Carus Monographs		102.10
Delegate 50th anniv. Soc. Sc. Bruxelles		15.00
Expense acct. Chace publication		37.44
Pd. expense La.-Miss Sec.		18.00
Total expenditures		$10,828.75

		Cash on hand	$ 29.88
Total expenditures	10,828.75	Checking account	392.31
Balance to the end of 1926 business	$6,749.66	Oberlin Savgs. Bk. acct	3,495.66
		Peoples Bkg. Co. acct	2,596.81
Received on 1927 business	765.0	Liberty Bonds	1,000.00
Book balance Dec. 16, 1926	$7,514.6	Bank balance Dec. 16, 1926	$7,514.66

Of the funds on hand, $333.00 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1927; $20.00 is set aside as the first installment toward the Chauvenet Prize Fund; $201.40 belongs to the CARUS MONOGRAPH FUND (not yet transferred). Aside from the above mentioned funds on hand, the sum of $1000, received in 1917 from the previous owners of the MONTHLY, is held in reserve (in liberty bonds) as the beginning of a GENERAL ENDOWMENT FUND; also the CARUS MONOGRAPH FUND, increased by sales and interest, now amounting to $2,824.42, is carried as a separate fund in the form of a certificate of deposit which bears 4%, compounded quarterly.

When the accounts were closed December 16, 1926, there remained on the total business for the year 1926 the following items:

BILLS RECEIVABLE (partly estimated)		BILLS PAYABLE (partly estimated)	
1926 individual dues	$100.00	Publisher's bills (Nov., Dec. '26)	$1,250.00
1926 institutional dues	20.00	President's office	40.00
Advertising	50.00	Manager's office	40.00
		Editor-in-Chief's office	150.00
	$170.00	Other editors' postage	50.00
		Committee on Membership	50.00
		Secretary-Treasurer's office	450.00
		Annals subvention	150.00
		Printing annual programs, ballots, etc.	200.00
		Initiation fees due to sections	600.00
		Life Membership Fund	333.00
		Carus Monograph Fund	201.40
			$3,514.40

If to the balance on 1926 business shown in this report, $6,749.66, there be added the bills receivable, $170.00, and there be subtracted the estimated bills payable, $3,514.40, there results an estimated final balance on 1926 business of approximately $3,400. This favorable advance in our resources is due to the decrease in printing costs of the past year, to a steadily increasing number of members, to a continuation of the gratifying amount of advertising secured, and to the careful administration of office expenses by our various officials.

W. D. CAIRNS, *Secretary-Treasurer*.
Presented at the meeting by
H. E. Slaught, *Manager*.

FREDERICK THE GREAT ON MATHEMATICS AND MATHEMATICIANS

By FLORIAN CAJORI, University of California

1. Frederick William I. of Prussia ordered that his son, known later as Frederick the Great, should "learn no Latin"; "let him learn arithmetic, mathematics, artillery,—economy to the very bottom."[1] The old king allowed the Berlin "Society of Sciences," the favorite child of Leibniz, to languish and almost to pass away. His son, Frederick, on the other hand, secretly acquired some Latin, shunned the study of mathematics beyond its rudiments, and brought the Berlin "Academy"[2] to great splendor. Frederick looked upon mathematical study with disfavor. As crown prince, he wrote, on January 26, 1738, to Voltaire his plan of study,[3] "to take up again philosophy, history, poetry, music. As for mathematics, I confess to you that I dislike it; it dries up the mind. We Germans have it only too dry; it is a sterile field which must be cultivated and watered constantly, that it may produce."

2. **Bantering the mathematicians.** D'Alembert once wrote to Frederick the Great:[4] "It is the destiny of your majesty to be always at war; in summer with the Austrians, in winter with mathematics." The king himself put it in these words:[5] "I love to wrangle with mathematicians, that I may know whether, without understanding $xx+y$, it is not possible to be in the right." The king presents an argument why D'Alembert should not decline his invitation to settle in Berlin, ending with the remark:[6] "Such is my refutation. I hold myself to be victorious, and erect a trophy to myself, for having vanquished a great mathematician, wholly to his disgrace." Another time, he commentated on some essays of D'Alembert:[7] "I read that part of the work in which you condescend to sink the science of *the sublime geometry* to the level of my ignorance."

[1] Thomas Carlyle, *History of Frederick* II *of Prussia*, vol. 2, London, 1870, p. 19, 20.

[2] The institution commonly known as the Berlin Academy was founded by Leibniz in 1700 under the name of *Societas regia scientiarum.* After his death the organization was neglected for about a quarter of a century and it nearly passed out of existence. It received new life under the patronage of Frederick the Great who reorganized it under the name of *Académie royale des sciences et des belles lettres de Berlin*, adopting French, instead of Latin, as the language in which its memoirs should be published. Men of note were invited to the Academy and were given a salary enabling them to devote their time to research. Among the mathematicians thus invited were Maupertuis, Euler, Lambert and Lagrange. At the beginning of the nineteenth century the Academy was again reorganized under the stimulus mainly of Wilhelm von Humboldt, and it was given a strictly German stamp.

[3] *Oeuvres de Voltaire*, vol. 53 (Paris, 1831), p. 28.

[4] *Posthumous Works of Frederick* II, Letters between Frederick II and M. D'Alembert. Translated from the French by Thomas Holcroft, vol. 11 (London, 1789), p. 4. Letter of December 23, 1762.

[5] Loc. cit., p. 27. Letter of August 20, 1765.

[6] Loc. cit., p. 28.

[7] Loc. cit., p. 65, Letter of May 5, 1767.

He ventured upon physical subjects:[1] "Is it not true that electricity, and all the prodigies it has hitherto discovered, have only served to excite curiosity? Is it not true, that the doctrine of attraction and gravity, has done nothing more than astonish the imagination? Is it not true that all the operations of chemistry are in the same predicament? But are robbers less numerous or contractors less covetous?" To which D'Alembert replies:[2] "Your majesty is pleased to treat the sublime geometry a little cavalierly. I allow that it frequently is, as your majesty well observes, a luxury in which idle learning indulges; but to prove that it has often been useful we need only recollect the system of the world, the phenomena of which it so well explains." But the king persists:[3] "An algebraist, who lives locked up in his cabinet, sees nothing but numbers, and propositions, which produce no effect in the moral world. The progress of manners is of more worth to society than all the calculations of Newton." Nor did he hesitate to make sport of the Germans:[4] "Our geometrician of Berlin is in excellent health, and rather lives in the planet Venus than on this terraqeous globe. The people, who have heard speak of Venus and her passage over the sun's disk (transit of Venus of 1769), have been two nights on the watch, to observe the phenomenon. You will laugh at the expense of my good countrymen; but it is all the wit they have." Once D'Alembert took occasion to observe:[5] "I perceive your majesty has always a lash in store for geometry." Quite natural is the king's dictum relating to the telescopes of Beguelin:[6] "I suppose the calculations according to which they are constructed are admirable, but the fact is that I wished to make use of them, but could not see through them."

3. **The King and Euler.** During the first thirteen years at the court of St. Petersburg, Euler gained general recognition throughout Europe as a mathematician of the first rank. Frederick, even before ascending the throne, had determined upon an eager search for men of genius for his academy. How he settled upon the name of Euler is not known. Perhaps through his friend von Suhm who had been in St. Petersburg since 1737, and was buying books for Frederick, including certain memoirs of the St. Petersburg academy. On June 14, 1740, two weeks after ascending the throne, Frederick wrote von Suhm,[7] "do what you can to engage Mr. Euler, the great algebraist, and if you can, bring him with you. I shall give him 1000 or 1200 écus salary." On July

---

[1] Loc. cit., p. 79. Letter of January 7, 1768.
[2] Loc. cit., p. 82. Letter of January 29, 1768.
[3] Loc. cit., p. 145. Letter of January 4, 1770.
[4] Loc. cit., p. 127. Letter of July 2, 1769.
[5] Loc. cit., p. 235. Letter of March 6, 1771.
[6] Loc. cit., vol. 12, p. 166. Letter of January 29, 1779
[7] *Œuvres de Frédéric Le Grand*, vol. 16 (Berlin, 1850), p. 391.

15, the king repeated:[1] "Bring Euler, if you can." But von Suhm died and the Prussian ambassador took up the matter.[2] Euler accepted and reached Berlin, July 25, 1741. On September 4, the king wrote him from the field of the Silesian war, and set his salary at 1600 Taler.[3] The war delayed the organization of the new academy, but at its close a volume of *Miscellanea* of the old society came from the press, containing five papers by Euler. Anxious that the new academy should be organized as soon as possible, Euler wrote the king in 1743,[4] suggesting that the revenues of the academy resulting from the sales of the annual almanacs would be sufficiently increased by the sale in the newly acquired territory of Silesia, "to support an academy of sciences of rank equal to that of St. Petersburg or Paris." Two days later came the king's inappreciative reply:[5] "But I believe that, being accustomed to the abstractions of magnitudes in algebra, you have sinned against the ordinary rules of calculation. Otherwise you would not have imagined such a large revenue from the sale of almanacs in Silesia." Euler instantly replied[6] that he had been prompted by a desire "to be worthy of the favors which the king had bestowed upon him." The king's delay was due, apparently, to his plan of making Maupertuis the ruling head of the academy, which was delayed by war conditions until 1746. Meanwhile Euler and high officials in Berlin organized informally a scientific society which for six months held weekly meetings, and then fused with the old society into a new academy which held its first meeting on January 24, 1744, the king's birthday.[7] This provisional organization comprised four classes: Physics, mathematics, speculative philosophy and philology. Euler became director of the mathematics class, and retained this place until his return to St. Petersburg in 1766. The king's inquiry for the best book on artillery led to Euler's translation into German in 1745, of the treatise by Benjamin Robins. Euler's important commentaries thereon were translated into French and English.[8] In a practical way Euler was of service to the king also for information on lotteries and aid in planning a canal connecting the Oder and Havel.[9] The king was aware of the high esteem in which Euler was held for his researches. Before 1753 Euler had won seven prizes offered by the Paris academy.[10] He was elected foreign associate of that academy. In 1775

[1] Loc. cit., p. 394.

[2] G. Valentin, *Festschrift zur Feier des 200. Geburtstages Leonhard Eulers*, (Leipzig und Berlin, 1907), p. 4.

[3] *Œuvres de Frédéric Le Grand*, vol. 20 (Berlin, 1852), p. 199.

[4] Loc. cit., pp. 199, 200. Letter of January 19, 1743.

[5] Loc. cit., p. 200.

[6] Loc. cit., p. 201.

[7] G. Valentin, loc. cit., p. 8.

[8] Nicolaus Fuss, *Lobrede auf Leonhard Euler* (Basel, 1797), p. 47.

[9] Op. cit., p. 48.

[10] P. H. Fuss, *Correspondance mathématique et physique de quelques célèbres géomètres du* XVIII$^{\text{ème}}$ *siècle*, vol. 1 (St. Petersburg, 1843), p. 608.

the king mentioned Euler as having been an ornament of the Berlin academy.[1] Nevertheless, he did not appreciate and admire Euler. At one time, when commenting on the lack of ability of mathematicians in literature and art, the king said:[2] "A certain geometer who has lost one eye in calculating presumed to compose a minuet by *a* plus *b*. Were it to be played before Apollo, the poor geometer would run the risk of being flayed alive, as was Marsyas." In a letter to Voltaire, the king speaks[3] of a "huge cyclops of mathematician" (un gros cyclope de géomètre). The king had no knowledge of mathematics and no appreciation of its value in civilization. "Euler en vains calculs met sa philosophie," he wrote.[4] Moreover, he missed in Euler the power of light, witty, brilliant conversation and correspondence, in which his French correspondents, Voltaire, D'Alembert, Maupertuis, and D'Argens so eminently excelled. When Maupertuis left Berlin in 1756, broken in health, no regular successor was appointed, not even after his death. In 1763 the king offered the presidency to D'Alembert, but he could not be persuaded to leave France. Thereupon the king made himself head, but D'Alembert in Paris was the secret president,[5] the one upon whose judgment the king depended almost entirely in the selection of new members to the academy.[6] The real burden of routine administration fell for the ten years following 1756 upon Euler who, according to Harnack,[7] "was conscientious, economical, but hardly less violent and self-willed than the old president (Maupertuis); just he was to be sure, but not without prejudice." When the presidency was offered to D'Alembert, Euler felt that he was not appreciated and wrote Goldbach[8] that if the plan persisted to make this a French academy, he might feel constrained to move elsewhere. On December 2, 1763, Euler asked[9] the king's consent to the marriage of one of his daughters to a cornet in the army, which was promptly denied because of the low rank of the officer. There arose another source of irritation. The king and some members of the academy criticized Euler for allowing an official, Köhler, excessive profits from the sales of the calendar upon which the academy was financially dependent. A commission of investigation was appointed, of which Euler was a member. But, disregarding the commission, Euler made proposals

---

[1] *Œuvres*, vol. 3, p. 25.

[2] Loc. cit., vol. 9, p. 64.

[3] Loc. cit., vol. 11, p. 128.

[4] Loc. cit., vol. 10, p. 138.

[5] Adolf Harnack, *Geschichte der Königlichen preussischen Akademie der Wissenschaften zu Berlin*, vol. 1 (1900), p. 359.

[6] After the death of D'Alembert, it was Condorcet who for 16 months acted as adviser to Frederick the Great in matters relating to the Berlin academy. See Loc. cit., p. 390.

[7] Loc. cit., pp. 351, 352.

[8] P. H. Fuss, op. cit., vol. 1, p. 667.

[9] *Œuvres*, vol. 20, p. 208.

to the king directly. The latter replied:[1] "I cannot compute any curves, but I know quite well that 16000 Taler are more than 13000 Taler." Euler addressed a second letter to the king and received a sharp reply.[2] These events caused Euler to ask the king for permission to leave Berlin, but not till the third request did the king write a gracious note:[3] Euler "would do him pleasure if he would desist from request for leave and would not revert to this again." But Euler persisted and on May 2, 1766, received leave in a note of two lines. With derision, the king expressed himself to D'Alembert:[4] "Mr. Euler, who is in love even to madness with the great and little bear, has travelled northward to observe them more at his ease. The ship which bore him and his xz and yy has been wrecked. All is lost, which is a pity, for there were materials enough to have formed six volumes, in folio, of memoirs in figures from beginning to end: and Europe, in all probability, will be deprived of the agreeable amusement which such a course of reading would have afforded." Thus parted the two great men in discord. But ten years softened the feelings. When both were in the autumn of their lives, Euler sent to the king two papers relating to widow's pensions, which the king was introducing, and received in reply gracious acknowledgments. The king also expressed appreciation of his election to honorary membership of the Petersburg academy.[5]

4. **The King and Maupertuis.** As the famous earth flattener, Maupertuis had become the social lion in Parisian circles. Voltaire recommended him to the crown prince Frederick. Probably at the suggestion of Voltaire, Maupertuis sent to the crown prince a copy of his *La figure de la terre*, 1738, and received in reply an appreciative letter:[6] "Nature cannot but unveil herself to persons who study her with such great attention. Although the subject treated in this book demands a profound knowledge of mathematics and of speculative astronomy, I shall read it with pleasure, reserving the right to ask for explanation of points which I do not understand." Immediately after ascending the throne, Frederick expressed in a letter,[7] "the desire of having you here, that you might put our Academy into the shape you alone are capable of giving it. Come then, come and insert into this wild crabtree the graft of the sciences, that it may bear fruit. You have shown the figure of the Earth to mankind; show also to a King how sweet it is to possess such a man as you." Maupertuis visited Prussia in 1740 and 1741 during wartime, but returned to France, to await the

[1] Loc. cit., vol. 20, p. 209.
[2] Adolf Harnack, op. cit., vol. 1, p. 364.
[3] *Œuvres*, vol. 20, p. 210. Letter of March 17, 1766.
[4] *Posthumous Works* (Ed. Th. Holcroft), vol. 11, p. 43.
[5] *Œuvres*, vol. 20, p. 211; A. Harnack, op. cit., p. 365.
[6] Loc. cit., vol. 17, p. 335.
[7] Loc. cit., vol. 17, p. 335. Letter of June, 1740.

end of the war. In 1746, Maupertuis assumed the presidency and, in the words of Frederick was to be "the pope of the Academy." During the first four years his administration was very successful. In 1750, Maupertuis communicated to the academy two papers on a new principle in science, that of *least action*. It was developed in a book, the *Cosmologie*, 1750. The Swiss mathematician Samuel König, then residing in Holland, criticized the principle as enunciated by Maupertuis, and pointed out also that it was not new, since long ago Leibniz had advanced the idea in more correct form. There arose a quadrangular controversy involving Maupertuis, König, Voltaire and Frederick II, which for sensational features and brilliancy of satire has never been equalled in scientific circles. The Newton-Leibniz controversy on the invention of the calculus pales in comparison.

5. The King and Lambert. The Swiss Sulzer, who was director of the philosophy class of the academy, strongly recommended his countryman Lambert. At the first meeting, the king found Lambert very crude and conceited. "What sciences are you pursuing?" asked the king. "All." "Are you also a skilled mathematician?" "Yes." "What professor has taught you mathematics?" "I myself." "Are you therefore a second Pascal?" "Yes, your majesty." The king looked upon him as a simpleton and could not be persuaded to appoint him to the academy until the year following when it became known that the Russian ambassador in Berlin was considering him for the St. Petersburg academy.[1] Euler thought well of him; so did the king later. On March 1, 1765 D' Alembert wrote the king:[2] "I am only acquainted with one work of Mr. Lambert, which is good, but which does not seem to me comparable to any of the works of Euler; and, if the latter be on his knees before Mr. Lambert, as your majesty has done me the honor to inform me he is, we must say of Mr. Euler as has been said of LaFontaine, that he was silly enough to believe Aesop and Phaedrus had more wit than himself · · ·. I should think him (Lambert) tolerably well provided for when he should be, to speak mathematically, in the same ratio to Euler · · · as Bayle is to Descartes and Newton." On October 5, 1777 the king wrote:[3] "Alas! We have recently lost poor Lambert, one of our best members. I know not who could treat the subject ("whether it be useful to deceive the people") philosophically; Beguelin I believe to be the only person."

6. The King and Lagrange. It was D'Alembert who suggested Lagrange as the successor to Euler. On May 19, 1766, D'Alembert wrote:[4] "I take the

[1] A. Harnack, op. cit., p. 366, who quotes Sulzer's *Lebensbeschreibung, von ihm selbst aufgesetzt*, herausgegeben von Merian und Nicolai, Berlin, 1809, p. 381.

[2] *Posthumous Works* (Ed. Th. Holcroft), vol. 11 p. 20.

[3] Op. cit., vol. 12, p. 107.

[4] Op. cit., vol. 11, p. 36.

earliest oportunity to inform your majesty that Mr. de la Grange has received your proposal with equal respect and gratitude. · · · I therefore believe" he will come "to replace Mr. Euler; and I dare to assure your majesty that, both with respect to ability and industry, he will be an excellent substitute; and that, in character and conduct, he will never excite the least division, or disturbance in the Academy." Lagrange had some difficulty in securing from the ministers of the king of Sardinia permission to leave Turin. D'Alembert secured Frederick's consent for Lagrange to take his trip to Berlin by way of Paris and London.[1] He will be[2] "much more useful to the academy than I could be. This is no false modesty, but the simple truth. Mr. de la Grange is young, and I am almost old; his ardor is rising, mine is on the decline; he is getting up, and I am going to bed." The king replied, July 26, 1766:[3] "To your care and recommendation am I indebted for having replaced a half-blind mathematician by a mathematician with both eyes, which will especially please the anatomical members of my academy." D'Alembert recommended Lagrange for the position of director of the mathematical class:[4] "If however your majesty should have other views, relative to the place of Director, Mr. de la Grange, well satisfied with the fifteen hundred crowns · · · will not insist upon this point; he only intreats your majesty would be kind enough to name a Director before his arrival, in order that the court of Turin · · · should not imagine that Mr. de la Grange, at his first coming, should meet an apparent kind of disgrace." Once the king invited the members of his academy to meet him afternoons socially.[5] "I then saw Mr. de la Grange, who endeavored to temper the sublimity of his language in an inverse ratio to the square of my ignorance. He led me from abstraction to abstraction, into labyrinths so dark that my poor understanding would have lost its road, had not our good Swiss Mr. Merian conducted me from these high infinitesimal regions, and safely landed me on the abject and crude globe on which I vegetate." On January 23, 1782, the king wrote to D'Alembert:[6] "Mr. de la Grange calculates, calculates and calculates, curves as many as you please." Without appreciation of the great rôle which mathematics was playing in the intellectual life of Europe, the king tolerated the mathematicians at his court simply as ornaments. "Ages must pass away before nature shall produce another Voltaire,"[7] said he once, but no such appreciation was ever applied to Lagrange, Euler or Lambert.

---

[1] Op. cit., pp. 39–50.
[2] Op. cit., p. 40. Letter of July 11, 1766.
[3] Op. cit., p. 43.
[4] Op. cit., p. 45.
[5] Op. cit., vol. 12, p. 198. Letter of 1780, undated.
[6] Op. cit., p. 316.
[7] Op. cit., p. 19. Letter of December 30, 1775.

After the death of the king, a situation arose so that Lagrange felt no more at home in Berlin than had Euler, twenty years previously. The notorious Mirabeau, who then happened to be on a secret mission to Berlin, wrote[1] to the French government, on November 28, 1786: "I find that at this moment a conquest could be made which would be worthy of the king of France · · ·. The celebrated Lagrange, the first mathematician that has arisen since Newton, is not happy here." In 1787 Lagrange left for Paris.

7. **The King and Castillon.** On May 26, 1766, D'Alembert wrote the king:[2] "Your majesty wishes to have an astronomer and I believe Mr. de Castillon to be a proper person; especially as he can educate his son in the same studies, and enable him to be his successor, should that be required." Castillon went to Berlin and took up the work at the observatory which at D'Alembert's suggestion had undergone repairs. Modern readers remember Castillon, chiefly, for his editions of the minor works of Newton and of the correspondence of Leibniz and John Bernoulli.

8. **Frederick, Fontenelle and Madame Châtelet.** Frederick had some correspondence with Fontenelle who was in doubt[3] as to the validity of the theory of gravitational "attraction." Through Voltaire, Frederick as crown prince, wrote to Madame du Châtelet who endeavored to persuade him to study physics and to win him over to Newtonian philosophy.[4]

9. **Conclusion.** Frederick the Great was desperately in love with poetry and philosophy, and wholly unsympathetic with mathematics. Yet, by his patronage at the Berlin academy, he contributed nothing substantial to poetry and philosophy, but achieved marvels in the advancement of mathematics. How did this happen? Partly because in selecting academicians Frederick received good advice from D'Alembert and Voltaire on matters mathematical, while in the fields of philosophy and belles-lettres he was not equally fortunate. Then besides, having himself received a French education, he was inappreciative of German talent. He made no attempt to draw to the Berlin academy either Kant or Goethe.[5] The mathematical creations of Euler, Lagrange, Lambert, Maupertuis, Beguelin, and Castillon, while these were at the Berlin academy, rank among the brightest pages in the eighteenth century annals of the exact sciences. Euler and Lagrange adorned the academy during forty-five years.

---

[1] A. Harnack, op. cit., vol. 1, pp. 505–506.

[2] *Posthumous Works* (Ed. Th. Holcroft), vol. 11, p. 38.

[3] *Œuvres*, vol. 16, p. 197. Letter of September 29, 1737.

[4] Loc. cit., vol. 17, p. 15. Letter of February 16, 1739.

[5] The Swiss and French were the ones who gave character to the Berlin academy in the time of Frederick II. Among the Swiss were L. Euler, Lambert, J. Bernoulli (1744–1807), Beguelin, A. Euler, Merian, de Catt, Passavant, Sulzer, Weguelin. The last five did not belong to the mathematical class. See Harnack, op. cit., p. 327–480.

Frederick the Great's controlling motive for his academy was splendor. This is expressed in the form of invitation he is said to have sent to Lagrange at Turin: "The greatest king of Europe" wishes to have "the greatest mathematician" at his court.

NOTE ON THE COMPUTATION OF ROOTS

By J. V. USPENSKY, Carleton College

The purpose of this note is to call attention to a certain method for the computation of roots which is especially advantageous when the value of the root to many decimal places is required. The author does not venture to say positively whether this method is new or not, for it would be rash to make any positive statement concerning so elementary a matter; at least he knows of no work where this, or a similar method, can be found. Let a be a given positive number and θ its nth root, so that

$$\theta^n = a\,. \tag{1}$$

Denoting by s any number, the product

$$(s-\theta)^3(p_0+p_1\theta+p_2\theta^2+\cdots+p_{n-1}\theta^{n-1}),$$

by means of (1), may be reduced to the form

$$R_0+R_1\theta+R_2\theta^2+\cdots+R_{n-1}\theta^{n-1}$$

where

$$R_k = s^3p_k-3s^2p_{k-1}+3sp_{k-2}-p_{k-3}$$

for $k=3, 4, \cdots, n-1$, but

$$R_2 = s^3p_2-3s^2p_1+3sp_0-ap_{n-1}$$
$$R_1 = s^3p_1-3s^2p_0+3sap_{n-1}-ap_{n-2}$$
$$R_0 = s^3p_0-3s^2ap_{n-1}+3sap_{n-2}-ap_{n-3}.$$

This holds true whatever may be the values attributed to $p_0, p_1, \cdots, p_{n-1}$. Now we shall determine these values in such a way that all the coefficients $R_2, R_3, \cdots, R_{n-1}$ vanish. This gives us a system of $n-2$ linear and homogeneous equations involving $p_0, p_1, \cdots, p_{n-1}$, to which, in order to attain the greatest possible simplicity, we adjoin an additional equation $p_{n-1}=0$.

It is easy to see that all the equations

$$R_3=0,\qquad R_4=0,\cdots, R_{n-1}=0,\qquad p_{n-1}=0$$

are satisfied if we assume

$$p_k=(b+ck)(n-1-k)s^{n-k-2}$$

for every $k=0, 1, 2, \cdots, n-1$, b and c being arbitrary constants. Introducing the values thus determined into the remaining equation $R_2=0$, we find that it is satisfied when $b=c$. Accordingly, putting for simplicity $b=c=1$, we reach the following conclusion: p_k for $k=0, 1, 2, \cdots, n-2$ being determined by the equation

$$p_k = (k+1)(n-k-1)s^{n-k-2}, \tag{2}$$

the product

$$(s-\theta)^3(p_0 + p_1\theta + p_2\theta^2 + \cdots + p_{n-2}\theta^{n-2})$$

can be reduced to the binomial form $R_0+R_1\theta$ where

$$R_0 = s[(n+1)a + (n-1)s^n] \text{ and } -R_1 = (n-1)a + (n+1)s^n.$$

That is, we have

$$\begin{aligned}(s-\theta)^3(p_0 + p_1\theta + p_2\theta^2 + \cdots + p_{n-2}\theta^{n-2})\\ = s[(n+1)a + (n-1)s^n] - [(n-1)a + (n+1)s^n]\theta\end{aligned} \tag{3}$$

provided that the p_k are determined by the equation (2).

The equation (3) will be our starting point. Supposing s positive and putting

$$s_1 = s\frac{(n+1)a + (n-1)s^n}{(n-1)a + (n+1)s^n}$$

we have first

$$s_1 - s = 2s\frac{a - s^n}{(n-1)a + (n+1)s^n} \tag{4}$$

and second from (3)

$$s_1 - \theta = (s-\theta)^3 P \text{ where } P = \frac{p_0 + p_1\theta + p_2\theta^2 + \cdots + p_{n-2}\theta^{n-2}}{(n-1)a + (n+1)s^n}. \tag{5}$$

If $s<\theta$ or $s^n<a$ it follows from (4) that $s_1>s$; at the same time (5) shows that in this case $s_1<\theta$. On the contrary, if $s>\theta$, then $s_1<s$ and $s_1>\theta$. Hence it is obvious that, starting with $s<\theta$ as a first approximate value, we get s_1 as a new approximation which is always nearer to θ than s and is still $<\theta$. Starting with s_1 instead of s from (4) we obtain a still better approximation $s_2>s_1$ and so on. Thus starting with an arbitrary positive value $s<\theta$ we can form an ascending sequence of numbers $s, s_1, s_2, s_3, \cdots$ which all remain less than θ and converge to θ as a limit. In the same way, starting with $s>\theta$ we obtain a descending sequence of numbers $s, s_1, s_2, s_3, \cdots$ all remaining greater than θ and converging to θ as a limit The convergence in both cases is very rapid. To see this it is

necessary to have an upper limit for the quantity denoted by P. Putting $\theta/s=\xi$ we have

$$P = \frac{1}{s^2} \cdot \frac{\sum_{k=0}^{n-2} (k+1)(n-k-1)\xi^k}{(n-1)\xi^n + n + 1}$$

For $\xi \leqq 1$, it is obvious that

$$(n-1)\xi^n + n + 1 > n + 1,$$

$$\sum_{k=0}^{n-2} (k+1)(n-k-1)\xi^k \leqq \sum_{k=0}^{n-2} (k+1)(n-k-1) = \frac{n(n-1)(n+1)}{6},$$

so that in this case

$$P < (n^2 - n)/(6s^2).$$

For $\xi > 1$ and $n > 2$, it is easy to show that

$$(n-1)\xi^n + n + 1 > \frac{n(n-1)}{n-2}\xi^{n-2}$$

which gives the following upper limit for P:

$$\begin{aligned} P &< \frac{1}{s^2}\,\frac{n-2}{n(n-1)} \sum_{k=0}^{n-2} (k+1)(n-k-1)\xi^{k-n+2} \\ &< \frac{1}{s^2}\,\frac{n-2}{n(n-1)} \sum_{k=0}^{n-2} (k+1)(n-k-1) \\ \text{or} \quad P &< (n^2 - n - 2)/(6s^2). \end{aligned}$$

In the excluded case $n=2$ we have $P < 1/(3s^2)$ and hence we can draw the conclusion that the inequality $P < (n^2-n)/(6s^2)$ holds true in every case independently of the value of $\xi = \theta s^{-1}$.

With this upper limit for P we have

$$(s_1 - \theta)/(s-\theta)^3 < (n^2 - n)/(6s^2)$$

which shows that the new approximate value has about three times as many correct decimal places as the preceding one. Besides the rapidity of convergence, this method presents another very precious advantage in that it enables us to approach the required root from either side and consequently to form a clear idea as to the accuracy attained at every step of the process. Before giving

numerical illustrations of this method it is desirable to write down the final formulas for certain special cases.

Case $n=2$

$$s_1 = s + 2s\frac{a - s^2}{a + 3s^2}$$

$$\frac{s_1 - \theta}{(s - \theta)^3} < \frac{1}{3s^2}$$

$$\theta = a^{1/2}$$

Case $n=3$

$$s_1 = s + s\frac{a - s^3}{a + 2s^3}$$

$$\frac{s_1 - \theta}{(s - \theta)^3} < \frac{1}{s^2}$$

$$\theta = a^{1/3}$$

Case $n=4$

$$s_1 = s + 2s\frac{a - s^4}{3a + 5s^4}$$

$$\frac{s_1 - \theta}{(s - \theta)^3} < \frac{2}{s^2}$$

$$\theta = a^{1/4}$$

Case $n=5$

$$s_1 = s + s\frac{a - s^5}{2a + 3s^5}$$

$$\frac{s_1 - \theta}{(s - \theta)^3} < \frac{10}{3s^2}$$

$$\theta = a^{1/5}.$$

As a first example let us find the approximate value of $(100)^{1/3}$. Taking first $s=4$ we have

$$s_1 = 4 + \frac{144}{228} = 4.631 \cdots.$$

Again, starting with $s=5$ we get

$$s_1 = 5 - \frac{125}{350} = 4.6429 \cdots$$

and hence we infer that the difference between $(100)^{1/3}$ and 4.63 is certainly less than 1.2×10^{-2}. Starting again with $s=4.63$ we may be sure that the new approximation s_1, being less than $(100)^{1/3}$, will differ from it by less than

$$\frac{(1.2)^3\times 10^{-6}}{(4.6)^2} < \frac{1.8 \times 10^{-6}}{21} < 10^{-7}.$$

Performing the computation we find $s_1 = 4.6415917 \cdots$ whence it follows that $(100)^{1/3} = 4.6415917$ correct to 7 decimal places. For the second example let us find the approximate value of $(64)^{1/5}$. Here we start first with $s=2$ and then with $s=3$; the corresponding values of s_1 are

$$s_1 = 2 + \frac{64}{224} = 2.285 \cdots \quad \text{and} \quad s_1 = 3 - \frac{537}{857} = 2.373 \cdots$$

and it is certain that the number 2.28 differs from $(64)^{1/5}$ by less than 9.3×10^{-2}. Starting with $s=2.28$ and performing the computation we find $s_1=2.297394\cdots$ and hence we infer that the difference between $(64)^{1/5}$ and 2.28 in reality is less than 1.8×10^{-2}, so that we can conclude $(64)^{1/5}=2.297395$ correct to 6 decimal places. If we started again with this number the next approximate value would be correct to about 18 places, and this remark may convey an adequate idea of the rapidity of the convergence in the above described process.

The fundamental idea underlying this method can be generalized, but the resulting formulas, on account of their complexity, do not seem practically useful.

ON AN APPLICATION OF BOUGUER'S THEOREM

By JAMES PIERPONT, Yale University

Poincaré in his *La Science et L'Hypothèse* p. 83 seq., has given an elegant realization of hyperbolic space. The points of this space lie within an e-sphere[1]

$$\lambda = 4R^2 - r^2=0, \qquad r^2 = x^2 + y^2 + z^2.$$

The absolute temperature at a point $P=(x,y,z)$ is given by $T=\lambda/4R^2$. All bodies when moved about in this sphere acquire instantly the temperature prevailing at this point and contract or expand in such a way that a small unit measuring rod has the length

$$ds = d\sigma\cdot T^{-1},\ d\sigma^2 = dx^2 + dy^2 + dz^2.$$

Thus the length of the radius of the sphere $\lambda=0$ is infinite in this enclosure, and to an H-observer his space seems to be of infinite extent. Now H-straights appear to an e-observer as e-circles cutting the sphere $\lambda=0$ orthogonally. As a ray of light should in H-space move along an H-straight, the question arises, can we fill the λ-sphere with a medium of varying index of refraction n such that light will satisfy this condition. Poincaré asserts this to be the case if we take

$$(1) \qquad n=g/\lambda, \qquad g \text{ constant}.$$

This "monde non-euclidien" of Poincaré is often cited but the writer has failed to find a proof.

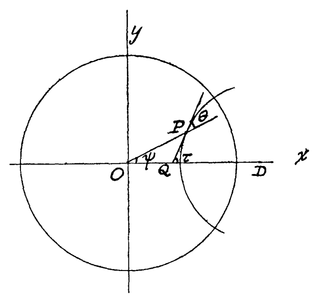
Let us ask: Can we find an index depending on r alone such that a ray of light will travel along an e-circle cutting $\lambda=0$ orthogonally. As the ray must move in a plane thru O, let us take this plane to be $z=0$. It cuts $\lambda=0$ in the circle

$$\lambda=4R^2-r^2=0, \qquad z=0, \qquad r^2=x^2+y^2.$$

[1] For e—read euclidean, for H—read hyperbolic.

We will take the x-axis so that it passes thru the center D of the orthogonal circle whose radius we call ρ. The equation of this circle is

$$(x - c)^2 + y^2 = \rho^2, \quad c^2 = 4R^2 + \rho^2, \quad OD = c.$$



The tangent to this circle at P makes the angle $\tau = PQD$ with the x-axis. Let $POD = \psi$, while $\theta = OPQ$ is the angle that the tangent at P makes with the radius vector $OP = r$. Then $\cos\psi = x/r$, $\sin\psi = y/r$, $\cos\tau = x/\rho$, $\sin\tau = (c-x)/\rho$, $\sin\theta = \sin(\tau - \psi) = \sin\tau\cos\psi - \cos\tau\sin\psi = (cx - r^2)/r\rho = \lambda/(2r\rho)$.

Hence (2) $$r\sin\theta = \lambda/2\rho.$$

Now Bouguer's theorem asserts, that when the index n depends on r only, the ray moves so that

(3) $$nr\sin\theta = k, \quad \text{a constant.}$$

Comparing (2) and (3) gives $n = 2k\rho/\lambda$ which is (1) if we take the constant $g = 2k\rho$. *Q.E.D.*

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. ON THE RELATIVE ACCURACY OF SIMPSON'S RULES AND WEDDLE'S RULE[1]

By J. B. SCARBOROUGH, U. S. Naval Academy

1. Introduction. When a continuous function is approximated over an interval by a polynomial, it is well known that a polynomial of high degree generally gives a closer approximation than one of low degree. This fact has sometimes led to the belief that Simpson's three-eighths rule is more accurate than his one-third rule, because the former is derived by passing a third-degree parabola through four consecutive points whereas the latter is usually derived by passing a second-degree parabola through three consecutive points. That the contrary is really the case, that the three-eighths rule is less accurate than the one-third rule, was first pointed out and proved by C. W. Merrifield[2] in

[1] For a statement and a derivation of these rules see Lipka's *Graphical and Mechanical Computation*, page 233.

[2] *Transactions of the Institution of Naval Architects*, Vol. VI (1865), p. 41.

1865. Merrifield divided a given interval into six equal subintervals of width h and then applied both rules to this subdivided interval. By using the calculus of finite differences and comparing the results for the two rules with the more accurate value given by Cotes' formula for six sub-intervals, he found that the principal part of the inherent error in the three-eighths rule was 9/4 times that of the one-third rule.

In 1909 W. W. Johnson[1] made extensive numerical tests of Simpson's two rules and found that the one-third rule gave a more accurate result in every case. In a later paper[2] Professor Johnson stated that he compared the two rules with an assumed power series

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

and found that the errors began with terms in a_4 in each case. He gave no expressions for the errors, but stated that their ratio was 4 : 9 in favor of the one-third rule, as previously found by Merrifield.

F. A. Willers[3] has also found expressions for the errors of the two Simpson rules in a given interval, but since the number of sub-intervals was different in each case the results furnish no comparison of the relative accuracy of the two rules.

The only correct way to compare the accuracy of quadrature formulas is to apply them all to the *same interval* and divide this interval into the *same number of sub-intervals*, as pointed out by Merrifield. The object of the present paper is to derive definite expressions for the errors inherent in Simpson's two rules and in Weddle's rule[4] when all are applied to the same interval and to the same number of sub-intervals. The expressions here obtained for the errors are believed to be new.

2. Expressions for the Error in the three Rules. THEOREM. *The errors inherent in Simpson's two rules and in Weddle's rule, when applied under identical conditions to the quadrature of a continuous function $f(x)$ over the interval $(k-3h, k+3h)$, are:*
For Simpson's one- hird rule,

$$E_{1/3} = -\frac{h^5}{30} f^{\text{iv}}(k) - \frac{17h^7}{420} f^{\text{vi}}(k) - \cdots$$

---

[1] *U. S. Naval Institute Proceedings*, Vol. 35 (1909), p. 759.

[2] *U. S. Naval Institute Proceedings*, Vol. 37 (1911), p. 871.

[3] *Graphische Integration* (1920), pp. 29 and 33.

[4] Weddle's rule for seven equidistant ordinates is $W=(3/10)h(y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)$, where h is the distance between ordinates. In applying this rule, the number of equal intervals must be a multiple of 6. See Lipka, loc. cit.

For Simpson's three-eighths rule,

$$E_{3/8} = -\frac{3h^5}{40}f^{\text{iv}}(k) - \frac{53h^7}{560}f^{\text{vi}}(k) - \cdots$$

For Weddle's rule,

$$E_w = -\frac{h^7}{140}f^{\text{vi}}(k) - \cdots$$

where k is the mid-point of the interval considered and h is the width of the sub-interval.

PROOF. The smallest number of sub-intervals to which all three rules can be applied is six. We therefore divide the interval from $x=k-3h$ to $x=k+3h$ into six equal sub-intervals of width h. The division points are then $k-3h$, $k-2h$, $k-h$, k, $k+h$, $k+2h$, and $k+3h$.

Assume that $f(x)$ and all its derivatives are continuous in the interval considered. Also let

$$\int_k^x f(x)dx = F(x) + C.$$

Then the true value of the integral

$$\int_{k-3h}^{k+3h} f(x)dx \quad \text{is} \quad I = \int_{k-3h}^{k+3h} f(x)dx = F(k+3h) - F(k-3h).$$

For a sufficiently small value of h the given function can be represented in the given interval by a Taylor series. Hence, since $F'(x)=f(x)$, $F''(x)=f'(x)$, etc., we have

$$F(k+3h) = F(k) + 3hf(k) + \frac{9h^2}{2}f'(k) + \frac{27h^3}{6}f''(k) + \cdots$$

$$F(k-3h) = F(k) - 3hf(k) + \frac{9h^2}{2}f'(k) - \frac{27h^3}{6}f''(k) + \cdots - \cdots$$

Therefore

$$(1)\quad I = \int_{k-3h}^{k+3h} f(x)dx = 6hf(k) + 9h^3f''(k) + \frac{81}{20}h^5f^{\text{iv}}(k) + \frac{243}{280}h^7f^{\text{vi}}(k) + \cdots$$

The value of this integral by Simpson's one-third rule is

$$S_{1/3} = \frac{h}{3}\{f(k-3h) + f(k+3h) + 4[f(k-2h) + f(k) + f(k+2h)] + 2[f(k-h) + f(k+h)]\}.$$

Replacing the functions $f(k-3h)$, $f(k+3h)$, etc. by their Taylor expansions, we get

$$(2)\qquad S_{1/3} = 6hf(k) + 9h^3f''(k) + \frac{49}{12}h^5f^{\text{iv}}(k) + \frac{329}{360}h^7f^{\text{vi}}(k) + \cdots$$

Subtracting (2) from (1), we have

$$(3)\qquad E_{1/3} = I - S_{1/3} = -\frac{h^5}{30}f^{\text{iv}}(k) - \frac{17h^7}{420}f^{\text{vi}}(k) - \cdots$$

Simpson's three-eighths rule gives for the integral (1)

$$S_{3/8} = \frac{3h}{8}\{f(k-3h) + f(k+3h) + 3[f(k-2h) + f(k+2h) + f(k-h) + f(k+h)] + 2f(k)\}.$$

Replacing the several functions on the right by their Taylor expansions, we have

$$(4)\qquad S_{3/8} = 6hf(k) + 9h^3f''(k) + \frac{33}{8}h^5f^{\text{iv}}(k) + \frac{77}{80}h^7f^{\text{vi}}(k) + \cdots$$

Subtracting (4) from (1),

$$(5)\qquad E_{3/8} = I - S_{3/8} = -\frac{3h^5}{40}f^{\text{iv}}(k) - \frac{53}{560}h^7f^{\text{vi}}(k) - \cdots$$

The value of the integral (1) by Weddle's rule is

$$W = \frac{3h}{10}\{f(k-3h) + f(k+3h) + 5[f(k-2h) + f(k+2h)] + f(k-h) + f(k+h) + 6f(k)\}.$$

Replacing the functions by their Taylor expansions as before, we get

$$(6)\qquad W = 6hf(k) + 9h^3f''(k) + \frac{81}{20}h^5f^{\text{iv}}(k) + \frac{7}{8}h^7f^{\text{vi}}(k) + \cdots$$

Subtracting (6) from (1), we find

$$(7)\qquad E_w = I - W = -\frac{h^7}{140}f^{\text{vi}}(k) - \cdots$$

as the error in Weddle's rule.

3. Comparison of the Errors. When the errors for the Simpson rules are written in the forms

$$E_{1/3} = -\frac{4h^5}{120}\left(f^{\mathrm{iv}}(k) + \frac{51}{42}h^2 f^{\mathrm{vi}}(k)\right)$$

$$E_{3/8} = -\frac{9h^5}{120}\left(f^{\mathrm{iv}}(k) + \frac{53}{42}h^2 f^{\mathrm{vi}}(k)\right),$$

it is evident that the parenthetical expressions are nearly equal; and since the coefficients of the parentheses are in the ratio of 4 to 9, it follows that the error inherent in the one-third rule is generally about 4/9 that of the three-eighths rule. Moreover, since the second term in each parenthesis is multiplied by h^2, it is also evident that the principal part of the error is contained in the first term. This consideration makes it all the more apparent that the errors of the two rules are generally in the ratio of 4 to 9. The only case in which $E_{1/3}$ could ever exceed $E_{3/8}$ would be that in which both $f^{\mathrm{iv}}(k)$ and $f^{\mathrm{vi}}(k)$ were different from zero and of opposite sign and in which the expression $f^{\mathrm{iv}}(k)+53/42h^2f^{\mathrm{vi}}(k)$ were zero or nearly so. Such a case would be of rare occurrence.

The error in Weddle's rule is so much smaller than $E_{1/3}$ and $E_{3/8}$ that it can hardly be compared with them. As a matter of fact, when appropriate values are taken for h the error in Weddle's rule is practically negligible.

4. **Conclusion.** The results obtained in this paper confirm the previous results of Merrifield and W. W. Johnson as to the relative accuracy of the two Simpson rules. Both rules will give exact, and therefore identical, results when the given function is a polynomial of the third degree or less; but in all other cases, with rare possible exceptions, the one-third rule will give the more accurate result.

It is generally agreed that the one-third rule is simpler in form and more flexible and convenient in its application than the three-eighths rule. It is also more accurate than the latter, as we have already seen. Hence there seems to be no good reason for the continued existence of the three-eighths rule in mathematical textbooks, and the writer is of the opinion that all space devoted to this rule in the textbooks is wasted.

The expression for the error in Weddle's rule shows that this quadrature formula is extremely accurate. The rule is simpler in form and application than Simpson's three-eighths rule (because of the coefficient 3/10 instead of 3/8) and is far more accurate than either of the Simpson rules. Its only disadvantage is that the number of sub-intervals used must be a multiple of six. Nevertheless, this rule should, in the writer's opinion, receive more attention in undergraduate textbooks.

II. The Error in an Approximate Division of the Circle

By E. J. Mc Shane, Tulane University

In the report of the 1926[1] meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America, an approximate construction was given for the division of a circle into n equal parts, and it was remarked that it would be interesting to know the limits of error. The construction is this: On a diameter AB construct an equilateral triangle ABP. Divide AB into n equal parts; let C be the second point of division from A. Let PC produced intersect the circle in D. Then arc AD is very nearly an nth part of the circumference.

To investigate the error, we will assume axes and scale of drawing such that the radius is 1, the center the origin, and BA the x-axis. The coordinates of A are $(1,0)$; of P, $(0, -3^{1/2})$; of D, $(\cos\theta, \sin\theta)$, where $\theta=\angle AOD$. Let $x=OC$. We will now permit θ to vary continuously from 0 to $\pi/2$.

If the construction were everywhere exact, we should have $x=1-(2\theta/\pi)$. We find however that $x=3^{1/2}\cos\theta/(3^{1/2}+\sin\theta)$.

$$\text{Set } e = f(\theta) = 1 - \frac{2\theta}{\pi} - \frac{3^{1/2}\cos\theta}{3^{1/2}+\sin\theta}\cdot \quad \text{Then } f'(\theta) = -\frac{2}{\pi} + \frac{3^{1/2}+3\sin\theta}{(3^{1/2}+\sin\theta)^2}\cdot$$

Both are continuous in $(0, \pi/2)$. $f'(\theta)$ has two zeros in $(0, \pi/2)$, which we find with no difficulties other than computational to be $\Theta=\text{arc sin } .29210$, $\Theta'=\text{arc sin } .95619$.

We find $f(0)=f(\pi/3)=f(\pi/2)=0$; that is, for these values the error vanishes. Moreover, for no other values in $(0, \pi/2)$ is the construction exact, for $f'(\theta)$ has but two zeros in that interval.

We will henceforth restrict ourselves to the interval $(0, \pi/3)$, thereby leaving out of consideration the case of division into five equal parts; for this there is an exact geometrical construction.

$$f(\Theta) = -.007082, \text{ whence } |e| \leqq .007082.$$

It is easily shown that $f''(\theta)$ is continuous and has only one zero in $(0, \pi/2)$, which is arc sin $3^{-1/2}$. Hence in the interval $(0, \text{arc sin } 3^{-1/2})$ and a fortiori in the interval $(0, \Theta)$, $f'(\theta)$ is monotonic, and if $0\leqq\theta\leqq\Theta$, $0\geqq de/d\theta\geqq f'(0)$. Integrating and remembering that $f(0)=0$,

$$0 \geqq e \geqq f'(0)(\theta) \text{ and } \left|\frac{e}{\theta}\right| \leqq |f'(0)| = .0592694.$$

---

[1] This Monthly, vol. 33 (1926), p. 434.

But

$$\left|\frac{dx}{d\theta}\right| = \frac{3^{1/2}\sin\theta + 3^{1/2}}{(3^{1/2}+\sin\theta)^2} = \frac{2}{\pi} + \frac{de}{d\theta}.$$

Let the error in θ be η. If $\Theta < \theta \leqq \pi/3$,

$$\frac{dx}{d\theta} \geqq \frac{2}{\pi}, \quad \left|\frac{d\theta}{dx}\right| \leqq \frac{\pi}{2}, \quad |\eta| \leqq \frac{\pi|e|}{2} \leqq .011123, \text{ and } \left|\frac{\eta}{\theta}\right| \leqq \frac{|\eta|}{\Theta} \leqq .037525.$$

If $0 \leqq \theta \leqq \Theta$, $\quad \dfrac{dx}{d\theta} \geqq \dfrac{2}{\pi} + f'(0) = 3^{-1/2}$,

$$|\eta| \leqq 3^{1/2}|e| \leqq .012265, \text{ and } \left|\frac{\eta}{\theta}\right| \leqq 3^{1/2}\left|\frac{e}{\theta}\right| \leqq 3^{1/2}|f'(0)| = .10265.$$

Hence the error cannot be as great as $43'$, nor the percentage of error greater than 10.265; and the latter figure is the limit of the percentage error as θ approaches 0.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB TOPICS

1927 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Whitman College, Walla Walla, Washington

In 1925 and again in 1926 lists of important events in the history of mathematics for which those years were centennials were published in this MONTHLY. These were printed as suggestions to program committees of college and high school mathematics clubs, teachers of courses in the history of mathematics, and others who might find in these events suggestive topics for programs and special investigations or reports.

The year 1927 proves to be less fruitful as a centennial year than either of the preceding years. The 1925 list[1] contained 24 important events; the 1926 list[2] had 20. In contrast with these, the 1927 list has but eight. However, it includes the death of two of the greatest mathematicians of all the centuries, to each of whom several programs might be devoted, Newton and Laplace. The list follows:

A.D.

727. Death of I-hsing, inventor of a new Chinese calender. Chinese calendars.

---

[1] This MONTHLY, vol. 32 (1925), p. 258.

[2] This MONTHLY, vol. 33 (1926), p. 274.

1527. Birth of John Dee, who wrote the preface to the first English translation of Euclid ("a very fruitfull Praeface"), and possibly made the translation also.

1727. Appointment of Isaac Greenwood as Hollis professor of mathematics at Harvard College, first man to occupy a collegiate chair of mathematics in New England.

1727. Death of Sir Isaac Newton, "the greatest genius that ever lived" (Lagrange); "probably the greatest mathematical mind of all time" (Cajori); who "in mathematical power has never been surpassed" (Ball); inventor of the calculus and discoverer of the law of gravitation as set forth in the *Principia*, "the greatest production of the human mind" (Lagrange), a book which, according to Laplace will always be assured "a preeminence above all the other productions of human genius."

1827. Publication of "The New Arithmetic" by Daniel Adams, one of the three "great arithmeticians" of the early nineteenth century in America. Comparison with Pike and Daboll.

1827. Death of Fresnel. Mathematical theory of light.

1827. Death of Woodhouse, "apostle of the new movement" in England (Ball). Relation of the differential calculus to the calculus of fluxions.

1827. Death of Laplace, author of the *Mécanique Céleste* which, according to the author, was intended to "offer a complete solution of the great mechanical problem presented by the solar system." The nebular hypothesis; celestial mechanics.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Linear Integral Equations. By W. V. LOVITT. New York, McGraw-Hill Book Co., 1924. 253 pages. Price $3.00.

This book on linear integral equations is substantially a reprint of its author's notes of lectures on integral equations by Bolza, to whom the author acknowledges indebtedness in the preface. The lectures were delivered at the University of Chicago for the Summer Quarter of 1913. Even at that date the publication might have been regarded as a fragmentary presentation of the theory, and of course in 1924 the discussion is still less adequate. On the theoretical side, in fact, although there is an excuse in the endeavor to make

the treatment elementary, the book does not stand comparison with the briefer presentations of Lalesco and Heywood-Fréchet, whom the author mentions in his brief list of references, nor with a certain *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen,* which the author does not mention. The applications of the theory are well chosen both from pure and applied mathematics, and constitute about half of the text.

The "notes" have been slightly rearranged, and one or two new sections have been added, for example, sections 25 and 71, to the former of which we will revert below. But the principal addition is a collection of examples, useful for the student. As he becomes proficient he may be interested in devising methods of solution for them which are not mentioned in the text, e. g., by means of Fourier series or differential equations.

The contents are arranged with four main divisions. First the integral equation is considered as a generalization of an algebraic problem, the integral replacing a finite sum; and the Fredholm theory is given. In the second part are given applications, notably to the Dirichlet and Neumann problems. The theoretical treatment is then resumed with a study of orthogonal functions and symmetric kernels (they are called "kerns" in one place), and in the last part there are given applications to boundary problems in ordinary differential equations, including a reference to the calculus of variations, and in partial differential equations. There are occasional slips, as for instance when the author carefully defines *index* of a root of $D(\lambda)$, as distinguished from *multiplicity* (page 52), and then (page 55) uses the word *order* for the one of these that the reader is least likely to guess. There are one or two instances of awkward typography, as on page 46, although McGraw-Hill deserves credit for a generally satisfactory job of difficult typing.

The principal fault for the students for which the book is designed, is the lack of employment of the theory of functions of a complex variable—a fault which is not compensated for by giving in the very last section a single instance of the theory in the method for calculating successive characteristic values. Vivanti starts his book off with a chapter on analytic functions. In the text under review the student is shown that the resolvent kernel is the ratio of two entire functions of the parameter λ, and will wonder why none of the customary methods of dealing with them are of any use. Of course, they *are* of use, and employment of them will not only broaden the point of view, but will also make the reader feel that the subject is an open and a live one rather than one that is entirely closed and finished. To take a simple illustration, given two kernels $K_1(x, y)=K(x, y)$ and $K_2(x, y)=K(y, x)$, what are the relations between their resolvent kernels $k_1(x, y, \lambda)$ and $k_2(x, y, \lambda)$, their Fredholm determinants $D_1(\lambda)$ and $D_2(\lambda)$, and the minors $D_1(x, y, \lambda)$ and $D_2(x, y, \lambda)$? These relations are established algebraically in the text (page 56), by means

of simple relations among determinants, but they may also be established simply by means of the analyticity of the functions involved.

From the relations (in accord with Lovitt's notation)

$$k_1(x, y, \lambda) - K_1(x, y) = \lambda\int_a^b k_1(x, t, \lambda)K_1(t, y)dt = \lambda\int_a^b K_1(x, t)k_1(t, y, \lambda)dt,$$

which deserve a prominent place, the function $k_1(x, y, \lambda)$ is uniquely determined for small values of λ, and thus for all values of λ except those for which $D_1(\lambda)=0$. Hence also, by interchanging x and y,

$$k_1(y, x, \lambda) - K_2(x, y) = \lambda\int_a^b k_1(y, t, \lambda)K_2(x, t)dt = \lambda\int_a^b K_2(t, y)k_1(t, x, \lambda)dt$$

It follows therefore immediately that $k_2(x, y, \lambda) = k_1(y, x, \lambda)$ for all except the characteristic values of λ.

But with the same exception

$$\frac{d}{d\lambda}\log D_1(\lambda) = -\int_a^b k_1(x, x, \lambda)dx$$

which is therefore the same as $d\log D_2(\lambda)/d\lambda$, so that $D_1(\lambda) \equiv D_2(\lambda)$, since $D_1(0) = D_2(0) = 1$. Likewise since $\lambda k_i(x, y, \lambda) = D_i(x, y, \lambda)/D_i(\lambda)$ we have $D_1(x, y, \lambda) \equiv D_2(y, x, \lambda)$. Since however these $D_i(\lambda)$, $D_i(x, y, \lambda)$ are entire functions, the relations just established hold not only for all values of λ except the characteristic values, but for the characteristic values as well. In the same way similar relations for the Fredholm minors of higher order may be established. Moreover, with this point of view the student may suspect that there are other transformations on the kernels worth studying besides the mere interchange of the order of the variables.

A striking illustration, but more advanced in theory, would be a discussion of the *genus* of the entire functions involved and the distribution of zero's of $D(\lambda)$. This subject is discussed by Lalesco and Vivanti.

The author has been criticized elsewhere for not including material in his discussion other than merely of the early days of the subject. The Moore generalizations, the Volterra generalizations, the relation to functional equations are all much studied nowadays. Perhaps it might have been worth while in the treatment of the Hilbert-Schmidt theory to bring out clearly the connection of that theory with the theory of quadratic forms. But I should like to dwell a little on the subject of calculation, which is related to section 25 of the text.

There seems to be a conviction in the minds of many, very often expressed, that one cannot calculate the terms in the Fredholm expansion of the solution

of an integral equation, without evaluating determinants and performing integrations of successively increasing order. Let us first dispel that illusion. We write the equation, in order to avoid alternating signs in the development, in the form

$$(1) \qquad u(x) = f(x) - \lambda \int_a^b K(x, s)u(s)ds$$

with solution in the form

$$(2) \qquad u(x) = f(x) - \lambda \int_a^b k(x, s, \lambda)f(s)ds$$

where the resolvent kernel

$$k(x, y, \lambda) = D\left(\begin{matrix} x \\ , \\ y \end{matrix} \lambda\right)/D(\lambda)$$

satisfies and is determined by the equations

$$(3)\ K(x, y) - k(x, y, \lambda) = \lambda \int_a^b K(x, s)k(s, y, \lambda)ds = \lambda \int_a^b k(x, s, \lambda)K(s, y)ds.$$

If now we write

$$D\left(\begin{matrix} x \\ , \\ y \end{matrix} \lambda\right) = A_0(x, y) + \lambda A_1(x, y) + \cdots + \lambda^p A_p(x, y) + \cdots$$

$$D(\lambda) = 1 + \lambda a_1 + \cdots + \lambda^p a_p + \cdots$$

the equation (3) yields the relations

$$(4) \qquad \begin{aligned} A_0(x, y) &= K(x, y) \\ A_p(x, y) &= a_p K(x, y) - \int_a^b K(x, s)A_{p-1}(s, y)ds, \quad p = 1, 2, \cdots . \end{aligned}$$

As is well known, the equations (4) taken with the further relation

$$(5) \qquad pa_p = \int_a^b A_{p-1}(s, s)ds,$$

determine precisely the Fredholm solution of the integral equation.[1]

In other words the pair of equations (4), (5) constitute recurrent relations to determine successively a_1, $A_1(x, y)$, $\cdots$ a_p, $A_p(x, y)$, $\cdots$ if the integrations can be performed. It is for the theory that the determinants

$$K\begin{pmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & \cdots & K(x_1, y_n) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K(x_n, y_1) & \cdots & K(x_n, y_n) \end{vmatrix}$$

[1] Lalesco, *Introduction à la théorie des équations intégrales*, (Paris 1912), p. 26.

and their integrals have their uses. They seem to be necessary, for instance, in order to prove the convergence of the two series.

For purposes of calculation, special forms of the kernel may be handled in special ways. Thus if the kernel is of the form $K(x, y) = K_1(x-y) + K_2(x+y)$, where $K_1(t)$ and $K_2(t)$ are periodic with period $b-a$, the resolvent kernel is of the same kind and may be determined directly by expansion in Fourier series. The student will do well to study a kernel of the form $K(x-y)$ for an interval $(-\pi, \pi)$ in this way; and if he uses complex numbers when he can, a multitude of results will flow forth with such a simplicity and directness that it would be a pity to spoil his delight by a printed exposition of the subject. Whittaker and Robinson, in their book on interpolation, handle kernels related to these in another fashion.

A second special form, important because it contains the general polynomial and any continuous kernel may be represented to an arbitrary approximation ρ by a polynomial, is the form already mentioned with reference to section 25:

$$K(x, y) = a_1(x)b_1(y) + \cdots + a_n(x)b_n(y)\,, \tag{6}$$

where for simplicity we shall take the sets of functions $a_i(x)$, $b_i(y)$ each as linearly independent. Incidentally the part (b) of Lovitt's theorem on page 69 is partly false. If, with Lovitt (changing λ to $-\lambda$ for convenience), we write

$$K_i = \int_a^b b_i(t)u(t)dt, \quad C_{ki} = \int_a^b a_k(t)b_i(t)dt,$$

we shall obtain from (1) the explicit solution

$$u(x) = f(x) - \lambda(a_1(x)K_1 + \cdots + a_n(x)K_n)$$

if we can solve the equations

$$K_i + \lambda(C_{1i}K_1 + \cdots + C_{ni}K_n) = \int_a^b b_i(t)f(t)dt,\ i = 1, 2, \cdots, n,$$

for the constants $K_1 \cdots, K_n$. This can be done provided the determinant

$$H(\lambda) = \begin{vmatrix} 1+\lambda C_{11} & \lambda C_{12} & \cdots & \lambda C_{1n} \\ \lambda C_{21} & 1+\lambda C_{22} & \cdots & \lambda C_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \lambda C_{n1} & \lambda C_{n2} & \cdots & 1+\lambda C_{nn} \end{vmatrix}$$

is not zero. It is obviously not identically zero.

In the text, and in some others, this determinant is called $D(\lambda)$ without more ado. The fact is correct, but not by Revelation. It requires proof, since the expression of the resolvent kernel as a ratio of two entire functions of λ is not uniquely determined.[1]

We write
$$H(\lambda) = 1 + h_1\lambda + \cdots + h_n\lambda^n ; \quad h_n = \begin{vmatrix} C_{11} & \cdots & C_{1n} \\ \cdot & \cdots & \cdot \\ C_{n1} & \cdots & C_{nn} \end{vmatrix}.$$

On the other hand, by direct examination of the determinants in $a_{n+1}, \cdots$, it is seen that these are all zero, so that $D(\lambda)$ is itself a polynomial of degree not higher than n. It is not difficult to show that $a_n = h_n$. In fact, we have by definition

$$n!a_n = \int_a^b \begin{vmatrix} a_1(s_1)b_1(s_1) + \cdots + a_n(s_1)b_n(s_1) & \cdots & a_1(s_1)b_1(s_n) + \cdots + a_n(s_1)b_n(s_n) \\ \cdot\ \cdot\ \cdot & \cdots & \cdot\ \cdot\ \cdot \\ a_1(s_n)b_1(s_1) + \cdots + a_n(s_n)b_n(s_1) & \cdots & a_1(s_n)b_1(s_n) + \cdots + a_n(s_n)b_n(s_n) \end{vmatrix} ds_1 \cdots ds_n,$$

and this again is seen to be

$$\int_a^b \begin{vmatrix} a_1(s_1)b_1(s_1) + \cdots + a_1(s_n)b_1(s_n) & \cdots & a_1(s_1)b_n(s_1) + \cdots + a_1(s_n)b_n(s_n) \\ \cdot\ \cdot\ \cdot & \cdots & \cdot\ \cdot\ \cdot \\ a_n(s_1)b_1(s_1) + \cdots + a_n(s_n)b_1(s_n) & \cdots & a_n(s_1)b_n(s_1) + \cdots + a_n(s_n)b_n(s_n) \end{vmatrix} ds_1 \cdots ds_n,$$

by writing the first determinant as the product of two factor determinants $| a_i(s_j) |$ and $| b_m(s_k) |$, and then combining the rows and columns in another way to form a new product.

But this last integral is the sum of $n!$ identical integrals h_n, the rest being zero identically through the elementary properties of determinants. Since every root of $D(\lambda)$ is a pole of the resolvent kernel and is therefore a root of $H(\lambda)$, it follows that if all the roots of $D(\lambda)$ are distinct $D(\lambda)$ is a factor of $H(\lambda)$, and if $h_n \neq 0$ it follows therefore, since $h_0 = a_0 = 1$ and $h_n = a_n$, that $D(\lambda) \equiv H(\lambda)$. The exceptional cases may be taken account of by a familiar principle of continuity, although it is a continuity of functionals which is involved. Thus, if for given functions $b_i(y)$, h_n were zero for all $a_i(x)$, it would follow that the $b_i(y)$ were linearly dependent. There cannot be identically multiple roots of $D(\lambda)$ for kernels of the form (6), since the contrary is known for kernels of special types.[2]

[1] An extended proof is given for instance by E. Goursat, Bulletin de la Societé Mathématique de France, vol. 35 (1907), p. 163.

[2] See for instance, Lalesco, loc. cit., p. 36.

Given any kernel $K(x, y)$ we can calculate the resolvent kernel, the characteristic values, and so on, approximately by replacing $K(x, y)$ by a uniformly approximate polynomial. It is important to establish this fact completely, and to know the order of the error and of the shift in the characteristic values—since obviously all but a finite number of these move off to infinity. The quantities mentioned are functionals of $K(x, y)$; the problem therefore is one of the differentials of functionals, and depends on finding the differential of the determinant

$$K\begin{pmatrix} x_1 \cdots x_n & s_1 \cdots s_p \\ y_1 \cdots y_n & s_1 \cdots s_p \end{pmatrix}.$$

This is of course the sum of $n+p$ determinants, each linear in $\delta K(x, y)$, and is thus easily written down; it is hardly worth while here to give a symbolic form for it. By means of it the following formulae may be verified formally, remembering that the $s_1, \cdots, s_p$ are merely arguments of integration.

We shall have, in fact,

$$\begin{aligned} \delta D\begin{pmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{pmatrix}, \lambda\Big) &= \sum_1^n{}_{ij}(-1)^{i+j}\,\delta K(x_i, y_j) D\begin{pmatrix} x_1 \cdots x_{i-1}x_{i+1} \cdots x_n \\ y_1 \cdots y_{j-1}y_{j+1} \cdots y_n \end{pmatrix}, \lambda\Big) \\ &+ \lambda D\begin{pmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{pmatrix}, \lambda\Big) \int_a^b \delta K(\xi, \xi)\,d\xi \\ &-\lambda \sum_1^n{}_i \int_a^b \Big[\delta K(x_i, \xi) D\begin{pmatrix} x_1 \cdots x_{i-1}\xi x_{i+1} \cdots x_n \\ y_1 \cdots \cdots \cdots y_n \end{pmatrix}, \lambda\Big) \\ &+ D\begin{pmatrix} x_1 \cdots \cdots \cdots x_n \\ y_1 \cdots y_{i-1}\xi y_{i+1} \cdots y_n \end{pmatrix}, \lambda\Big)\delta K(\xi, y_i)\Big]\,d\xi \\ &- \lambda^2 \int_a^b \delta K(\xi, \eta) D\begin{pmatrix} x_1 \cdots x_n, \eta \\ y_1 \cdots y_n, \xi \end{pmatrix} d\xi d\eta, \end{aligned} \tag{7}$$

of which special cases are the following :

$$\begin{aligned} \delta D\begin{pmatrix} x \\ y \end{pmatrix}, \lambda\Big) &= \delta K(x, y) D(\lambda) + \lambda D\begin{pmatrix} x \\ y \end{pmatrix}, \lambda\Big) \int_a^b \delta K(\xi, \xi)\,d\xi \\ &- \lambda \int_a^b \Big[\delta K(x, \xi) D\begin{pmatrix} \xi \\ y \end{pmatrix}, \lambda\Big) + D\begin{pmatrix} x \\ \xi \end{pmatrix}, \lambda\Big) \delta K(\xi, y)\Big]\,d\xi \\ &- \lambda^2 \int_a^b \delta K(\xi, \eta)\, D\begin{pmatrix} x & \eta \\ y & \xi \end{pmatrix}, \lambda\Big) d\xi d\eta. \end{aligned} \tag{8}$$

$$\delta D(\lambda) = \lambda D(\lambda) \int_a^b \delta K(\xi, \xi)\,d\xi - \lambda^2 \int_a^b \delta K(\xi, \eta) D\begin{pmatrix} \eta \\ \xi \end{pmatrix}, \lambda\Big) d\xi d\eta .$$

From (8) we obtain the value of $\delta u(x)$ if λ is not a characteristic value, and from (7) the approximations to the new fundamental solutions if λ is a characteristic value. For purposes of calculation it will be convenient to consider the quantities in this equation as relating to the approximate kernel, which may be taken as $K(x, y)$, rather than the given kernel.

The new characteristic values, which correspond to a given λ_0 where $D(\lambda_0)=0$, will be the roots of the equation

$$(D + \delta D)(\lambda_0 + d\lambda) = 0 . \tag{9}$$

If λ_0 is a root of multiplicity n and of index q (i. e., with q corresponding characteristic functions), we shall have $D^{(n)}(\lambda_0)\neq 0$, and usually

$$D^{(q-1)}\binom{n}{\xi}, \lambda_0\Big) \not\equiv 0$$

as the derivatives of lowest order which are not zero, since

$$D^{(q-1)}\left(\begin{matrix}\eta\\ \xi\end{matrix}, \lambda_0\right) = \int_a^b D\left(\begin{matrix}\eta s_1 \cdots s_{q-1}\\ \xi s_1 \cdots s_{q-1}\end{matrix}, \lambda_0\right) ds_1 \cdots ds_{q-1}$$

and

$$D\left(\begin{matrix}x_1 \cdots x_q\\ y_1 \cdots y_q\end{matrix}, \lambda_0\right) \not\equiv 0.$$

Hence (9) becomes

$$\frac{1}{n!} D^{(n)}(\lambda_0) d\lambda^n + \frac{\lambda_0}{n!} D^{(n)}(\lambda_0) d\lambda^n \int_a^b \delta K(\xi, \xi) d\xi$$
$$- \frac{\lambda_0^2}{(q-1)!} d\lambda^{q-1} \int_a^b \delta K(\xi, \eta) D^{(q-1)}\left(\begin{matrix}\eta\\ \xi\end{matrix}, \lambda_0\right) d\eta = 0$$

or

$$d\lambda^{n-q+1}\left\{1 + \lambda_0 \int_a^b \delta K(\xi, \xi) d\xi\right\}$$
$$= \frac{\lambda_0^2}{D^{(n)}(\lambda_0)} \frac{n!}{(q-1)!} \int_a^b \delta K(\xi, \eta) D\left(\begin{matrix}\eta s_1 \cdots s_{q-1}\\ \xi s_1 \cdots s_{q-1}\end{matrix}, \lambda_0\right) d\xi d\eta ds_1 \cdots ds_{q-1} . \tag{10}$$

This relation shows that $d\lambda^{n-q+1}$ is an infinitesimal at least of the first order in $\rho = \max. |\delta K(x, y)|$. If the index is equal to the multiplicity, as with a symmetric kernel, we have $q=n$, and $d\lambda$ itself is of the first order, or, in exceptional cases, higher order.

Further discussion of these questions is evidently desirable. The formulae given above depend for their justification on some inequalities obtained by Tricomi, who thus gives upper bounds for the changes in $D(\lambda)$ and $D(x, y, \lambda)$ although he does not give any of the differential formulae or consider the

characteristic values.[1] He notices the fundamental fact that the evaluation of the bounds which he gives depends upon the properties of the single transcendental function

$$\Omega(x) = \sum_{n=0}^{\infty} (n^{n/2}/n!)x^n < (1 + x)e^{ex^2/2},$$

whose values and differences he tabulates. Here again further work is desirable since this function and its derivatives are dominating functions in the formulae given above.

Much earlier work was done by Mrs. Rachel B. Adams.[2] In this, inequalities are given for the change in $D(\lambda)$ and for the shift of parametric values, yielding a different kind of information from that in (10). Also a theorem is stated on the relation between the fundamental functions of the approximate and the given kernels for the corresponding values of λ, that the normalized fundamental *solutions* for one kernel are approximated to by linear combinations of the normalized fundamental *functions* of the other.[3]

The length of the present article probably needs an apology. But the text is one that has been and will be read by many a reader of this MONTHLY, and it is hoped that this review will, in connection with the text, orient him in respect to the subject. In praise of the book it may be said that it is clear and easy to read; the student must however follow the maxim that no book on mathematics should be read without pencil and paper.

G. C. EVANS.

Automorphe Funktionen. By L. SCHLESINGER. Göschens Lehrbücherei, I Gruppe, Band 5. Berlin, Walter de Gruyter & Co., 1924. x+205 pages.

This book gives a readable, systematic account of a considerable portion of Poincaré's theory of the automorphic functions. It is a much briefer work than the Klein-Fricke treatise, and does not, as do the books of Fubini and Giraud, start with functions of several variables. For these reasons it will possibly be more useful to beginners than the books on automorphic functions already available.

---

[1] Rendiconti della Reale Accademia dei Lincei, vol. 23 (1924) 1° sem., p. 483 and 2° sem., p. 26. From Mrs. R. B. Adams I have the information that a result equivalent to that expressed by (10) is given, and differential formulae obtained, also differentials of higher order, by H. Block, Acta Universitatis Lundensis, Nova Series, Med. Matem. och Natur, vol. 7 (1911). We may regard either kernel, of course, as an approximation to the other.

[2] Thesis in candidacy for the degree of Doctor of Philosophy submitted to Radcliffe College (1921), not yet published. Mrs. Adams kindly has sent a summary of a portion of this thesis to the writer of this review.

[3] For the terminology, see Lalesco, loc. cit., p. 55 and p. 57.

Schlesinger starts with the simplest examples of the automorphic functions, considering in turn x^n, e^x and certain elliptic functions. He then develops the theory of Fuchsian functions of genus zero, using Poincaré's theta series to establish the existence of such functions. The book concludes with a chapter on the uniformization problem. A detailed treatment of the Riemann mapping problem, from the most modern point of view, is contained in this chapter.

Schlesinger seems to accomplish well what he sets out to do. His book will facilitate the movement of students through a theory in which great results stand behind masses of thorny details.

J. F. Ritt

Lehrbuch der Mathematik: Eine Einführung in die Differential- und Integralrechnung und in die Analytische Geometrie. By G. Scheffers. Sixth edition. Berlin und Leipzig, Walter de Gruyter & Co., 1925. 743 pages.

This book represents a thorough revision of Professor Scheffer's Lehrbuch, which first appeared in 1905 and has since that time gone through six editions. It was written primarily, as the author says, for students of the natural sciences and engineering, who wish to acquire a sound working knowledge of higher mathematics by themselves. And it may be doubted whether any book could fulfill this avowed purpose more completely, especially if used by students as mature as those usually found in the first semesters of continental universities and technical schools.

The voluminous character of the work is due to the fact that very little mathematical equipment on the part of the reader is presupposed, it being taken for granted that he has forgotten a good deal of his school mathematics. Accordingly, one finds such things as linear functions, quadratic equations, logarithms, and trigonometry treated. However, the presentation of these elementary topics is not of the perfunctory kind so often met with, but contains much that is novel and interesting. This is especially true of the chapter on logarithms, which clearly reveals, as F. Klein remarked in his "Elementarmathematik vom höheren Standpunkte aus," the eminent pedagogical ability of the author. He starts with the integral $\int_a^x dx/x$, and then develops the properties of the logarithmic function from this point of view.

In the chapters on analytics, both plane and solid, it is unusual to find the author stating explicitly in each instance whether or not the validity of a theorem is dependent upon the choice of the same unit for all coördinates involved.

The topics from the differential and integral calculus which are taken up, are those usually found in a standard treatise, but the customary separation of the subject matter into two distinct parts is not adhered to. The proofs given are lucid, and all assumptions as to continuity, existence of derivatives,

etc., are carefully stated, but unnecessary generality is avoided where it might confuse the student. A wealth of excellent figures accompanies the text. There are almost no formal exercises for the student, but a great profusion of illustrative examples are worked out in complete detail, and taken from the most varied fields of application. In some instances five or more pages are devoted to a single example. A case in point is the derivation of the probability integral. As an illustration of periodic functions, Fourier's series are studied at considerable length.

As an amusing detail, the reviewer notes that on p. 59 the author criticises the practice of speaking of "positive" and "negative" sign, instead of "plus" and "minus" sign. On p. 230, in the statement of Theorem 5, we find him using the very phrase he objects to.

There seem to be very few misprints. The reviewer noted only one of any consequence, namely at the foot of p. 80, where a Δ is misplaced. The typography and paper of the work are excellent.

H. BETZ

The Fourth Dimension and the Bible. By W. A. GRANVILLE. Boston, R. G. Badger, 1922. 119 pages. Price $1.50.

In the preface to this work Dr. Granville has materially assisted the critics by frankly reviewing in two sentences his own production. He says: "The author's chief aim will be to point out the remarkable agreement which exists between numerous Bible passages and some of the concepts which follow quite naturally from the mathematical hypothesis of higher spaces. In making this attempt the author is well aware that he is on a 'no man's land' exposed to fire from the mathematical trenches on the one side and the theological trenches on the other."

The statement is perfectly true; he will receive little favorable comment from the mathematician, because his treatment of higher spaces is, as he doubtless intended it should be, for those who know no mathematics. From the theologian he will receive scant praise because his treatment of theology is, as he doubtless intended it should be, for those who know no theology. As to the layman with respect to both fields, this individual will be inclined to look upon the book as one which, as should be the case with the product of a mathematician's pen, draws conclusions,—but the chances are that he will question all of the latter, or at any rate that he ought to do so. When a man reads the fourth dimension into the phrase, "When I consider thy heavens," he opens himself to the possible immortality of being embalmed in a new "Budget of Paradoxes" by some future De Morgan, and similarly when he sees higher orders of space in such phrases as "caught up into the third heaven," "Behold, I create new heavens," and "the heaven of heavens cannot contain thee."

It would be rather unbecoming to seek to ridicule an earnest effort to explain certain passages in the Scriptures by postulating dimensions higher than what we commonly conceive to be our own, just as it would show poor taste to smile at old Father Bongo's voluminous treatise on the mystery of biblical numbers. Nevertheless, it must frankly be said that Dr. Granville has shown no originality in his treatment of the fourth dimension, that his essay is written in an unfortunate style, and that the conclusions which he seems to attempt to draw from certain poetic phrases in the Bible are far-fetched and serve rather to render them meaningless than to give them new and inspiring significance.

DAVID EUGENE SMITH

Hauptzätze der Elementarmathematik. Ausgabe B. By F. G. MEHLER and A. SCHUELTE-TIGGES. Berlin, Walter de Gruyter & Co., 1925. (1) Text, 254 pages. (2) Problems, bound separately, 89 pages.

This is the second volume of the work of Dr. F. G. Mehler, revised by A. Schuelte-Tigges, and the accompanying problem book. It is designed for use as a text-book. The student would need about the same quantity of preparation as is required for admission to our American colleges, but the quality would need to be higher. The individual chapters are developed as units in orderly fashion, but there is no attempt apparently to weld the chapters into a larger unit. This is probably done deliberately, so that teachers desiring to give briefer courses can omit entire chapters without any difficulty. The discussions in the text are brief, but suggestive, and lead up to the theorems naturally. Much more is left to the labor and to the imagination of the student than is done in the American text of the same grade. In trigonometry, for example, no numerical examples are worked out in the text. The content of the usual high school or freshman text in trigonometry is covered in a chapter of sixteen pages. It contains the usual goniometry, trigonometric equations, and similar topics.

As compared with a survey book like that of Lennes[1] it contains many times the material and is diametrically different in its aims. Lennes aims to make mathematics alive and to show how it connects up with life. Mehler develops an amazingly large amount of mathematical theory with an even more surprisingly small amount of references to, or illustrations by, applied problems. The most marked exceptions are the section on interest, annuities, and life insurance as illustrations of series, and the brief reference to the application of plane trigonometry in surveying and of spherical trigonometry in astronomy.

The very multiplicity of topics treated renders a brief summary impossible. A statement of the general content of each of the ten chapters should be sugges-

[1] N. J. Lennes, *A Survey Course in Mathematics* (New York, Harper and Brothers, 1926).

tive. The order of the chapters will be interesting to one who has not seen the earlier editions or possibly some similar foreign text.

Chapter I is a brief summary of Modern Geometry based on harmonic points and pole and polar properties of the circle. Chapter II is Descriptive Geometry. All the ordinary central and parallel projections, including Mercator's, are discussed. Chapter III is the synthetic treatment of the conic sections. This is worth reading by any young teacher of mathematics. The Conics are developed (a) from the locus definitions, (b) as sections of a cone, (c) as projections of a circle, and (d) from the modern projective view point,—all within the compass of twenty-five pages.

Chapter IV—entitled Arithmetic and Algebra—covers all the topics of the usual College Algebra, plus an extended discussion of the nature of the different kinds of number, and the laws of combination and operation in arithmetic, including the complex number.

Chapters V, VI, and VII are devoted to Plane Trigonometry, Solid Geometry, and Spherical Trigonometry respectively. The trigonometric functions are developed as projections in a unit circle, and the formulas for angles differing from x by multiples of 90 and 180 degrees are derived at once from the figure.

Chapters VIII and IX are devoted to Differentiation and Integration respectively. There are the usual applications to maxima and minima, equations of tangents to curves, curvature, plane areas, volumes, etc. The study is not limited to algebraic functions.

Chapter X is Plane Analytic Geometry. No study is made of higher plane curves. The conic sections are treated in both cartesian and polar coordinates.

Although the book is not likely to be adopted as a textbook in any American schools under the tendencies now prevailing, it should prove interesting to the teacher or any other reader interested in the content or teaching of elementary mathematics, if the reading of the German is not prohibitive.

The problem book is a collection of over one thousand geometrical examples and problems of varying degrees of difficulty. The problems are classified to correspond to the chapters and paragraphs of the text. There are no general review lists. There is, as might be expected, very little of novelty in the list, but the teacher looking for new examples will find in this one volume a collection of suitable examples or suggestions for reviews and tests covering a wide range of topics. There are also some originals which should prove useful as outside work for the better students. There are no answers.

DAVID D. LEIB

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2)

titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by Americal authors published in foreign journals.

Annals of Mathematics, Volume 27, No. 4, September 1926: "On classes of ideals in a quadratic field" by H. H. Mitchell, 297–314; "Developments in Legendre polynomials" by M. H. Stone, 315–329; "On certain two-dimensional varieties of order $\frac{1}{2}r(r-1)$ in r space and certain curves on them" by B. C. Wong, 330–336; "On class number relations implied by representations as sums of an odd number of squares" by E. T. Bell, 337–348; "Table of quadratic residues" by A. A. Bennett, 349–356; "Anharmonic groups" by A. S. Hathaway, 357–362; "Some general determinant theorems in tensor notation" by C. M. Cramlet, 373–380; "Some conditions under which a continuum is a continuous curve" by H. M. Gehman, 381–384; "Successive derivatives of a function of several functions" by L. S. Dederick, 385–394; "Plane cubic curves in the Galois fields of order 2^n" by A. D. Campbell, 395–406; "On certain determinant relations" by W. H. Metzler, 407–420; "On the decomposition of tensors" by J. W. Alexander, 421–423; "Flow in a Möbius strip" by B. O. Koopman, 424–426; "A class of reciprocal functions" by E. Hille, 427–464; "Some extensions of the generalized Kronecker symbol" by J. J. Nassau, 465–470; "On the indeterminate equation $t^2-p^2Du^2=1$" by D. H. Lehmer, 471–476; "The operations of Boolean algebras" by O. Frink, 477–490; "A proof of Petersen's theorem" by O. Frink, Jr., 491–493; "Geometry of a set of n functions" by L. Ingold, 494–510; "Irregular fields" by E. T. Bell; "On the approximate solution of the integro-differential equations of mathematical physics" by N. Kryloff, 537–540; "A new proof of Parseval's identity for trigonometric functions" by J. Tamarkin, 541–547; "Tensors whose components are absolute constants" by T. Y. Thomas, 548–550; "Some convergence proofs in the vector analysis of function space" by D. Jackson, 551–567; "Harmonic cubics" by T. R. Hollcroft, 568–576.

Bulletin of the American Mathematical Society, Volume 32, No. 5, September-October 1926: "The Borel theorem and its generalizations" by T. H. Hildebrandt, 423–474, "Some recent work in the calculus of variations" by A. Dresden, 475–521; "Note on the Mersenne number $2^{139}-1$" by D. H. Lehmer, 522; "On the serial relation in Boolean algebras" by B. A. Bernstein, 523–524; "Note on horospheres by J. Pierpont, 524–528; "Some theorems concerning measurable functions" by L. M. Graves, 529–532; "A general theory of representation of finite operations and relations" by B. A. Bernstein, 533–536; Isolated singular points of harmonic functions" by G. E. Raynor, 537–544 "A list of errors in tables of the Pell equation" by D. H. Lehmer, 545–550.

Mathematische Annalen, Volume 96, Nos. 3–4, October 1926: "Uber die Entwicklung einer analytischen Funktion nach Polynomen" by J. L. Walsh, 430–436, "Über die Entwicklung einer Funktion einer komplexen Veränderlichen nach Polynomen" by J. L. Walsh, 437–450.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3244. Proposed by V. M. Spunar, Chicago, Illinois.

Divide a triangle by two straight lines into three parts, which when properly arranged shall form a parallelogram whose angles are of given magnitudes.

3245. Proposed by A. A. Bennett, Lehigh University.

Let $r=f(x+y+z)$, $s=f(x-y-z)$, $t=f(-x+y-z)$, $u=f(-x-y+z)$. Obtain in symmetrical form with as little computation as possible the algebraic identity in r, s, t, u, for each of the three cases, $f(x) \equiv \sin x$, $\cos x$, $\tan x$.

3246. Proposed by Daniel Kreth, Wellman, Iowa.

From a point within a square, lines drawn to three of the vertices are 30, 40, and 50. Required the side of the square. A geometrical solution is desired.

3247. Proposed by Nathan Altshiller-Court, University of Oklahoma.

When can two conics be projected into two orthogonal circles by a real projection?

3248. Proposed by F. A. Foraker, University of Pittsburgh.

Construct a triangle, given the sum of two sides, the third side, and the bisector of the angle opposite the third side.

3249. Proposed by Otto Dunkel, Washington University.

If $i!f_i = d^if(x)/dx^i$ and

$$D=\begin{vmatrix} f_r & f_{r+1} & \cdots & f_{r+n-1} \\ f_{r+1} & f_{r+2} & \cdots & f_{r+n} \\ \vdots & \vdots & & \vdots \\ f_{r+n-1} & f_{r+n} & \cdots & f_{r+2n-2} \end{vmatrix}$$

prove that

$$\frac{dD}{dx} = (r + 2n - 1)D'$$

where D' is the determinant D with the subscripts of the last row increased by unity.

3250. Proposed by Norman Anning, University of Michigan.

Give, in a form suitable for high school students, a proof of the theorem that when the perimeter of a triangle is given the area is greatest when the triangle is equilateral.

SOLUTIONS

3175 [3171; 1926, 105]. Proposed by Philip Fitch, Denver, Colorado.

A right circular cylinder with a diameter d is composed of wood and metal. The wooden part is l cm long and has a density s. The metal part is l' cm long and has a density s'. If it is allowed to float in still water, what angle will the axis of the cylinder make with the surface of the water?

Solution by Horace S. Uhler, Yale University.

Case 1. The plane of the water surface intersects the lateral surface of the compound cylinder only, so that the lower metal base is entirely submerged while the upper wooden base is entirely above water.

In any case two conditions must be fulfilled in order to maintain equilibrium: (a) the algebraic sum of the vertical forces must equal zero, and (b) the algebraic sum of the moments of these forces about any horizontal axis must vanish.

(a) Total force vertically downward $=\pi a^2(ls+l's')g$, where a equals $d/2$. Let λ denote the length of the submerged portion of the geometrical axis of the cylinder. Then the total force vertically upward $=\pi a^2\lambda\rho g$, where ρ symbolizes the density of the water (or of any suitable liquid). For equilibrium of translation:

(1) $$\lambda = l\sigma + l'\sigma' \text{ where } \sigma \equiv s/\rho \text{ and } \sigma' \equiv s'/\rho,$$

that is σ and σ' are specific gravities.

(b) Take moments with respect to the horizontal diameter of the lower base of the metal cylinder. Let θ denote the angle required in the problem. The weight of wood equals $\pi a^2 lsg$, with the lever arm $(\frac{1}{2}l+l')\cos\theta$, so that the moment is $\pi a^2 lsg(\frac{1}{2}l+l')\cos\theta$. Similarly the moment of the weight of the metal is $\pi a^2 l's'g(\frac{1}{2}l')\cos\theta$. Total moment tending to decrease θ equals

$$\pi a^2 g(\tfrac{1}{2}l^2 s + ll's + \tfrac{1}{2}l'^2 s')\cos\theta.$$

The displaced water may be conveniently analyzed into the volume of a right circular cylinder of altitude $\lambda - a\cot\theta$ plus the volume of the remaining cylindrical ungula truncated by the plane of the water surface.

Weight of water in right cylinder is $\pi a^2(\lambda - a\cot\theta)\rho g$, with lever arm $\frac{1}{2}(\lambda - a\cot\theta)\cos\theta$, hence the corresponding moment tending to increase θ equals $\frac{1}{2}\pi a^2\rho g(\lambda - a\cot\theta)^2\cos\theta$.

It will be shown below, as a lemma, that the coördinates of the center of mass of the ungula are $\bar{x}=5a/4$ and $\bar{y}=5a\cot\theta/8$. The origin of coördinates is here taken as the point common to the circular and elliptic sections of the ungula. $\bar{x}$ lies along a diameter of the circular section and $\bar{y}$ is parallel to the generatrices of the lateral surface of the truncated cylinder. The orthogonal projection on the surface of the water of the vector sum $\bar{x}+\bar{y}$ equals $\bar{x}\sin\theta+\bar{y}\cos\theta$ so that the lever arm for the weight of water in the ungula is $\lambda\cos\theta+\bar{x}\sin\theta+\bar{y}\cos\theta-a\csc\theta$ or $\lambda\cos\theta+(a/8)(5\sin\theta-3\csc\theta)$. The volume of the ungula, being half that of a right cylinder of altitude $2a\cot\theta$, equals $\pi a^3\cot\theta$. The corresponding moment tending to increase θ becomes $\pi a^3\rho g[\lambda\cos\theta+(a/8)(5\sin\theta-3\csc\theta)]\cot\theta$. Combining the preceding results conformably to the condition for rotational equilibrium gives:

(2) $$a^2\cot^2\theta = 4(l^2\sigma + 2ll'\sigma + l'^2\sigma') - 2a^2 - 4\lambda^2.$$

Substituting λ from equation (1) in (2), and reducing, the following final formula is obtained:

(3) $$\tan\theta = a\left\{2[2l^2\sigma(1-\sigma) - 2l'(2l\sigma + l'\sigma')(\sigma'-1) - a^2]\right\}^{-1/2}$$

Case 2. The plane of the water surface intersects one, or both, of the ends of the given compound cylinder. The two conditions for equilibrium can be expressed by equations without excessive labor, but it does not seem possible to eliminate λ and to obtain a single equation in θ. The difficulty arises from the occurrence (explicitly or implicitly) in the equations of the angle(s) subtended at the geometric axis of the cylinder by the water line(s), or horizontal chord(s), across the partially submerged base(s) as well as the sine and cosine of the same angle(s).

The results for the case where the lower metal base is wholly submerged while the water line across the upper wooden base subtends an angle 2α at the center of the base will be given without proof. Let $c\equiv\cos\alpha$, $e\equiv\sin\alpha$, $t\equiv\tan\theta$ then the equations corresponding to (1) and (2) above are respectively:

(1′) $$3\pi[l(1-\sigma) - l'(\sigma'-1)]t - a[3c(\pi-\alpha) + e(2+c^2)] = 0,$$

(2′) $$\begin{aligned}&24\pi a[l(1-\sigma) - l'(\sigma'-1)]t^3 - 6\{a^2(\pi + 4\pi c + 3\alpha - 3ce - 2ce^3)\\ &\quad + 2\pi[l'(2l+l')(\sigma'-1) - l^2(1-\sigma)]\}t^2 + 24\pi a(1-c)[l(1-\sigma) - l'(\sigma'-1)]t\\ &\quad - a^2[3(\pi-\alpha)(1+8c-4c^2) + e(16-3c+8c^2-6c^3)] = 0.\end{aligned}$$

The expression for t, as given by equation (1′), may be substituted in equation (2′) thus producing a single equation in α, $\sin\alpha$, and $\cos\alpha$. When appropriate numerical values are given for the other symbols the resulting equation can be solved approximately for α. Then θ may be found most conveniently by putting the value of α in equation (1′).

When the water surface intersects both bases of the cylinder the equation for rotational equilibrium becomes more complicated than (2′).

Illustration. The following numerical example brings out the salient features of the given problem very clearly. Let l' vary while keeping $\sigma=0.4$, $\sigma'=8.5$, $l=3.31$ cm, and $a=\sqrt{(1.1928)}$ cm.

The upper base of the cylinder will be "awash" when

$$l' = l(1-\sigma)/(\sigma'-1) \text{ or } l' = 2.648 \text{ cm.}$$

As l' decreases from this value the axis of the cylinder will remain vertical and the denominator of equation (3) imaginary, since the equation does not apply, until l' attains the value which makes this denominator vanish:

$$l' = \frac{(2l^2\sigma(\sigma'-\sigma)-\sigma'a^2)^{1/2} - l\sigma(2\sigma'-2)^{1/2}}{\sigma'(2\sigma'-2)^{1/2}} \text{ or } l' = 1 \text{ cm., exactly.} \tag{3}$$

When vertical, the height of the upper base above the water level is given by $z=l(1-\sigma)-l'(\sigma'-1)$ and this equals 12.36 cm for $l'=1$ cm.

Decreasing l' farther causes formula (3) to be valid and θ to decrease until the water surface just touches the lowest point of the upper base. Then

$$l' = \frac{1-\sigma}{\sigma'-1}l - \frac{a}{(\sigma'-1)t}. \tag{4}$$

Eliminating l' between equations (3) and (4) gives

$$2[2l^2(1-\sigma)(\sigma'-\sigma)+a^2(\sigma'-1)]t^2 - 8al(\sigma'-\sigma)t + a^2(5\sigma'-1) = 0,$$

or $21316.5504t^2 - t2144.88\sqrt{(1.1928)}+49.5012=0$; whence $t_1 \doteq 0.08134573030$, $\theta_1 \doteq 4°39'\ 1.9''$, $l_1' \doteq 0.857857$ cm. The root $t_2 \doteq 0.02854723429$, $\theta_2 \doteq 1°38'\ 6.7''$, leads to the inadmissible *negative* value of $l_2' \doteq -2.453037$ cm.

The submerged length of the uppermost generatrix of the cylinder is now $z_1 = l+l_1'-2a/t_1$ or $z_1 \doteq 7.10571$ cm.

As l' continues to decrease equation (3) no longer holds and the domain of equations (1′) and (2′) has been entered. Eventually the lower base of the cylinder will have risen until its highest point lies in the plane of the water surface. Then $t=a(1+\cos\alpha)/(l+l')$ and the boundary of the domain has been reached. After this both bases of the cylinder will be partly above the water surface until finally, when $l'=0$, the level condition $\theta=0$ will be attained.

Lemma (*vide supra*). As element of volume $d\tau$ take a plane lamina normal to the axis of x, and let the angle subtended at the center of the circular base by the intersection of the plane of the lamina with this base be symbolized by 2ξ;

$$d\tau = (dx)(x\cot\theta)(2a\sin\xi), \quad x = a + a\cos\xi.$$

Hence
$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,d\tau}{\int d\tau} = \frac{2a}{\pi}\int_0^\pi (1+\cos\xi)^2\sin^2\xi\,d\xi = \frac{5}{4}a.$$

To obtain $\bar{y}$ take a plane lamina parallel to the circular base as element of volume $d\tau'$.

$$d\tau' = (dy)a^2(\xi - \sin\xi\cos\xi), \quad y = a(1+\cos\xi)\cot\theta.$$

Hence
$$\bar{y} = \frac{\int y\,d\tau'}{\int d\tau'} = \frac{a}{\pi}\cot\theta\int_0^\pi (1+\cos\xi)(\xi-\sin\xi\cos\xi)\sin\xi\,d\xi$$
$$= (5a\cot\theta)/8.$$

3177 [3173; 1926, 159]. Proposed by Samuel Beatty, University of Toronto.

If X is a positive irrational number and Y its reciprocal, prove that the sequences

$$(1+X)\,, \qquad 2(1+X)\,, \qquad 3(1+X)\,, \cdots$$
$$(1+Y)\,, \qquad 2(1+Y)\,, \qquad 3(1+Y)\,, \cdots$$

contain one and only one number between each pair of consecutive positive integers.

I. Solution Jointly by Dr. A. Ostrowski, Göttingen, and Dr. J. Hyslop, Glasgow.

No member of either sequence is integral. Beneath any positive integer N, lie $[N/(1+X)]$ members of the first sequence and $[N/(1+Y)]$ members of the second sequence. Now

$$(1) \qquad \frac{N}{1+X}+\frac{N}{1+Y}=\frac{N}{1+X}+\frac{XN}{1+X}=N.$$

Since X is irrational, $N/(1+X)$ and $N/(1+Y)$ contain non-zero fractional parts, whose sum, being by (1) integral, must be unity.

Hence $[N/(1+X)]+[N/(1+Y)]=N-1$, and this is the total number of members of the sequences beneath N. Similarly, there are altogether N members of the sequences less than $N+1$; and so exactly one member lies between N and $N+1$.

II. Solution by A. C. Aitken, University of Edinburgh.

We shall first prove a similar theorem concerning rational numbers.

Theorem. *Let $x=p/q$, $y=q/p$, where p and q are positive integers prime to each other, and $p<q$. Then the sequences*

$$(1) \qquad \begin{aligned} &a_1, a_2, \cdots, a_{q-1}, \qquad a_i = i(1+x),\\ &b_1, b_2, \cdots, b_{p-1}, \qquad b_j = j(1+y), \end{aligned}$$

are such that the corresponding sequences $I(a_i)$, $I(b_j)$ are the integers $1, 2, \cdots, p+q-2$ in some order without repetitions or omissions, where $I(c)$ means the largest integer in c.

Proof. The omitted integers in the sequence $I(a_i)$ may be determined as follows: $a_m=m+mpq^{-1}=m+k+rq^{-1}$, where k and r are positive integers such that $mp=kq+r$, $r<q$. Then $a_{m-1}=m-1+k+(r-p)q^{-1}$. Hence the integer $m+k-1$ is omitted, if and only if $r<p$. But if this omission occurs, $I(b_k)$ is precisely the omitted integer, for $b_k=k+kqp^{-1}=k+m-rp^{-1}$. Form by division the sequence of equations,

$$(2) \qquad \begin{aligned} m_1p &= q+r_1,\\ m_2p &= 2q+r_2,\\ &\;\;\vdots \\ m_kp &= kq+r_k,\\ &\;\;\cdots \end{aligned} \qquad \begin{aligned} 1 \leqq r_k < p < q,\\ k \leqq p-1. \end{aligned}$$

It is clear that $m_1 \geqq 2$ and that $m_k \geqq m_{k-1}+1$. Hence the omitted integers are $m_1, m_2+1, m_3+2, \cdots, m_{p-1}+p-2$, which is precisely the sequence $I(b_j)$. We shall now use this result to prove the theorem of the problem.

If X is an irrational number, $0<X<1$, $Y=1/X$, we can approximate to X by an ordinary continued fraction for which p/q is a convergent. Then $X-pq^{-1}=\epsilon_1=\theta_1q^{-2}$, $Y-qp^{-1}=\epsilon_2=\theta_2p^{-2}$, where $|\theta_i|<1$. By the theorem just proved the integral parts of the terms of the sequences

$$i(1+X+\epsilon_1), \qquad 1 \leqq i \leqq q-1,$$
$$(1+Y+\epsilon_2), \qquad 1 \leqq j \leqq p-1,$$

are the integers $1, 2, \cdots, p+q-2$ in some order. Since $|i\epsilon_1|<q^{-1}$, $|j\epsilon_2|<p^{-1}$, by increasing the order of the convergent p/q we can increase p and q indefinitely. These sequences approach, therefore, the sequences of the problem and the desired result is thus established.

Also solved by L. R. FORD, HARRY LANGMAN, and F. L. WILMER.

3178 [3174; 1926, 159]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Construct a triangle given an angle, the difference of the including sides, and the altitude to the third side.

SOLUTION BY THEODORE BENNETT, University of Illinois.

Let the given angle be $2\theta=\angle QOP$, the altitude h, and the difference of the including sides $d=OQ-OP$. Let ϕ be the angle between h and the bisector of the angle QOP. Then $h=OQ\cos(\phi+\theta)$ $=OP\cos(\phi-\theta)$. Hence

$$d = h[\sec(\phi+\theta) - \sec(\phi-\theta)] = 2h\sin\theta\sin\phi/(\cos^2\theta-\sin^2\phi).$$

Setting $z=d\sin\phi$, we have: $z^2+2h\sin\theta\, z-d^2\cos^2\theta=0$. This equation has a positive root z_1 and a negative root $-z_2$, where $z_1z_2=d^2\cos^2\theta$, $z_2-z_1=2h\sin\theta$.

The construction is as follows. Bisect the given angle QOP and lay off on OP the lengths $OH=h$, $HD=d$. Drop the perpendiculars HF and DG on the bisector, draw the circle with center H and radius HF, and the line GH cutting the circle in C_1 and C_2, in the order GC_1HC_2. Then $z_1=GC_1=d\sin\phi_1$. It is clear from the construction that $GC_1<d$. Construct on the bisector the angle ϕ_1 so that $\sin\phi_1=GC_1/d$, the other side of the angle being in the direction of OP. On this side of the angle ϕ_1 lay off $OK=h$. The perpendicular to OK at K will cut the sides of the angle QOP in the required points P and Q.

The root z_2 does not furnish an angle ϕ_2 unless $2h\leqq d\sin\theta$. If this condition is satisfied ϕ_2 may be constructed, but it does not furnish a solution of this problem.

If P and Q denote the other two angles of the triangle $\phi_1=(P-Q)/2$, and hence the above formulae furnish means for solving for the angles P and Q and then for the sides OP and OQ.

II. SOLUTION BY THE PROPOSER.

Let ABC be the required triangle, so that angle $A=A$, $AC-AB=d$, $AH=h$, where A, d, and h are given. Let P, Q be the ends of the circumdiameter of ABC passing through the midpoint M of BC, the points P, A lying on the same side of BC, and let R, S be the feet of the perpendiculars from P upon AB, AC. We have

$$(1) \qquad AR + AS = (BR - AB) + (AC - CS) = (AC - AB) + (BR - CS).$$

Now P is equidistant from B, C; moreover AP is the external bisector of the angle A. Hence, from the two pairs of congruent triangles APR, APS and PBR, PCS, we have $AR=AS$, and $BR=CS$, and (1) gives

$$AR = AS = (AC - AB)/2 = d/2.$$

Thus in the quadrilateral $ASPR$ we know the angle $SAR=180°-A$, the two sides AS, AR, and the angles R, S are right angles; hence this quadrilateral may be constructed.

The point P lying on the circumcircle of ABC, the line RS is the Simson line of P with respect to ABC; hence RS passes through the foot M of the perpendicular PM from P upon BC. Let W, T be the traces of AP on BC and RS respectively. From the similar triangles WMP, WHA we have

$$(2) \qquad PM : AH = PW : AW = PW : (PW - PA)$$

and from the triangle PMW we have

$$(3) \qquad PM^2 = PT \cdot PW.$$

Eliminating PW from these two relations and noticing that $PA \cdot PT = PS^2$, we have

$$PM^2 - AH \cdot PM - PS^2 = 0, \tag{4}$$

where AH is given, and PS is known from the quadrilateral $ASPR$ already constructed.

To construct PM from (4), draw a circle on $XY = AH = h$ as diameter and on the tangent at X lay off $XZ = PS$. The line joining Z to the center of the circle will meet the circle in two points U, V, and ZU, ZV are the two sought for values of PM (the shorter segment ZU is to be taken negatively).

Now with P as center and ZV as radius draw a circle meeting RS in M. The perpendicular to PM at M will meet the sides AR, AS of the quadrilateral $ASPR$ in the two other vertices B, C of the required triangle ABC.

The point M must lie outside of the segment RS; hence only the longer segment ZV of the two segments ZU, ZV may be used as radius. The second point of intersection M' of the circle with RS will yield another solution of the problem congruent to the first and symmetric to it with respect to the line AP.

Remarks. I. Instead of PW we could have eliminated PM from (2) and (3), and obtained an equation in PW. We would thus determine the point W, and the base BC is then the tangent drawn frôm W to the circle having A as center and h as radius.

II. A trigonometric solution of the proposed problem appeared in Mathesis for June, 1926 (p. 286).

III. The above method of solution may also be followed in the case when instead of the difference, d, the sum s of AB and AC is given. The rôle of the external bisector of A is then played by the internal bisector of this angle. Thus we can solve the problem: $b+c$, A, h.

IV. The problem $b+c$, A, h was solved in this MONTHLY several years ago (1917, 329–330). The well known French mathematician Emile Lemoine solved this problem in 1885 in the "Journal de Mathématiques Élémentaires" (de Longchamps), p. 199. The problem was again discussed in "Mathesis" (1905, pp. 79–80), where several solutions are given.

NOTE BY THE EDITORS. In the above solution $2PM = d \cot\theta \cos\theta \csc\phi$ and $2PS = d \cot\theta$, where the angles θ and ϕ are those used in the first solution.

Also solved by DANIEL KRETH, HARRY LANGMAN, W. J. PATTERSON, and A. PELLETIER.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Title pages for this volume (34) and the preceding volume (33), to be used in binding, are printed as the first pages of this issue of the Monthly.

The Third Carus Mathematical Monograph on "Mathematical Statistics," by Professor H. L. Rietz, is promised by the printer for April 10, and distribution will begin immediately thereafter.

Owing to unexpected and unavoidable delays in the final preparation of the plates for the Rhind Mathematical Papyrus, it has become necessary to postpone the date of publication by at least two months. The first volume will soon be off the press, but the second volume, containing the plates, will probably not be ready before June. It is thought best to withhold the distribution until both volumes can be sent out together.

Associate Professor ALAN D. CAMPBELL of the University of Arkansas has been appointed associate professor of mathematics in Syracuse University.

Associate Professor CLIFFORD B. UPTON of Teachers College, Columbia University, has been promoted to a full professorship of mathematics.

The following courses in mathematics are announced for the summer of 1927.

University of Chicago, first term, June 20 to July 27; second term, July 27 to September 2. In addition to the usual courses in college algebra, plane analytical geometry and differential and integral calculus, the following advanced courses are announced. By Professor G. A. BLISS: Calculus of variations; Thesis work in analysis. By Professor H. E. SLAUGHT: Differential equations; Theory of definite integrals. By Professor E. T. BELL: Theory of functions of a complex variable; Theory of modular systems; Reading and research in the theory of numbers. By Professor H. H. MITCHELL: Introduction to higher algebra; Analytic theory of numbers; Reading and research in algebra and the theory of numbers. By Professor W. C. GRAUSTEIN: Analytic projective geometry; Metric differential geometry; Reading and research in differential geometry; By Professor MAYME I. LOGSDON: Differential calculus; Algebraic geometry of hyper-spaces; Reading and research in algebraic geometry. By Professor WARREN WEAVER: Electro-dynamics; Integral calculus; Reading and research in electro-dynamics. By Dr. WALTER BARTKY: Celestial mechanics; Descriptive astronomy.

Columbia University, July 11 to August 19. In addition to courses in trigonometry, solid geometry, college algebra, analytic geometry, and calculus, and a series of courses for teachers of secondary mathematics, the following advanced courses are offered. By Professor E. R. HEDRICK: Theory of functions of a real variable; Fundamental concepts of mathematics. By Professor W. B. FITE: Higher algebra. By Professor G. A. PFEIFFER: Projective geometry. By Professor K. W. LAMSON: Differential equations.

University of Illinois, June 20 to August 13. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered. By Professor J. B. SHAW: Synoptic course in higher mathematics. By Professor R. D. CARMICHAEL: Relativity. By Professor ARNOLD EMCH: Geometric transformations; Mathematics of statistics. By Dr. M. G. CARMAN: Advanced calculus. By Dr. F. E. JOHNSTON: Theory of equations.

University of Iowa, first term, June 13 to July 22. In addition to courses in algebra, solid geometry, trigonometry, analytic geometry, and calculus, the following courses are offered. By Dr. M. A. NORDGAARD: Subject matter and

teaching of mathematics. By Dr. L. E. WARD: Ordinary differential equations; Differential geometry. By Professor C. C. WYLIE: Astronomy; Mathematics of finance; Celestial mechanics. By Professor ROSCOE WOODS: Projective geometry. By Professor J. F. REILLY: Algebra for high school teachers. Second term, July 25 to August 26. By Dr. M. A. NORDGAARD: History of mathematics. By Dr. N. B. CONKWRIGHT: Differential equations. By Professor E. W. CHITTENDEN: Matrices and determinants; Introduction to the analysis of continua.

Johns Hopkins University, June 28 to August 5. In addition to elementary courses the following graduate course will be given. By Professor J. R. MUSSELMAN: Projective Geometry.

University of Michigan, June 27 to August 19. In addition to courses in algebra, plane and solid geometry, trigonometry, analytic geometry, elementary calculus, mathematical statistics, and the mathematical theory of interest and insurance, the following advanced courses are offered. By Professor W. B. FORD: Advanced calculus; Infinite series with special reference to Fourier series. By Professor L. C. KARPINSKI: Teaching of geometry; History of mathematics. By Professor PETER FIELD: Vector analysis. By Professor T. R. RUNNING: Empirical formulas. By Professor T. H. HILDEBRANDT: Theory of functions of a complex variable. By Professor H. C. CARVER: Theory of probability; Advanced mathematical theory of statistics. By Professor L. A. HOPKINS: Elements of mechanics. By Professor V. C. POOR: Differential equations. By Professor C. J. COE: Analytic mechanics; Integral equations. By Professor NORMAN ANNING: Solid analytic geometry; Finite differences. By Professor J. A. NYSWANDER: Higher algebra. By Professor G. Y. RAINICH: Quadratic forms and quadratic numbers. By Professor R. L. WILDER: Differential equations; Analysis situs. By Mr. S. E. FIELD: Projective geometry.

University of Minnesota, first term, June 17 to July 29; second term, July 30 to September 3. In addition to the usual elementary work, the following courses will be offered. First term: By Professor DUNHAM JACKSON: History of ancient and modern mathematics. By Professor W. L. HART: Differential equations. By Professor HART and Professor JACKSON: Reading in advanced mathematics. Second term. By Professor R. W. BRINK: Selected topics in advanced mathematics.

Ohio State University, first term, June 18 to July 23; second term July 25 to August 31. In addition to courses in algebra, analytic geometry, calculus, and the teaching of secondary mathematics, the following advanced courses are announced. By Professor H. BLUMBERG: Introduction to the theory of functions of a complex variable; Problems in analysis; Reading and research

in analysis. By Professor H. W. KUHN: Analytic projective geometry; Finite groups; Reading and research in group theory.

University of Pennsylvania, July 5 to August 13. In addition to the usual courses in solid geometry, trigonometry, college algebra, analytic geometry, and calculus, the following courses are offered. By Professor M. J. BABB: Theory of numbers. By Professor J. R. KLINE: Theory of functions of a real variable.

Stanford University. By Professor Manning: Modern algebra; Groups of finite order. By Dr. HOTELLING: Analysis situs; Probability and statistics. By Dr. BRINKMANN: Differential equations; Theory of functions of a real variable. By Miss WEISS: Theory of functions of a complex variable.

University of Texas, first term, June 8 to July 20. By Dean H. Y. BENEDICT: Advanced calculus. By Professor E. L. DODD: Functions of real variables Probability. By Professor R. L. MOORE: Theory of sets; Foundations of geometry. By Professor H. J. ETTLINGER: Mathematical physics. By Professor H. S. VANDIVER: Calculus. By Professor C. D. RICE: Advanced calculus. By Professor P. M. BATCHELDER: Theory of equations. By Professor MARY DECHERD: Calculus. Second term, July 20 to August 21. By Professor A. C. LUNN (Chicago): Relativity; Crystallography. By Professor H. J. ETTLINGER: Calculus of variations; Mathematical physics. By Professor H. S. VANDIVER: Finite groups; Definite integrals. By Professor GOLDIE HORTON: Advanced calculus. By Mr. GORDON WHYBURN: Calculus. By Mr. W. M. WHYBURN: Calculus; All freshman courses, both terms.

University of Wisconsin, General session, June 25 to August 5. By Professor L. W. DOWLING: Projective geometry; Analytic geometry. By Professor J. H. TAYLOR: Differential equations; Theory of equations. By Professor R. W. BABCOCK: Analytical mechanics; Vector analysis. By Dr. J. S. GEORGES: Teaching of high school and junior high school mathematics; Teaching and supervision of arithmetic. Special nine weeks course for graduates, June 25 to August 26. By Professor E. B. SKINNER: Linear algebras; Theory of finite groups.

Oberlin College, June 16 to August 4. By Professor W. D. CAIRNS: Teachers' course in mathematics; Mathematics of finance, especially for teachers. June 22 to August 10. By Professor M. E. SINCLAIR: Introduction to calculus; College algebra (for teachers or engineering students).

Bucknell University, July 5 to August 12. By Professor J. S. GOLD: Integral calculus; Advanced algebra; College geometry. By Professor H. S. EVERETT: Analytic Mechanics; Theory of numbers; Modern higher algebra.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.
BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 6-7, 1927.
Twelfth Annual Meeting, Nashville, Tenn., December, 1927.
The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.
INDIANA, De Pauw University, April 29-30.
IOWA, University of Iowa, May 6-7.
KANSAS, Topeka, Kan., February 5.
KENTUCKY, Lexington, May 7.
LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, April.
MINNESOTA, St. Peter, Minn., May 21.
MISSOURI, St. Louis, Mo., November 25-26.
NEBRASKA, Lincoln, May 14.
OHIO, Columbus, Ohio, April 8.
PHILADELPHIA, Philadelphia, Pa., November.
ROCKY MOUNTAIN, Colorado College, April 22-23.
SOUTHEASTERN, March.
SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.
TEXAS, Not yet determined.

AFFILIATED ORGANIZATION: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.
Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 4, APRIL

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN.

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE FALL MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fourth regular meeting of the Southern California Section of the Mathematical Association of America was held at Pomona College, Claremont, California, on Saturday, November 6, 1926, Professor H. C. Willett presiding.

There were forty-seven present, including the following thirty-two members of the Association:

O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, H. Bateman, G. E. Berry, W. N. Birchby, E. T. Bell, F. P. Brackett, P. H. Daus, H. H. Gaver, H. E. Glazier, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, M. N. Keith, G. R. Livingston, W. E. Mason, C. B. Millikan, B. Podolsky, W. P. Russell, G. E. F. Sherwood, H. M. Showman, M. Skarstedt, D. V. Steed, H. C. Van Buskirk, L. E. Wear, M. G. Whiting, H. C. Willett, Clyde Wolfe, E. R. Worthington.

Professor Theodor von Karman, head of the aerodynamical laboratories at Aachen, Germany, who is visiting this country under the auspices of the "Daniel Guggenheim Fund for Promotion of Aeronautics", was the guest for the day.

The following papers were presented:

1. "On the upper limit of the roots of an equation" by Professor GLENN JAMES, University of California at Los Angeles.

2. "A method of representing the aerodynamical characteristics of staggered biplanes", by Professor H. BATEMAN, California Institute of Technology.

3. "The arithmetic of logic", by Professor E. T. BELL, California Institute of Technology.

4. "A fundamental enumerative problem in hyperspace geometry" by Professor D. V. STEED, University of Southern California.

5. "Theory of earth pressure" by Professor THEODOR VON KARMAN, University of Aachen. (By invitation).

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. Professor James presented a method for determining an upper limit for the roots of an equation, which depended upon the evaluation of a formula by means of successive approximations.

2. In connection with an experiment in a wind tunnel to find the effect of staggering the planes of a biplane, the following method of representing and compounding forces was used. A force was represented by a pair of points, whose distance apart was proportional to the magnitude of the force, and the points were images of each other with respect to the line of force. Two forces

were compounded by means of certain circles related to the points, the actual mechanical composition being facilitated by the use of two pairs of doubled dividers.

3. Professor Bell discussed the arithmetic of logic, distinguishing it from the algebra of logic by the exclusion of the operation corresponding to algebraic division. He indicated how the theory of classes was abstractly the same as the theory of integers, and how a complete set of postulates for the latter might be formulated.

4. Professor Steed explained a fundamental enumerative problem of hyperspace and explained how as the dimension of the space increased, the difficulty of solving the problem was partly overcome by solving a related enumerative problem.

5. The theory of earth pressure, in contrast to the theory of water pressure, presents several mathematical difficulties, since there is friction between the particles in the former case. The solution of the problem depends upon the solution of a second order partial differential equation, but the difficulty is increased by the fact that the solution is sought in the neighborhood of the singularities. Professor von Karman explained how this was done and how the gleit lines or characteristics must be modified in the neighborhood of the retaining wall.

P. H. DAUS, *Secretary*

ORGANIZATION MEETING OF THE PHILADELPHIA SECTION

On Saturday, November 27, 1926, several members of the Mathematical Association living in the eastern part of Pennsylvania, southern New Jersey and Delaware, met for the purpose of engaging in a mathematical meeting and of organizing a new section of the Association. The meeting was held at Lehigh University, Bethlehem, Pa. Many members who had planned to be present expressed regret at their inability to attend. There were twenty persons present, including the following members of the Association: A. A. Bennett, R. L. Charles, V. H. Doushkess, J. A. Foberg, M. S. Knebelman, P. A. Knedler, K. W. Lamson, W. F. Long, H. H. Mitchell, J. B. Reynolds, L. L. Smail, W. M. Smith, F. M. Weida, A. H. Wilson.

After the close of the morning session, luncheon was served at the University cafeteria; the wives of the local members were present at this time. Immediately at the close of the luncheon, a brief business meeting took place, resulting in the following action. Those present unanimously agreed to request permission of the trustees of the Mathematical Association for the organization of a new section to be known as the Philadelphia Section. It is intended to serve

members living in the eastern part of Pennsylvania, southern New Jersey, and Delaware, the meetings usually to take place in Philadelphia. Professor H. H. MITCHELL was elected chairman for the coming year, and Professor A. A. BENNETT vice-chairman and secretary-treasurer. It was voted to defer official adoption of by-laws until the next meeting, which is scheduled for November 26, 1927. After a vote of appreciation for the courtesies extended by Lehigh University, the session adjourned.

The following papers were presented:

1. "The evolutes of a certain type of symmetrical plane curves," by Professor J. B. REYNOLDS, Lehigh University.

2. "The analogue for ideals of the Lagrange-Gauss theory of quadratic forms," by Professor H. H. MITCHELL, University of Pennsylvania.

3. "A new treatment of exponentials and logarithms on the basis of a modified Dedekind theory of irrationals," by Professor L. L. SMAIL, Lehigh University.

4. "The derivation and solution of certain ordinary differential equations," by Professor W. M. SMITH, Lafayette College.

5. "The state course of study in mathematics," by J. A. FOBERG, State director of mathematics and science.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. The author demonstrated nine theorems concerning a curve and its evolutes when the curve is an analytic symmetrical plane curve having continuous evolutes, one infinite branch and no point singularities. The parabola and its successive evolutes are a case of the type of curves discussed.

2. Professor Mitchell discussed the determination of the number of classes of ideals in the quadratic field by methods that follow closely those used in the theory of binary quadratic forms. He pointed out that in this case these methods lead to more satisfactory results than those based on the Minkowski theorems concerning linear forms.

3. In the published literature on the theory of irrational numbers on either the Dedekind or the Cantor definition, the four rational operations are discussed by most writers on the basis of the original definition of irrational numbers. Before taking up the higher operations of exponentials and logarithms, however, they all introduce the theory of limits with more or less of its definitions, terminology and rules of manipulation, and then base the treatment of the rational and the irrational exponents and logarithms on this foundation. From several points of view it would seem desirable to avoid this break in the discussion of the rational and irrational operations caused by interpolating the theory of limits at this point. The object of the paper by Professor Smail is to show how, by use of a modified form of the Dedekind definition of irrationals,

it is possible to treat the rational and irrational exponents and logarithms without explicit use of the theory of limits, using only the original definition of irrational numbers.

4. This expository paper discussed the derivation and solution of the Riccati equation; of the equation of forced, damped vibrations; and of a certain linear equation with discontinuous solutions arising from a kind of maintained vibration in which the force of restitution or "spring" of the body, susceptible of vibration, is subject to an imposed periodic variation.

5. The attention consistently given by the Mathematical Association to the interests of mathematical instruction in the secondary school makes it appropriate that discussion of mathematics instruction in the public schools should form part of the program of this initial meeting of the Philadelphia Section. The state program of studies in Pennsylvania contemplates a continuous twelve-year program in such major subjects as mathematics, science, English, social studies, art. Mathematics is a required study through the first nine school years—thereafter it is elective. Opportunity is afforded in the junior high school for presenting such topics in the whole field of elementary mathematics as can be justified on the basis of their values in training for citizenship. Obviously, the design of the course of study involves much experiment and study on the part of teacher and supervisor. Such special objectives as meeting college requirements are taken care of in the senior high school. A number of colleges and universities in Pennsylvania now admit applicants upon a showing of twelve units of work done in the three-year senior high school. It is hoped and expected that this plan will become general in the near future.

A. A. BENNETT, *Secretary*

THE FIFTEENTH MEETING OF THE IOWA SECTION

The fifteenth regular meeting of the Iowa Section of the Mathematical Association of America was held at Coe College, Cedar Rapids, Iowa, on April 30 and May 1, 1926, in affiliation with the annual meeting of the Iowa Academy of Science.

Among those attending were the following twenty-nine members of the Association: E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, I. S. Condit, Marian E. Daniells, R. D. Daugherty, C. W. Emmons, Annie W. Fleming, C. Gouwens, D. L. Holl, Dora E. Kearney, F. M. McGaw, J. V. McKelvey, C. A. Messick, E. E. Moots, E. A. Pattengill, J. F. Reilly, F. Reusser, H. L. Rietz, E. R. Smith, G. W. Snedecor, J. Theobald, J. S. Turner, L. E. Ward, C. W. Wester, W. H. Wilson, R. Woods, C. C. Wylie, J. L. Yothers.

The chairman of the section, Professor I. S. Condit, presided at both the Friday afternoon and the Saturday morning sessions. Dinner was enjoyed together Friday evening at the Y. M. C. A. building, at which time Professor Condit gave his retiring address on the subject "Progress of the Perry Movement". At the business meeting the following were elected officers for 1926–1927: Chairman, J. V. McKelvey, Iowa State College; Secretary-treasurer, J. F. Reilly, University of Iowa.

The next regular meeting will be held at Iowa City, May 6, 7, 1927.

The following papers were presented:

1. "Expansions of a function in the neighborhood of branch points" by Professor C. Gouwens, Iowa State College.

2. "A preliminary report on freshman mathematics for social science students" by Professor C. W. Emmons, Simpson College.

3. "A general interpolation formula" by Professor J. F. Reilly, University of Iowa.

4. "On a certain group of two-squares" (by title) by Professor W. J. Rusk, Grinnell College.

5. "On certain three-squares that are also two-squares" (by title) by Professor Rusk.

6. "Note on the concepts of correlation and of stochastic connection" by Professor H. L. Rietz, University of Iowa.

7. "Some relativistic reflections" (by invitation) by Professor E. S. Allen, Iowa State College.

8. "Progress of the Perry movement" by Professor I. S. Condit, Iowa State Teachers College.

9. "The torsion problem and its analog in curvilinear coordinates" by Dr. D. L. Holl, Iowa State College.

10. "Note on problem solving in elementary mathematics" by Professor C. W. Wester, Iowa State Teachers College.

11. "Note on ciphering matches" by Professor Wester.

12. "Fundamental relations for sin z and cos z defined by infinite products" by Professor E. R. Smith, Iowa State College.

13. "Certain objectives in the teaching of calculus" by Professor W. H. Wilson, University of Iowa.

14. "Solutions of a certain functional equation" by Professor Wilson.

15. "The sunspot cycle in the flow of rivers" by Professor C. C. Wylie, University of Iowa.

16. "The solar surface during the past five years" (by title) by Mr. D. E. Hadden, Alta, Iowa.

17. "Geometric interpretation of median and quartiles" (by title) by Professor E. E. Watson, Iowa State Teachers College.

18. "A slide rule for statisticians" by Professor G. W. SNEDECOR, Iowa State College.

19. "An anabasis to signed quantities" by Professor R. D. DAUGHERTY, Iowa State Teachers College.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. In his paper Professor Gouwens gave an exposition and discussion of the Newton polygon method for the expansion of an algebraic function.

2. The report gives the purpose and the plan of a new course in mathematics which Professor Emmons proposes to offer at Simpson College. The topics included are selected from the traditional courses in college mathematics with a view to their utility for students of the social sciences.

3. In his paper Professor Reilly developed a third order difference interpolation formula containing two parameters. By assigning special values to the parameters a number of formulas already known were obtained, in particular Sprague's osculatory formula. A method for determining the parameters so as to produce the best interpolated value at any point in a limited range was given by the use of least squares.

4. In his first paper Professor Rusk, assuming that the square of each of two numbers can be expressed as a three square, obtains in a very elementary fashion a set of forty-eight functions of their elements that are two squares. From these he is able to give explicit form to the product of any two numbers as a four-square.

5. In his second paper Professor Rusk assumes that the two numbers are the same and is able to get a set of three-squares that are two-squares, in particular the theorem that if $u^2=x^2+y^2+z^2$, then $(x\pm y)^2+(y\pm z)^2+(z\pm x)^2$ is a two-square for any combination of the signs.

6. The paper by Professor Rietz discusses Tschuprow's conception of stochastic connection[1] in comparison with various definitions of correlation given in the literature. It is shown that certain definitions make the concepts of correlation and of stochastic connection equivalent, but that with some of the usual definitions, as pointed out by Tschuprow, there may exist stochastic connection when there is no correlation.

7. Several authors have seen in the doctrine of relativity evidence for philosophical theories, in particular for idealism and for freedom in nature. In the opinion of Professor Allen these interpretations of the implications of relativity seem unjustified; at most, there must be a revision of the definition of causation and of the formulation of idealism.

---

[1] For a definition of stochastic connection, see A. A. Tschuprow's *Grundbegriffe und Grundprobleme der Korrelationstheorie* (1925); p. 20.

8. In his address Professor Condit discussed the progress of the Perry movement first as to the gains that have been made during the last quarter century, and second as to objectives. These may be enumerated as follows:

Gains of the quarter century 1900–1925 (a) Motivation of secondary mathematics, (b) Development of scientific spirit in determining standards and measuring results of teaching, (c) Efforts to unify mathematical subjects, and mathematics and science, (d) Development of general mathematics for the junior high school, (e) Efforts to coordinate secondary and college mathematics, (f) Efforts to determine purpose and essential subject matter of college mathematics, (g) More harmonious relations between secondary and college teachers, (h) Determining standards of teacher preparation, (i) Rapid growth of associations of teachers of mathematics and science.

Objectives (a) A greater willingness on the part of teachers to investigate and experiment with all constructive criticism of present practice in teaching mathematics, (b) Preparation of teachers of mathematics who can teach general mathematics efficiently, or who can apply the results of scientific investigation to stratified courses, (c) Adequate training of teachers of mathematics, (d) Better adjustment of college entrance requirements, (e) Determination of essential subject matter in freshman mathematics.

9. The conjugate cylindrical coordinates characterized by $x+iy=F(\alpha+i\beta)$ offer an extension to the two-dimensional solution for the torsion problem of a cylindrical rod. Dr. Holl considers a more general case with circular cross sections, but with meridian curves $r=f(z)$, where the coordinate system $z+ir=F(\alpha+i\beta)$ is adaptable. In the former case the deflection and slope of a soap film suspended from a plane boundary whose contour is that of a cross section of the rod, offer an experimental determination of the direction and magnitude of the stress in the rod.

10. In his first paper Professor Wester gave a preliminary report of an experiment in the use of the equation in the lower grades.

11. The purpose of mathematical contests was stated by Professor Wester to be (a) to arouse interest in mathematics, (b) to furnish an added incentive for the mastery of the more difficult processes, (c) to discover the minds capable of developing a high efficiency and to give them a training that will develop them to their utmost, (d) to arouse school spirit and to create a tradition of achievement in mathematical work.

Attention also was called to the need for standard rules for conducting such contests and for judging results, to the desirability of stage managing them to make them interesting and even exciting to onlookers, and to the need on the part of coaches to study and teach the most efficient computation methods.

12. Professor Smith showed that trigonometry for complex values of z may be logically developed directly from the definition of $\sin z$ in terms of an infinite

product without the aid of series or other infinite expressions. In particular the addition formulas may be developed, and the usual relations for the functions found in trigonometry or calculus may be obtained.

15. By comparing the figures for the run-off of various rivers with the Wolf sunspot numbers, and by forming the mean curves for one cycle, smoothed by overlapping means, Professor Wylie finds that, in general, for rivers west of the Mississippi there is a positive correlation, that is, the most water flows down the river in years when sun spots are numerous. For those investigated in the East a negative coefficient was obtained. The highest positive coefficient was obtained in Iowa, the coefficient for the Iowa river being $+0.97$; and the highest negative coefficients were obtained for New England rivers, Lake Cochituate yielding -0.93, and the Merrimac river -0.90.

16. In his paper Mr. Hadden gave a review of sunspot observations made at Alta, Iowa, from 1921 to 1925.

18. In describing the slide rule that he had invented Professor Snedecor pointed out that it facilitated the calculation of many of the functions employed in statistics. In addition to the usual scales, it contains the Kelley scales of the logarithms of the reciprocals of $(1-r)^{\frac{1}{2}}$ and $(1-r^2)$, together with a scale beginning with log .6745.

19. The introduction of the complex plane near the beginning of algebra was advocated by Professor Daugherty. By the operations of extension and rotation the processes that signed quantities are usually subjected to could be performed.

J. F. REILLY, *Secretary*

TENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The tenth annual meeting of the Kentucky Section of the Association was held May 8, 1926 at Berea College, Berea, Kentucky. Professor T. A. Martin presided at both sessions. A twelve o'clock dinner was served by Berea College at Boone Tavern. At this dinner Dr. W. J. Hutchins, president of Berea College, welcomed the members and their friends to Berea. He related entertaining stories and told interesting facts about the college.

At the business meeting which followed the program, a committee was authorized to consider the advisability of affiliating with the Kentucky Academy of Science; and the section expressed its appreciation to Berea College for the hospitable reception and delightful dinner given to the attending members. Dean P. P. BOYD of the University of Kentucky was made chairman and Professor A. R. FEHN of Centre College was made secretary-treasurer for the year 1926–27. It was voted to have the next meeting at the University of Ken-

tucky. There were twenty-seven present at the meeting, including the following members of the Association: M. C. Brown, J. M. Davis, H. H. Downing, A. R. Fehn, J. M. Guilliams, M. R. Hutcherson, Flora E. LeStourgeon, T. A. Martin, J. N. Peck, R. I. Pepper, E. L. Rees, D. C. South, D. O. Streyffeler.

Papers were presented as follows:

1. "Some remarks on functional calculus" by Dr. FLORA E. LESTOURGEON, University of Kentucky.

2. "Line complexes associated with the motion of a rigid body" by Professor E. L. REES, University of Kentucky.

3. "Unique and non-unique factorization" by Professor W. R. HUTCHERSON, Berea College.

4. "Kreisverwandtschaft" by H. W. MOBLEY, Graduate student, University of Kentucky (by invitation).

5. "Groups of linear transformations in the ternary field" by Miss VEDA L. NELSON, Graduate student, University of Kentucky (by invitation).

6. "Improving scholarship in mathematics in secondary schools" by Principal F. A. SCOTT, Paris High School, Paris, Ky.

7. "Some points on teaching secondary mathematics" by Professor J. M. GUILLIAMS, Berea Normal School.

8. Discussion of the last two papers.

Abstracts of most of these papers are given below, the numbers corresponding to those in the list of titles.

1. This paper discusses functionals having continuity of order one, and derives for such functionals a first and second necessary condition for a minimum analogous to the Euler and Jacobi conditions of the calculus of variations.

2. This was an outline of the kinematical theory of line complexes. It was shown by simple geometric reasoning that orthogonal conjugate lines with respect to the linear complex consisting of lines normal to the trajectories of their points belong to the tetrahedral complex of tangents and that they are the vertex tangents of focal parabolas enveloped by lines of the tetrahedral complex and the characteristics of the planes of these parabolas.

3. Definitions of field, algebraic integer, unit of field were given. In field $K(-3)$ it was shown that factorization was unique, using the factoring of 21 as an example:

$$21=(\sqrt{-3})\ (-\sqrt{-3})\ (2+1\sqrt{-3})\ (2-1\sqrt{-3}),$$

where all of the four expressions are prime and factors of 21. Then in field $K(-5)$ it was shown that factorization was non-unique. Thus,

$$21=3.7=(1+2\sqrt{-5})\ \ (1-2\sqrt{-5}).$$

In this field both 3 and 7 are prime, as well as the other factors.

6. The purpose of the writer was to note criticisms of the teaching of mathematics in the secondary schools and then to offer suggestions which

might remedy the situation. (1) Teachers are not properly prepared. (2) There is a lack of thoroughness in teaching. Teachers should not be in too much of a hurry to cover a certain portion of the book. Suggestions for vitalizing their teaching can be had by reference to the report of the National Committee on the Reorganization of Mathematics in Secondary Schools. (3) Pupils are passed without knowing the subject. Not true in general, but, where true, a rearrangement of courses would correct the evil. Two years is not sufficient to lay a good foundation for collegiate mathematics. (4) New courses and extra-curricular activities crowd out the subject. This should appeal as a challenge to the teachers; they must really teach. Extra-curricular activities offer one of the best opportunities to the teacher to promote interest in mathematics, through the use of mathematics clubs. Finally every teacher should join the Kentucky Section, should exchange ideas with other teachers, should subscribe either to the *Mathematics Teacher* or to *School Science and Mathematics*.

7. An outline of Professor Guilliams' paper covers the following points: (1) Words do not convey ideas. (2) They are merely the symbols of ideas; the ideas must be presented objectively. (3) Sir William Hamilton's definition: "Thinking is nothing other than affirming one idea of another idea." (4) Max Müller's dictum that thought worthy of the name is impossible without language. In teaching secondary mathematics, stress should be placed in the expression in good English of every problem and demonstration. (5) Neatness and orderly arrangement of written work, with a clear statement of all the necessary data. (6) Dr. Glenn Frank's statement: "In the deepest sense, we cannot teach anybody anything; the best we can do is to help them learn for themselves." (7) The lecture method was condemned for mathematics teaching in secondary schools. Knowledge is an achievement by the learner for the learner.

He blamed the colleges for the poor work done in mathematics in the average high school of the state, stating that in some colleges of the state the only requirement for entrance was the statement that the applicant had a year's credit in algebra and a year's credit in plane geometry at Possum Trot High School, that no mathematics is required for the degree of A.B., but that when he graduates, the student can go out and teach mathematics in the high schools of the state. He urged that every mathematics teacher in the state high schools should have studied and have a good knowledge of algebra, plane and solid geometry, trigonometry, analytic geometry, and calculus, and should have had as well a careful course in the teaching of high school mathematics.

A. R. FEHN, *Secretary*

THE THIRD ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION OF THE ASSOCIATION

The third annual meeting of the Louisiana-Mississippi Section of the Mathematical Association was held at New Orleans by invitation of Newcomb College and Tulane University Friday and Saturday, March 12–13, 1926. The meeting formally opened with a dinner at six o'clock on Friday in the Newcomb Tea Room. The dinner was a splendid success and was well attended by the members and the friends of the section. President Dinwiddie of Tulane University extended a cordial welcome to all present.

Following the dinner, Professor Florian Cajori, formerly of Tulane University and now of the University of California, gave a most profitable and interesting lecture on "The history of the principles of the conservation of energy." This lecture was given before a joint session of the Section and the New Orleans Academy of Science. Besides giving this lecture and paper no. 6 which is listed below, Professor Cajori addressed the faculty and students of Louisiana State University on March 11 upon "The growth of algebraic notations," and the Faculty and students of Tulane University and Newcomb College on March 12 upon "Women students prominent in mathematical fields."

The regular program was held on Saturday morning, the chairman, Professor H. E. Buchanan, being in charge. There was a large attendance at the sessions, including the following members of the Association:

H. E. Buchanan, Julia Dall, Winifred D. Daly, W. L. Duren, Jr., Dora M. Forno, Bertha Frankenbush, J. A. Hardin, Anna M. Howe, C. G. Killen, Odessa R. Lastrapes, C. G. Latimer, A. C. Maddox, J. R. Maguire, J. E. McShane, B. E. Mitchell, I. C. Nichols, G. B. Price, S. T. Sanders, C. D. Smith, Mary C. Spencer, Norma E. Touchstone, B. A. Tucker, W. P. Webber. Resolutions were adopted expressing the sincere thanks of the members to Professor Cajori, the New Orleans Academy of Science, Louisiana State University, Tulane University, Newcomb College, and to other organizations, as well as to those in charge of the dinner. Officers elected for the year were S. T. SANDERS, Louisiana State University, Chairman; ROBERT TORREY, University of Mississippi, Vice-Chairman; B. E. MITCHELL, Millsaps College, Secretary-Treasurer. Shreveport was selected as the place for the next annual meeting.

The following papers were presented:

1. "Some series resembling the theta functions of Jacobi" by H. E. BUCHANAN, Tulane University.

2. "A fifth degree curve with certain of its related curves" by J. A. HARDIN, Centenary College.

3. "Some curves associated with the linear transformation" by B. E. MITCHELL, Millsaps College.

4. "An application of Fourier series to a problem in electricity" by E. J. McSHANE, Tulane University.

5. "Ceva's theorem and its application in solving certain examples given in Wentworth-Smith plane geometry" by I. C. NICHOLS, Louisiana State University.

6. "Origins of fourth dimension concepts" by FLORIAN CAJORI, University of California.

7. "Certain extensions of a theorem by Markoff" by C. D. SMITH, Louisiana College.

8. "On certain indefinite quarternary forms representing all integers" by C. G. LATIMER, Tulane University.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. In this paper Professor Buchanan discussed the functions defined by

$$Q_0(x, k) = \sum_{n=-\infty}^{\infty} (-1)^{n-1} e^{-k\rho}, \quad \text{where} \quad \rho = \left(\frac{2n-1}{2} - \frac{ix}{k}\right)^4.$$

Their periodicity is established and an infinitude of zeros found.

2. Professor Hardin reported on an investigation of a one parameter curve of the fifth degree, whose equation is

$$x^5 - 2x^4 + 5ax^3y - 5axy^3 + 2y^4 = 0.$$

Certain of the related curves were investigated, including the Hessian and polar curves. Particular attention was given to the relation between the curve and its Hessian, singular points of each curve, and the relation of each curve to the line at infinity.

3. The orthogonal quadrilateral arising from the dissection of the linear transformation is used as a nexus between the transformation and lines, circles conics, and higher plane curves. Since this transformation is frequently referred to as "circle transformation" Mr. Mitchell considered in particular those lines (constituting a pencil) which preserve their linear characteristic under the transformation. Again, all those circles (constituting a pencil) which lose their circle identity under the transformation. All these configurations and the others considered bear necessarily an intimate relation to the orthogonal quadrilateral which is the link between them and the linear transformation.

4. Mr. McShane discussed a certain problem in electricity wherein the apparent disagreement of theory and practice was explained.

5. The discussions by Professor Nichols were the outgrowth of his teaching of a course in Modern Geometry based upon the text by Altshiller-Court. His aim was to enlist further interest in this important study. Ceva's Theorem was stated and proved, and then its converse stated and used in proving that

(1) the medians of a triangle are concurrent, (2) the internal bisectors of a triangle are concurrent, and (3) the altitudes of a triangle are concurrent.

6. Professor Cajori began with Aristotle and traced very carefully the concept of the 4th dimension through the successive periods of scientific thought and investigation to our present time, the cravings for generalization, the efforts at occultation, the influence of the ether theory, the time concept in theoretical mechanics and in electromagnetic phenomena, and the space-time concept.

7. In treating this subject, Mr. Smith stated as a theorem of A. A. Markoff that $P \leqq 1/t^2$ where P is the probability that $U > M_1 t^2$, M_1 being the mean value of U and t an arbitrary number. By a method similar to Markoff's proof the theorem was put in the form, $P \leqq M_1/\lambda$, where P is the probability that $U > \lambda$. A geometrical interpretation of the meaning of the theorem was given and the theorem was extended to various forms.

8. In this paper it was shown that if α is a positive integer $\leqq 163$ in the form $4n+3$, or the double of such an integer, the form $x^2+y^2-\alpha z^2-\alpha w^2$ represents all integers.

I. C. NICHOLS, *Secretary*

EXTENSIONS OF WARING'S THEOREM ON NINE CUBES

By L. E. DICKSON, University of Chicago

1. Introduction. In 1770, Waring conjectured[1] that every positive integer p is the sum of the cubes of nine integers $\geqq 0$. It will be shown in §2 that three of these cubes may be taken to be equal when $p<40{,}000$. Again, six of the cubes may be taken equal in pairs if $p<1400$ (end of §5). These and all possible similar theorems are corollaries to

THEOREM I. *Every positive integer p can be represented by $x^3+y^3+2z^3+2u^3+3v^3$ with $x \geqq 0, \cdots, v \geqq 0$.*

This is verified in §5 for $p \leqq 1200$. It was verified by direct trial for $1200 < p \leqq 1400$.

2. THEOREM II. *Every positive integer $p<40{,}000$ can be expressed in the form $S+3v^3$, where S is a sum of six cubes $\geqq 0$, and $v \geqq 0$.*

At the request of Jacobi,[2] the famous computer Dahse constructed a table showing the least number of positive cubes whose sum is any $p<12000$. There are exactly 138 values of p:

(T) $\qquad 7, 14, 15, 21, 22, 23, 42, \cdots, 5818, 8042$

[1] The first proof was given by Wieferich in 1909. For a report on the literature, see the writer's *History of the Theory of Numbers*, vol. 2, pp. 717–725.

[2] Journal für Mathematik, vol. 42 (1851), p. 41; Werke, vol. 6, p. 323.

which are not sums of six cubes $\geqq 0$. If p is a number in this table T such that $p-3$ is not in T, then $p-3$ is a sum S of six cubes, and $p=S+3$, as desired. The p's for which $p-3$ is in T are

$$50,\ 106,\ 114,\ 178,\ 295,\ 303,\ 340,\ 393,\ 1239.$$

We subtract $24=3\cdot 2^3$ from each. No difference is in T. This completes the proof[1] that every $p\leqq 12000$ can be expressed in the form $S+3v^3$ (in fact, with $v=0$, 1, or 2). By von Sterneck's[2] table showing the number of cubes needed for the representation of all numbers $p\leqq 40{,}000$, six cubes suffice when $p>8042$.

3. **All forms of weight 9.** The cubic form

$$(1) \qquad (a_1, \cdots, a_n) = a_1x_1^3 + \cdots + a_nx_n^3 \qquad (a_i>0,\ x_i\geqq 0)$$

shall be said to be of *order* n and *weight* $a_1+a_2+\cdots+a_n$. Since ax^3 is a sum of a cubes, a form of weight w is a sum of w cubes. Since 23 is not a sum of eight cubes, 9 is the minimum weight of a form (1) which represents all positive integers. Only when a form f is of weight 9 will expressibility of p by f imply expressibility of p as a sum of nine cubes. Hence if we seek theorems which imply Waring's theorem as a corollary, we must restrict attention to forms (1) of weight 9.

Without loss of generality, we may take $a_1\leqq a_2\leqq a_3 \cdots$ in (1). Let the resulting form f represent all positive integers. Then $f=1$ implies $a_1=1$; $f=2$ implies $a_2=1$ or 2. If $a_2=1$, $f=3$ implies $a_3<4$. If $a_2=2$, $f=4$ implies $a_3<5$. Let f be of weight 9. Then $n>3$. If $n=4$, all forms are excluded:

$$(1116) \neq 4,\quad (1125) \neq 12,\quad (1134) \neq 10,\quad (1224) \neq 11,\quad (1233) \neq 12.$$

If $n=5$, (11223) is the only form not excluded:

$$(11115) \neq 12,\quad (11124) \neq 46,\quad (11133) \neq 18,\quad (12222) \neq 11.$$

If $n=6$, the only forms are the two in Theorem III and

$$(111114) \neq 46,\quad 46 = 5\cdot 8 + 6 = 27 + 2\cdot 8 + 3.$$

THEOREM III. *If a form of weight* 9 *represents all positive integers, it is one of the following seven*:

(11223), (111222), (111123), (1111122), (1111113), (11111112), (111111111).

4. **Partition process.** If p is represented by (1) and if $a_1=r+s$, the first term may be replaced by $rx_1^3+sx_1^3$ to show that p is represented by $(r,s,a_2, \cdots, a_n)$. The converse usually fails since the cubes having the coefficients r and s may not be equal. Thus a proof of Theorem I would imply that every form in Theorem III represents all integers. Expressibility by any of the forms involving 3 implies expressibility by all subsequent forms. Expressibility by one not involving 3 implies expressibility by all subsequent ones not involving 3. By Theorem II, (11111112) represents all integers $<40{,}000$.

---

[1] It is much shortened by assuming Theorem I, which states that every $p\leqq 1400$ is expressible as $S+3v^3$.

[2] Akademie der Wissenschaften, Wien, *Sitzungsberichte*, vol. 112, IIa (1903) pp. 1627–1666.

5. **Numbers represented by** $q=(1122)$ **and** $q+kv^3$. We employ a list[1] of the 59 numbers <1200 which are sums of two cubes $\geqq 0$. The doubles of the first forty of them are <1200, and were added to each number of the first list when the sum is <1200. After deleting the sums from $1, \cdots, 1200$, we obtain the following list[2] L of all positive integers $\leqq 1200$ which are *not* represented by q:

7, 14, 15, 21–23, 36, 38, 42, 47, 49, 50, 52, 61, 73, 75, 77, 85, 87, 92, 94, 99–103, 106, 111–5, 122, 140, 148, 150, 159, 164, 166–7, 169, 174–8, 185–6, 188, 197, 201, 204, 211–5, 223, 225, 227, 229, 230–1, 237–9, 244, 262, 276, 281, 288–90, 292, 295, 300, 302–3, 309, 311, 318, 323, 326–7, 329, 335, 337, 340, 342, 349, 356, 363–6, 390, 393, 396, 401, 403, 412, 415–7, 419–20, 422, 427–8, 431, 438, 444, 446–7, 453–4, 458, 465, 489, 491–2, 505, 507, 510, 518–9, 534, 542, 548–9, 553, 556, 570, 579, 585, 604–6, 616, 618, 627, 631, 634–5, 644, 654, 660–2, 666, 668, 670–1, 676, 678–81, 692, 699, 700–1, 705, 708, 725–7, 735, 738, 757, 759, 771, 787, 790, 796, 801, 803, 806, 820, 833, 848, 850, 852–3, 861–2, 869, 876, 882–3, 885, 888, 894–7, 913–5, 920, 922, 927, 933–5, 943, 948, 950–1, 953, 958–9, 965, 969, 970, 972, 974, 976, 988, 1009, 1011, 1014, 1021, 1044, 1047, 1050, 1063, 1065, 1077, 1092, 1100–3, 1122, 1124, 1133, 1139–40, 1148, 1167, 1174–5, 1189, 1191, 1193, 1196.

We may now easily verify that $f=q+3v^3$ represents all positive integers $p\leqq 1200$ (Theorem I). If p is not in the preceding list L, we may take $v=0$. Evidently f with $v=1$ represents p if q represents $p-3$. Hence it remains to consider the 54 numbers p of L for which $p-3$ is in L. We list them in two sets:

p = 50, 52, 102, 103, 106, 114–5, 167, 169, 177–8, 204, 214–5, 230, 292, 295, 303, 329, 340, 393, 396, 415, 419, 422, 431, 447, 492, 510, 556, 634, 671, 679, 681, 708, 738, 790, 806, 853, 888, 897, 953, 1014, 1047, 1050, 1103, 1196,

for which $p-3\cdot 2^3$ is not in L and hence is represented by q, so that p is represented by f;

$$p=188,\ 326,\ 366,\ 420,\ 885,\ 951,\ 972,$$

for which $p-24$ is in L, but $p-3\cdot 3^3$ is not in L, whence p is represented by f. This proves Theorem I for $p\leqq 1200$ (and in fact with $v\leqq 3$).

THEOREM IV. (11224) *represents all integers* $\leqq 1200$.

The thirty numbers p in list L for which $p-4$ is in L are $p=42$, 77, 103, 106, 115, 178, 215, 227, 231, 292, 416, 419, 420, 431, 458, 553, 631, 635, 670, 680, 705, 969, 974, 976, 1193, for which $p-4\cdot 2^3$ is not in L, together with $p=201$, 229, 327, 666, 852, for which $p-4\cdot 3^3$ is not in L. Hence every integer $\leqq 1200$ is represented by $q+4v^3$ with $v\leqq 3$.

But for $k\neq 3$, $k\neq 4$, $f_k=q+kv^3$ fails to represent all small integers. If $k>7$, $f_k\neq 7$. By §§6, 7, $f_2\neq 23$, $f_1\neq 15$. Next, $f_5\neq 47$, $f_6\neq 21$, $f_7\neq 14$.

[1] Jacobi, Journal für Mathematik, vol. 42 (1851), p. 69; Werke, vol. 6, p. 354.

[2] Rechecked with complete agreement.

By §4, Theorem I implies that (111222) represents all positive integers $\leqq 1400$. This was verified up to 712 also by adding the numbers represented by (111) to those represented by 2(111).

6. Numbers represented by $g=(11122)$ **and** $g+kw^3$. We first find the numbers $\leqq 1200$ represented by $g=q+v^3$. Of the 73 numbers p in list L for which $p-1$ is in L, the following have also $p-2^3$ in L:

(ϵ) 22, 50, 100, 102, 114, 186, 212, 238, 364, 420, 428, 454;

(ω) 15, 23, 167, 175, 177, 231, 239, 289, 303;

$p=635$, 679, 700, 935, 951, 959, 1175 with $p-3^3$ not in L; and $p=662$ and 896 with $p-4^3$ not in L. For every p in (ϵ) and (ω), $d=p-v^3$ is in L for all v's for which d is positive. This proves[1]

THEOREM V. *$g=(11122)$ represents all positive integers $\leqq 1200$ except those in sets (ϵ) and (ω).*

By means of this result, we shall prove

THEOREM VI. *If $k=2$–6, 10–15, $(11122k)$ represents all positive integers $\leqq 1200$.*

First let k be odd and $\leqq 15$ (since the form evidently fails to represent 15 if $k>15$). For $w=1$, our form is $g+k$ and represents a chosen odd integer ω if the even integer $\omega-k$ is represented by g, and hence if $\omega-k$ is not one of the numbers ϵ. We find that when $\omega-\epsilon$ is positive, it exceeds 15 and hence is not equal to k except in the following two cases:

$$\omega = 23, \quad \epsilon = 22; \qquad \omega = 239, \quad \epsilon = 238 \ (\omega - \epsilon = 1).$$

The value 1 of k is excluded since no positive integer w gives $g+w^3=23$.

Similarly, $g+k$ represents a chosen even integer ϵ if the odd integer $\epsilon-k$ is represented by g, and hence if $\epsilon-k$ is not one of the numbers ω. But when $\epsilon-\omega$ is positive, it exceeds 15 except for

$$\epsilon = 22, \omega = 15; \quad \epsilon = 186, \omega = 175; \quad \epsilon = 186, \omega = 177; \quad \epsilon = 238, \omega = 231.$$

In the first and last cases, $\epsilon-\omega=7$ and $g+7w^3\neq 22$. In the second case, $g+11w^3=186$ for $w=2$. In the third case, $g+9w^3\neq 186$. The values 7 and 9 of k are therefore excluded.

Second, let k be even and $\leqq 14$. Then $g+k$ represents ϵ if the even integer $\epsilon-k$ is represented by g and hence is not an ϵ', whence $\epsilon-\epsilon'\neq k$. But the only differences $\leqq 14$ of two numbers (ϵ) are

$$102 - 100 = 2, \quad 114 - 102 = 12, \quad 114 - 100 = 14, \quad 428 - 420 = 8.$$

Similarly, $g+k$ represents ω if the odd integer $\omega-k$ is not an ω'. But the only differences $\leqq 14$ of two numbers (ω) are

$$23 - 15 = 8, \quad 175 - 167 = 8, \quad 177 - 167 = 10, \quad 177 - 175 = 2,$$
$$239 - 231 = 8, \quad 303 - 289 = 14.$$

[1] Verified up to 800 also by addition of the numbers represented by (111) and (22). Note that the numbers stop with 454.

The value $k=8$ is excluded since $g+8w^3\neq 23$. We see that none of

$$102 - 2\cdot 2^3 = 86,\ 114 - 12\cdot 2^3 = 18,\ 114 - 14\cdot 2^3 = 2,$$
$$177 - 10\cdot 2^3 = 97,\ 177 - 2\cdot 2^3 = 161,\ 303 - 14\cdot 2^3 = 191$$

are in (ϵ) or (ω), whence 102 is represented by $g+2w^3$, etc.

7. Numbers represented by $h=(11222)$ **and** $h+kw^3$. Consider $h=q+2v^3$. Of the numbers[1] p in the list L of §5 for which $p-2$ is in L, the following have also $p-2\cdot 2^3$ in L:

(ϵ) 38, 52, 166, 292, 342, 670;

(ω) 23, 77, 101, 103, 115, 213, 229, 231, 239, 365, 417;

$p=227$, 311, 678, 803, 885, 1191 with $p-2\cdot 3^3$ not in L;

$p=419$, 507, 950, 974 with $p-2\cdot 4^3$ not in L.

For every p in (ϵ) and (ω), $d=p-2v^3$ is in L for all v's for which d is positive. This proves[2]

THEOREM VII. *$h=(11222)$ represents all positive integers $\leqq 1200$ except those in sets (ϵ) and (ω).*

This is used to prove

THEOREM VIII. *If $k=1-13$, $17-23$, $(11222k)$ represents all positive integers $\leqq 1200$.*

The proof is like that of Theorem VI. We now have the following cases: $\omega = 365, \epsilon = 342, k = 23, \omega - 2^3k = 181; \epsilon = 38, \omega = 23, g + 15w^3 \neq 38; \epsilon = 52, \epsilon' = 38, \epsilon - \epsilon' = 14, g + 14w^3 \neq 52.$

The only positive differences $\leqq 23$ of two ω's are

$$103 - 101 = 2,\ 115 - 101 = 14,\ 115 - 103 = 12,\ 229 - 213 = 16,$$
$$231 - 229 = 2,\ 231 - 213 = 18,\ 239 - 231 = 8,\ 239 - 229 = 10.$$

But $g+16w^3\neq 229$. However, g represents

$$103 - 2\cdot 2^3 = 87,\ 231 - 2\cdot 2^3 = 215,\ 239 - 8\cdot 2^3 = 175,$$
$$239 - 10\cdot 2^3 = 159,\ 115 - 12\cdot 2^3 = 19,\ 231 - 18\cdot 2^3 = 87,$$

since 87, etc., are not in either (ϵ) or (ω).

8. Forms of order five. The forms $(1122k)$ were studied in §5. For $r=1$ or 3, the numbers represented by $(124r)$ were obtained from those represented by (124).

The form $f=(1234)$ represents all numbers $\leqq 800$ except 18, 22, 39, 60, 63, 74, 76, 77, 100, 103, 106–7, 117, 126, 178, 180, 201, 215, 228, 230, 245, 271, 289, 291, 295, 315, 341, 356–7, 393, 413, 419, 420, 480–1, 523, 559, 606, 611–2, 616, 671, 673, 705.

For $k=2$, 3, 4, 11, 13, 14, 16, 17, and $k>18$, the least integer not represented by $f+kv^3$ is 76, 63, 22, 74, 76, 74, 76, 39, 18, respectively. For the remaining k's, it suffices to use $v\leqq 2$ in view of the table for f. This proves

---

[1] Fifty have $p-16$ not in L.

[2] Verified up to 512 also by adding the numbers represented by (11) and (222).

THEOREM IX. *(1234k) represents all positive integers* $\leqq 800$ *if and only if* $k=1$, 5–10, 12, 15 *or* 18.

The form $l=(1124)$ represents all numbers $\leqq 600$ except 19, 23, 26, 38, 45, 46, 52, 53, 73, 77, 79, 90, 100–1, 103, 105, 115, 147, 166, 185, 194, 198, 202–3, 207–8, 213, 229, 231, 239, 242, 246, 268–9, 279, 292, 294, 305–6, 316, 331, 333, 335, 339, 342, 365, 368, 394, 396, 414, 416–7, 420–1, 426, 431, 446, 450, 457–8, 477, 481, 485, 487, 495, 521, 550, 583, 585–6.

For $k=1$, 4, 7, 8, 12, 13, 14, 15, 16, 17, 19, and $k>19$, the least integer not represented by $f+kv^3$ is 46, 23, 26, 46, 38, 90, 306, 38, 229, 90, 38, 19, respectively. For $k=3$, it suffices to use $v\leqq 4$. For the remaining k's, it suffices to use $v\leqq 2$. This proves

THEOREM X. *(1124k) represents all positive integers* $\leqq 600$ *if and only if* $k=2$, 3, 5, 6, 9, 10, 11, *or* 18.

Neither $(1133k)$ nor $(1244k)$ represents all integers $\leqq 57$ for any k. If $k\geqq 8$, $r=(1248k)$ and $s=(1249k)$ fail to represent some integer $\leqq 77$ with the following exceptions: $r\neq 194$ if $k=10$, $s\neq 315$ if $k=12$, $s\neq 206$ if $k=13$, $s\neq 350$ if $k=18$.

We readily verify that, whatever be its weight, *no form of order four represents all integers* $\leqq 19$.

9. All forms of weight ten. The following are excluded:

$$(111115) \neq 2\cdot 8+4,\ (11116)\neq 5,\ (11125)\neq 2\cdot 8+4,\ (11134)\neq 2\cdot 8+2,$$
$$(11233)\neq 39,\ (12223)\neq 39,\ 39=4\cdot 8+7=27+8+4.$$

By Theorem IV, $f=(11224)$ represents all integers $\leqq 1200$. Hence the same is true of the ten forms derived from it by partitioning one or more of 2, 2, 4 as in §4. The only further form of weight 10 and order >4 is $g=(111133)$. By means of Jacobi's table, we obtain a list of numbers which are sums of four cubes $\geqq 0$. To each such number was added the triples of sums of two cubes. In this maner it was verified that g represents all integers $\leqq 600$.

The form $h=(1111114)$ *represents all integers* $\leqq 40{,}000$. For, if p is in table T of §2, $p-4$ is in T only when $p=178$, and then $p-4\cdot 2^3=146$ is not in T. The same result holds for the forms derived from h by partitioning 4.

Of the above ten forms derived from f, let l be any one except h, (111124), and (112222). If we equate to zero one of the cubes having the coefficient unity in l, we obtain a form given in Theorem III. If the latter form represents all integers, the same is evidently true of l.

10. All forms of weight eleven. The following are excluded:

$$(11117)\neq 5,\ (11126)\neq 8+5,\ (11135)\neq 2\cdot 8+7,\ (11144)\neq 8+7,$$
$$(11225)\neq 47=5\cdot 8+7=27+2\cdot 8+4,\ (11333)\neq 13,\ (12224)\neq 8+5,$$
$$(12233)\neq 47,\ (111116)\neq 8+5,\ (111125)\neq 47,\ (122222)\neq 8+5,$$
$$(1111115)\neq 47.$$

There remain only $F=(11234)$ of Theorem IX and fifteen forms derived from F by partitioning one or more of 2, 3, 4. Let t be any one of the fifteen except (111134), (111233) and $G=(112223)$. If we equate to zero one of the cubes in t having the coefficient unity, we obtain g or f or one of the ten forms derived from f in §9. If we omit a 2 in G, we get the form in Theorem I.

11. Conclusion. We have found numerous cubic forms which represent all positive integers less than 40,000, 1400, 1200, 800, or 600. All of minimum weight 9 are derived from (11223) by partitioning 2, 3, 3. All of weight 9 or 10 are derived similarly from (11224), (11234), or (111133). If it could be proved that any one of these four actually represents all positive integers, the same would follow for all the forms derived from it. A proof would evidently be more difficult than the proofs of Waring's theorem on nine cubes, six of which are taken to be the cubes of $a \pm u$, $a \pm v$, $a \pm w$. In any case, the theorem would require verification for numbers below some limit of considerable size. Hence the facts established in this paper have an importance beyond making our empirical theorems seem probable.

Corresponding results for fourth powers will appear in the Bulletin of the American Mathematical Society. Also a proof that all sufficiently large integers are represented by the cubic form (1, 1, 1, 1, 1, 1, 1, 2) will appear there.

The Carnegie Institution has financed the employment of assistants to aid in the elaborate investigation of fifth and higher powers.

ELEMENTARY DERIVATION OF THE FUNDAMENTAL CONSTANTS IN THE POISSON AND LEXIS FREQUENCY DISTRIBUTIONS[1]

By HAROLD T. DAVIS, Indiana University

1. Introduction. Derivations of the arithmetic average and standard deviation for frequency distributions of Poisson and Lexis type are usually made by use of theorems on mean error. The proofs of these theorems cannot be considered as strictly elementary and the underlying significance of results obtained by their use is often masked by the apparent simplicity of the proof. This has seemed to the author to be particularly true with regard to the derivation of the fundamental constants in the frequency distributions already referred to where the proof is only a matter of two or three lines when the difficulties are thrown back upon the mean error theorems.[2]

---

[1] Contribution 26, Waterman Institute, Indiana University.

[2] For a comprehensive treatment of Bernoulli, Poisson and Lexis series see A. Fisher: The Mathematical Theory of Probabilities, N. Y., Macmillan (1922), pp. 117–126; consult also E. Czuber: Wahrscheinlichkeitsrechnung, Teubner, Leipzig, 4th ed. (1924), vol. 1, pp. 181–183; the Handbook of Mathe-

It is the object of the present paper to derive the fundamental constants of the Poisson and Lexis distributions from the definitions in much the same way as these constants are derived in standard texts for the Bernoulli distribution.

2. Poisson Distributions. In a Poisson distribution the probability of an event varies from trial to trial within a set of m trials, but the several probabilities for one set of m trials are identical with those of every other set.

For example, suppose that we have three urns, u_1, u_2, u_3, in which we have placed white and black balls in such ratios that the probabilities of drawing a white ball from the urn are 1/3, 1/2 and 2/3 respectively. If we draw a ball from each urn we have the following eight possibilities: *BBB; WBB, BWB, BBW; WWB, WBW, BWW; WWW*. The probabilities in these cases are respectively 1/9; 1/18, 1/9, 2/9; 1/18, 1/9, 2/9; 1/9. From these values we find that the probability that we shall draw no white ball is 1/9; one white ball, 7/18; two white balls, 7/18; and three white balls, 1/9. In 72 trials we should then expect in the ideal case the following frequencies:

No. of White Balls	0	1	2	3
Frequencies	8	28	28	8

In the example, the number of white balls drawn in 72 trials is 108 or $1\frac{1}{2}$ per trial; for the square of the standard deviation we have

$$\sigma_P^2 = \{(1\tfrac{1}{2}-0)^2\cdot 8 + (1\tfrac{1}{2}-1)^2\cdot 28 + (1\tfrac{1}{2}-2)^2\cdot 28 + (1\tfrac{1}{2}-3)^2\cdot 8\}/72 = 25/36.$$

In the general case we consider n urns, $u_1, u_2, \cdots, u_n$, which contain white and black balls in such ratios that the probabilities of drawing a white ball from the urns are $p_1, p_2, \cdots, p_n$ respectively. Now let us make N drawings of n balls each, i. e. one from each urn, and record the frequencies in the cases where we shall have no white ball, one white ball, $\cdots$, n white balls. We shall thus obtain a Poisson frequency table:

No. of White Balls	0	1	$2 \cdots n$
Frequencies	f_0	f_1	$f_2 \cdots f_n$

where $f_0+f_1+f_2+\cdots+f_n=N$.

The arithmetic average, A_P, for the series is defined as the most probable value of the arithmetic average of the number of white balls drawn per trial, so that

$$A_P = (f_0\cdot 0 + f_1\cdot 1 + f_2\cdot 2 + f_3\cdot 3 + \cdots + f_n\cdot n)/N.$$

We now prove the following theorem:

---

matical Statistics, Cambridge, Houghton Mifflin (1924), chap. VI by H. L. Rietz; C. V. L. Charlier, Vorlesungen über die Grundzüge der mathematischen Statistik (1920), p. 30.

THEOREM I. *The arithmetic average, A_P, of the Poisson series is equal to np, where $p=(p_1+p_2+\cdots+p_n)/n$.*

PROOF: It will be convenient to adopt the following abbreviations:

$$E_0=1\ ;\quad E_1=p_1+p_2+\cdots+p_n\ ;\quad E_2=p_1p_2+p_1p_3+\cdots\ :$$
$$E_3=p_1p_2p_3+p_1p_2p_4+\cdots\ ;\ \cdots\ ;E_n=p_1p_2p_3\cdots p_n.$$

We recall from algebra the following identity:

$$(x-p_1)(x-p_2)\cdots(x-p_n)\equiv E_0x^n-E_1x^{n-1}+E_2x^{n-2}-\cdots\pm E_n.$$

Since f_0 represents the number of drawings in which no white balls are obtained, we have

$$f_0=N[(1-p_1)(1-p_2)\cdots(1-p_n)]=N[E_0-E_1+E_2-\cdots\pm E_n].$$

Similarly, we have

$$\begin{aligned}f_1&=N[(1-p_1)(1-p_2)\cdots(1-p_{n-1})p_n\\&\qquad+(1-p_1)(1-p_2)\cdots(1-p_{n-2})(1-p_n)p_{n-1}+\cdots]\\&=N[E_1-2E_2+3E_3-\cdots\pm nE_n].\\f_2&=N[(1-p_1)(1-p_2)\cdots(1-p_{n-2})p_{n-1}p_n+\cdots].\end{aligned}$$

In order to express f_2 in terms of the E's, we observe that there are as many terms similar to the first one given as there are combinations of the p's taken two at a time, or ${}_nC_2$. Also, for any value of r up to $n-2$, each product of the form $(1-p_1)\ (1-p_2)\cdots(1-p_{n-2})$ has ${}_{n-2}C_r$ terms containing r letters each. There are, for example, ${}_{n-2}C_3$ terms of the form $p_1\,p_2\,p_3$. Hence, in f_2 there are altogether ${}_nC_2\cdot{}_{n-2}C_r$ terms of $r+2$ letters each. But E_{r+2}, which is the sum of products of $r+2$ letters each, contains ${}_nC_{r+2}$ terms, so that the coefficient of E_{r+2} in f_2 is equal to ${}_nC_2\cdot{}_{n-2}C_r/{}_nC_{r+2}=(r+2)!/(2!\,r!)$. It thus follows that we have

$$f_2=N\left[E_2-\frac{3!}{2!1!}E_3+\frac{4!}{2!2!}E_4-\frac{5!}{2!3!}E_5+\cdots\pm\frac{n!}{2!(n-2)!}E_n\right].$$

By a similar argument the other frequencies can be expressed in terms of the E's. Thus we get

$$f_3=N\left[E_3-\frac{4!}{3!1!}E_4+\frac{5!}{3!2!}E_5-\cdots\pm\frac{n!}{3!(n-3)!}E_n\right],$$

. . . .

$$f_n=NE_n\,.$$

From these values the arithmetic average is easily computed. Thus we have

$$\begin{aligned} A_P &= (f_0 \cdot 0 + f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + \cdots + f_n \cdot n)/N \\ &= E_1 + 2(1-1)E_2 + 3(1-1)^2E_3 + \cdots + n(1-1)^{n-1}E_n \\ &= p_1 + p_2 + \cdots + p_n = n(p_1 + p_2 + \cdots + p_n)/n = np. \end{aligned}$$

In a similar way we can calculate the standard deviation for the Poisson series.

THEOREM II. *The square of the standard deviation, σ_P, of a Poisson series is given by the formula:*

$$\sigma_P^2 = npq - \sum_{i=1}^{n} (p_i - p)^2, \textit{ where } p = (p_1 + p_2 + \cdots + p_n)/n \textit{ and } q = 1 - p.$$

PROOF: Since the arithmetic average is equal to $np = E_1$, we have from the definition of the standard deviation

$$\begin{aligned} \sigma_P^2 &= [f_0(E_1 - 0)^2 + f_1(E_1 - 1)^2 + \cdots + f_n(E_1 - n)^2]/N \\ &= [(f_0 + f_1 + \cdots + f_n)E_1^2 - 2E_1NA_P \\ &\qquad + (f_0 \cdot 0^2 + f_1 \cdot 1^2 + \cdots + f_n \cdot n^2)]/N \\ &= -E_1^2 + (f_0 \cdot 0^2 + f_1 \cdot 1^2 + \cdots + f_n \cdot n^2)/N. \end{aligned}$$

But we have $\quad (f_0 \cdot 0^2 + f_1 \cdot 1^2 + \cdots + f_n \cdot n^2)/N$

$$= E_1 - (2 - 4)E_2 + (3 - 3 \cdot 2^2 + 3^2)E_3 - \cdots.$$

It will now be seen that for $r > 2$, the coefficient of E_r is equal to

$$r \cdot 1^2 - \frac{r(r-1)}{2!}2^2 + \frac{r(r-1)(r-2)}{3!}3^2 - \cdots \pm \frac{r!}{r!}r^2 =$$

$$r\left[1 - (r-1) + \frac{(r-1)(r-2)}{2!} - \cdots \pm \frac{(r-1)!}{(r-1)!}\right]$$

$$- r(r-1)\left[1 - (r-2) + \frac{(r-2)(r-3)}{2!} - \cdots \mp \frac{(r-2)!}{(r-2)!}\right]$$

$$= r(1-1)^{r-1} - r(r-1)(1-1)^{r-2} = 0.$$

It thus appears that we have

$$\sigma_P^2 = -E_1^2 + E_1 + 2E_2 = np - \sum_{i=1}^{n} p_i^2 = npq - \sum_{i=1}^{n} (p_i - p)^2.$$

3. **Lexis Distributions.** A Lexis distribution is characterized by the fact that the probability attached to an event is constant from trial to trial within a set, but varies from set to set.

To illustrate this definition, suppose that we have three urns, u_1, u_2, and u_3, containing black and white balls in such ratios that the probabilities fo drawing a white ball are 1/3, 1/2 and 2/3 respectively. Suppose that twenty-four drawings of three balls each, with replacements each time, are made from each of the three urns and the number of white balls recorded. The following are the probable frequencies (to the nearest integer) for each urn since each set of 24 drawings forms a Bernoulli frequency:

No. of White Balls	0	1	2	3
u_1	7	11	5	1
u_2	3	9	9	3
u_3	1	5	11	7
Totals	11	25	25	11

The totals form a Lexis distribution. The arithmetic average, A_L, and the square of the standard deviation, $\sigma_L{}^2$, are 3/2 and 31/36 respectively.

In the general case we have r urns, $u_1, u_2, \cdots, u_r$ containing black and white balls in such ratios that the probabilities of drawing a white ball are $p_1, p_2, \cdots, p_r$ respectively. Then N drawings of n balls each, with replacements each time, are made from each of their urns and the number of white balls is recorded. The frequency distribution for each set of N drawings for any urn is clearly a Bernoulli distribution, but the frequency distribution obtained by adding the frequencies for all of the urns will form a Lexis distribution. We can represent this as follows:

No. of White Balls	0	1	2	...	n	A	σ^2
u_1	$f_0{}^{(1)}$	$f_1{}^{(1)}$	$f_2{}^{(1)}$	...	$f_n{}^{(1)}$	np_1	np_1q_1
u_2	$f_0{}^{(2)}$	$f_1{}^{(2)}$	$f_2{}^{(2)}$	...	$f_n{}^{(2)}$	np_2	np_2q_2
...	...	...	...	...	...	...	....
u_r	$f_0{}^{(r)}$	$f_1{}^{(r)}$	$f_2{}^{(r)}$	...	$f_n{}^{(r)}$	np_r	np_rq_r
Total (Lexis Distribution)	f_0	f_1	f_2	...	f_n		

where $f_0{}^{(i)}+f_1{}^{(i)}+f_2{}^{(i)}+\cdots+f_n{}^{(i)}=N$.

We now prove the following theorem:

THEOREM III. *The arithmetic average, A_L, of the Lexis distribution is equal to np, where $p=(p_1+p_2+\cdots+p_r)/r$.*

PROOF:

$$\begin{aligned} A_L &= (0\cdot f_0+1\cdot f_1+2\cdot f_2+\cdots+n\cdot f_n)/rN \\ &= (A_1+A_2+A_3+\cdots+A_r)/r \\ &= n(p_1+p_2+p_3+\cdots+p_r)/r = np. \end{aligned}$$

We now calculate the standard deviation for the Lexis series.

THEOREM IV. *The square of the standard deviation, σ_L, of a Lexis series, is given by the formula*:

$$\sigma_L^2 = npq + \frac{(n^2 - n)}{r}\sum_{i=1}^{r}(p_i - p)^2, \text{ where } p = (p + p_2 + \cdots + p_r)/r \text{ and } q = 1 - p.$$

PROOF: For the calculation of the square of the standard deviation, we consider the expression:

$$\begin{aligned}
\sigma_L^2 &= [f_0(np-0)^2 + f_1(np-1)^2 + \cdots + f_n(np-n)^2]/rN \\
&= [n^2p^2(f_0 + f_1 + \cdots + f_n)/rN] - [2np(0\cdot f_0 + 1\cdot f_1 + \cdots + n\cdot f_n)/rN] \\
&+ [(0^2 f_0 + 1^2 f_1 + 2^2 f_2 + \cdots + n^2 f_n)/rN] \\
&= -n^2p^2 + \sum_{i=1}^{r}(0^2 f_0^{(i)} + 1^2 f_1^{(i)} + 2^2 f_2^{(i)} + \cdots + n^2 f_n^{(i)})/rN.
\end{aligned}$$

From the fact that

$$\sum_{j=0}^{n} f_j^{(i)}(np_i - j)^2 = Nnp_iq_i,$$

we have, after squaring and collecting terms,

$$0^2 f_0^{(i)} + 1^2 f_1^{(i)} + 2^2 f_2^{(i)} + \cdots + n^2 f_n^{(i)} = Nnp_iq_i + Nn^2p_i^2.$$

Hence we get

$$\begin{aligned}
\sum_{i=1}^{r}[0^2 f_0^{(i)} + 1^2 f_1^{(i)} + \cdots + n^2 f_n^{(i)}]/rN &= \left(\sum_{i=1}^{r} np_iq_i + n^2\sum_{i=1}^{r}p_i^2\right)/r \\
= \left[n\sum_{i=1}^{r}p_i(1-p_i) + n^2\sum_{i=1}^{r}p_i^2\right]/r &= np + \frac{n^2-n}{r}\sum_{i=1}^{r}p_i^2.
\end{aligned}$$

Substituting this value above, we have the formula

$$\sigma_L^2 = -n^2p + np + \frac{n^2-n}{r}\sum_{i=1}^{r}p_i^2,$$

which is easily seen to be equivalent to the expression given in the theorem.

A PERIODIC SOLUTION FOR A CERTAIN PROBLEM IN MECHANICS[1]

By J. W. CAMPBELL, University of Alberta

Suppose a mass m resting on a smooth horizontal table, is connected by a light inextensible string, which passes through a small hole in the centre of the table, to a mass M hanging freely and suppose that m is given a blow in a di-

[1] Presented to the American Mathematical Society, Sept. 9, 1926.

rection at right angles to the string on the table. Then if the centrifugal force exerted by m is equal to the weight of M, m will revolve about the hole in a circle and M will remain in stationary equilibrium. If the blow imparted to m is less than this critical value then the centrifugal force exerted by m will be less than the weight of M and M will fall. But as M falls the distance from m to the hole will decrease, the centrifugal force exerted will increase, becoming equal to and then exceeding the weight of M. The downward motion of M will thus attain a maximum value and then decrease to zero, after which M will rise. The opposite effect would be obtained if the blow imparted to m were greater than the critical value. Therefore in general M will oscillate in a vertical line.[1]

The above physical phenomonon forms the basis of problems given in Lamb's *Dynamics*, page 148, and in Byerly's *Generalised Coordinates*, page 22. The problems as there stated are only to find the total range of the motion of M, and not to find a general solution of the motion, but a general periodic solution is here obtained when the total range of motion is not too great. The method of obtaining the solution is analogous to the methods used in the finding of periodic orbits[2] in mathematical astronomy, and the main purpose of the article is to illustrate the applicability of those methods to problems of physics and mechanics where periodic variation is involved. The solution also shows some peculiarities not common to astronomical problems.

Let us denote by r the distance of m from the hole, by x the distance of M below the position from which it starts, by θ the angular position of the string on the table, and by T the tension of the string at time t.

Then by the laws of motion we obtain the equations

$$m(\ddot{r} - r\dot{\theta}^2) + T = 0$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$
$$M\ddot{x} - Mg + T = 0.$$

Integrating the second of these equations and eliminating T between the first and third we obtain

$$r^2\dot{\theta} = c_1\,; \quad m(\ddot{r} - r\dot{\theta}^2) - M\ddot{x} + Mg = 0\,.$$

We now put $r = a - x$, and then, eliminating θ between these equations, we obtain

$$\ddot{x}(m + M) + mc_1^2/(a - x)^3 - Mg = 0\,.$$

[1] It is assumed that the length of the string and the initial conditions are such that neither m nor M reach the hole before the extreme positions of M are reached.

[2] F. R. Moulton and collaborators, *Periodic Orbits*, Carnegie Publication 161.

This equation may be written in the form

$$\ddot{x} + A/(a - x)^3 - g_1 = 0,$$

$$A = mc_1^2/(m + M), \quad g_1 = Mg/(m + M). \tag{1}$$

It follows from these equations that

$\ddot{x} = 0$ when $A/(a-x)^3 = g_1$, that is when $x = a - (A/g_1)^{1/3}$.

Then put

$$x = a - (A/g_1)^{1/3} + z, \tag{2}$$

and the equation of motion becomes

$$\ddot{z} + \frac{A}{\{(A/g_1)^{1/3} - z)\}^3} = g_1$$

or

$$\ddot{z} + g_1(1 - kz)^{-3} = g_1, \text{ where } k^3 = g_1/A.$$

Expanding by the binomial theorem this last equation may be written in the form

$$\ddot{z} + 3g_1k\left\{ z + 2kz^2 + \frac{10}{3} k^2z^3 + 5k^3z^4 - 7k^4z^5 \cdots \right\} = 0$$

provided $z < 1/k$. The k of this equation is a parameter of the problem but it is not a convenient parameter in which to express the solution without generalisation. To obtain a periodic solution we make the transformations

$$z = cy\ ; \ 3g_1k = K\ ; \ ck = \lambda\ ; \ t - t_1 = \tau\{(1 + \delta)/K\}^{1/2} \tag{3}$$

where t_1 is the time at which $x = a - 1/k$, or at which $z = 0$ and at which dx/dt is positive.

The differential equation then becomes

$$y'' + (1 + \delta)\{y + 2\lambda y^2 + 10/3\lambda^2y^3 + \cdots\} = 0 \tag{4}$$

where the primes denote differentiation with respect to τ.

The λ is now taken as the parameter of the solution. The δ is also a parameter which is to be determined so that the motion shall be periodic with period 2π in τ, the introduction of the K as above making this possible. The introduction of the c is to facilitate the discussion and the construction of the solution, as the c may be determined so that $y'(0) = 1$.
For

$$y' = z'/c = \dot{z}\frac{dt}{d\tau}/c = \dot{z}\{(1 + \delta)/K\}^{1/2}/c,$$

since $z = cy$. Also by (2) $\dot{x} = \dot{z}$, and the energy integral of (1) gives

$$\frac{\dot{x}^2}{2} + \frac{A}{2(a - x)^2} g_1x = c_2.$$

At $x=0$, $\dot{x}=0$ and therefore $c_2=A/2a^2$. Then at $x=a-1/k$ we have

$$\dot{x}^2 = A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right).$$

Therefore if $y'(0)=1$,

$$c = \left\{A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right)\right\}^{1/2}\{(1+\delta)/K\}^{1/2}. \tag{5}$$

It will be convenient for later use to denote the first factor of this expression by a single symbol, and accordingly we put

$$\gamma = \left\{A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right)\right\}^{1/2}. \tag{6}$$

Now equation (4) may be regarded as a differential equation in two parameters, δ and λ, and by the general theory of analytic differential equations it may be integrated as a power series in δ and λ, converging for δ and λ sufficiently small for any preassigned interval of time. It is true that the δ appears implicitly in λ but we shall generalise it and for the purpose of construction of the solution we shall regard it as a constant where it appears implicitly.

Let the power series solution in δ and λ be represented by

$$y = \sum_{i,j=0}^{\infty} y_{ij}\delta^i\lambda^j$$

$$\begin{aligned} y_{ij}(0) &= 0 \quad (i, j = 0, \cdots, \infty) \\ y_{00}'(0) &= 1 \\ y_{ij}'(0) &= 0 \quad (i, j = 0, \cdots \infty\ ;\ i+j \neq 0). \end{aligned} \tag{7}$$

Substituting (7) in (4) and equating coefficients of $\delta^i\lambda^j(i,j=0, \cdots \infty)$ we obtain a system of equations which may be integrated sequentially.

The coefficient of $\delta^0\lambda^0$ gives $y_{00}''+y_{00}=0$; and $y_{00}=\sin\tau$ is the solution satisfying the initial conditions stated.

The coefficient of $\delta\lambda^0$ gives $y_{10}''+y_{10}=-\sin\tau$.

The solution of this equation satisfying the initial conditions is readily found to be

$$y_{10} = -\frac{1}{2}\sin\tau + \frac{\tau}{2}\cos\tau.$$

These two steps are sufficient to establish the existence of a periodic solution of (4). For the conditions of such a solution with period 2π in τ are

$$y(2\pi) = y(0) = 0\ ;\ y'(2\pi) = y'(0) = 1. \tag{8}$$

These conditions are not independent however, as the first implies the second. For since $y(0)=0$ and $y'(0)=1$, the energy integral of (4) implies that if $y(2\pi)=0$, then

$$\{y'(2\pi)\}^2 = \{y'(0)\}^2 = 1, \quad \text{and therefore} \quad y'(2\pi) = \pm 1.$$

But since by the power series solution just obtained

$$y = \sin\tau + \left(- \tfrac{1}{2}\sin\tau + \frac{\tau}{2}\cos\tau\right)\delta + \cdots ,$$

it follows that

$$y(2\pi) = \pi\delta + q(\delta,\lambda), \quad q(0,0) = 0, \tag{9}$$

$$y'(2\pi) = 1 + p(\delta,\lambda), \quad p(0,0) = 0. \tag{10}$$

Equation (9) may be solved for δ as a power series in λ, vanishing with λ and converging for λ sufficiently small. Also (10) is true for λ sufficiently small. But when $\lambda=0$, by (10) $y'(2\pi)=1$. Therefore $y'(2\pi)=+1$, and not -1, for λ sufficiently small. Therefore the first of equations (8) implies the second.

Again changing the signs of y, λ and τ leaves the differential equation (4) unchanged and also the initial conditions $y(0)=0$, $y'(0)=1$ unchanged. Therefore if $y=f(\lambda, \delta; \tau)$ is a solution, $-y=f(-\lambda, \delta; -\tau)$.

And if $f(\lambda, \delta; \tau)$ is periodic with period 2π, then $f(-\lambda, \delta; -\tau)$ is also periodic with period 2π. Hence if δ is determined so that $f(\lambda, \delta; 2\pi)=0$, then

$$f(-\lambda, \delta ; -2\pi) = f(-\lambda, \delta ; 2\pi) = 0.$$

This means that $f(\lambda, \delta; 2\pi)$ is explicitly even in λ and therefore δ is expansible as a power series in even powers of λ.

Let such an expansion be represented by

$$\delta = \delta_2\lambda^2 + \delta_4\lambda^4 + \delta_6\lambda^6 + \cdots . \tag{11}$$

Now by (3), (5) and (6), $\lambda^2 = c^2k^2 = \gamma^2k^2(1+\delta)/K$,

and if we put $\mu = \gamma^2k^2/K$, then

$$\delta = \delta_2\mu(1+\delta) + \delta_4\mu^2(1+\delta)^2 + \delta_6\mu^3(1+\delta)^3 + \cdots . \tag{12}$$

But this is again an implicit equation in δ and μ which may be solved for δ as a power series in μ, vanishing with μ and converging for μ sufficiently small.

The solution is

$$\delta = \delta_2\mu + (\delta_2^2 + \delta_4)\mu^2 + (\delta_2^3 + 3\delta_2\delta_4 + \delta_6)\mu^3 + \cdots.$$

With this value for δ the solution (7) is periodic.

Direct Construction of the Solution

Instead of finding $f(\lambda, \delta; \tau)$ in the manner indicated and solving $f(\lambda, \delta; 2\pi)=0$ for δ to get equation (11), it is simpler to solve equation (4) directly as a power series in λ, determining the coefficients of the expansion (11) at each step so that the solution shall be periodic with period 2π in τ.

Accordingly we assume the solution

$$y = \sum_{i=0}^{\infty} y_i \lambda^i ; \qquad \delta = \sum_{i=1}^{\infty} \delta_{2i} \lambda^{2i}$$

$$y_i(0) = 0 \; (i = 0, \cdots, \infty)$$

$$y_0'(0) = 1, \;\; y_i'(0) = 0 \;\; (i = 1, \cdots, \infty).$$

This solution must satisfy the equation identically in λ, and therefore after the substitution we may equate the coefficients of λ^i separately to zero. There results a system of differential equations which may be integrated sequentially, the constants of integration being determined at each step so as to satisfy the above conditions.

The coefficient of λ^0 gives $y_0'' + y_0 = 0$. The solution is $y_0 = \sin\tau$.

The coefficient of λ gives $y_1'' + y_1 = -2y_0^2 = -1 + \cos 2\tau$.

The solution of this equation is

$$y_1 = -1 + \frac{4}{3}\cos\tau - \frac{1}{3}\cos 2\tau.$$

The coefficient of λ^2 gives

$$y_2'' + y_2 = -\left(4y_0y_1 + \frac{10}{3}y_0^3 + \delta_2 y_0\right)$$

$$= \left(-\delta_2 + \frac{5}{6}\right)\sin\tau - \frac{8}{3}\sin 2\tau + \frac{3}{2}\sin 3\tau.$$

For periodicity, $\delta_2 = 5/6$, and the solution of the equation gives

$$y_2 = -\frac{175}{144}\sin\tau + \frac{8}{9}\sin 2\tau - \frac{3}{16}\sin 3\tau.$$

The coefficient of λ^3 gives

$$y_3'' + y_3 = -\left(2y_1^2 + 4y_0y_2 + 10y_0^2y_1 + 5y_0^4 + \frac{5}{6}y_1 + \frac{5}{3}y_0^2\right)$$

$$= \frac{5}{6} - \frac{44}{9}\cos 2\tau + 6\cos 3\tau - \frac{35}{18}\cos 4\tau.$$

The solution of this equation is

$$y_3 = \frac{5}{6} - \frac{199}{108}\cos\tau + \frac{44}{27}\cos 2\tau - \frac{3}{4}\cos 3\tau + \frac{7}{54}\cos 4\tau.$$

This method may be carried on indefinitely and as many of the coefficients as desired obtained. The coefficients of odd powers of λ are cosine series in multiples of τ and the coefficients of even powers are sine series. It seems rather remarkable that while the determination of a δ_{2i} is necessary in order to eliminate the term in $\sin\tau$ in obtaining the coefficients of even powers of λ, in obtaining the coefficients of odd powers the term in $\cos\tau$ drops out automatically. There seems to be no simple direct proof of this fact, but that it is true follows from the existence and the uniqueness of the periodic solution.

Also, the values of the coefficients obtained may be checked by means of the energy integral of equation (4). That integral, after simplification, is

$$y'^2 + a\text{ constant} = -(1+\delta)\left(y^2 + \frac{4}{3}\lambda y^3 + \frac{5}{3}\lambda^2 y^4 + 2\lambda^3 y^5 + \frac{7}{3}\lambda^4 y^6 + \cdots\right).$$

This equation must be satisfied identically in λ, and therefore the substitution in it of the solution (12) requires that the following relations be satisfied.

$$\begin{aligned}
y_0'^2 + a\text{ constant} &= y_0^2 \\
2y_0' y_1' &= -\left(2y_0y_1 + \frac{4}{3}y_0^3\right) \\
y_1'^2 + 2y_0' y_2' &= -\left(y_1^2 + 2y_0y_2 + 4y_0^2y_1 + \frac{5}{3}y_0^4 + \delta_2 y_0^2\right) \\
2y_1' y_2' + 2y_0' y_3' &= -\left\{2y_1y_2 + 2y_0y_3 + 4y_0^2y_2 + 4y_0y_1^2 + \frac{20}{3}y_0^3y_1 \right. \\
&\qquad \left. + 2y_0^5 + \delta_2\left(2y_0y_1 + \frac{4}{3}y_0^3\right)\right\}
\end{aligned}$$

. . . .

Finally, the solution in terms of the original data is, on substitution,

$$\begin{aligned}
x = a - \frac{1}{k} + c\Big\{ &\sin\tau - \frac{1}{3}(3 - 4\cos\tau + \cos 2\tau)\lambda \\
&- \frac{1}{144}(175\sin\tau - 128\sin 2\tau + 27\sin 3\tau)\lambda^2 \\
&+ \frac{1}{108}(90 - 199\cos\tau + 176\cos 2\tau - 81\cos 3\tau + 14\cos 4\tau)\lambda^3 + \cdots\Big\}
\end{aligned}$$

where

$$\tau = (t - t_1)/\{(1+\delta)/K\}^{1/2}\,; \quad \delta = 5/6\mu + \cdots$$

$$c^2 = \gamma^2(1+\delta)/K\,; \quad \lambda = ck\,; \quad \mu = \gamma^2 k^2/K$$

$$\gamma^2 = A\left(\frac{1}{a^2} - k^2\right) + 2g_1\left(a - \frac{1}{k}\right); \quad A = \frac{ma^4\dot{\theta}_0^2}{m+M},$$

$$g_1 = \frac{Mg}{m+M}\,; \quad k = (g_1/A)^{1/3}\,; \quad K = 3g_1k.$$

The general solution shows that the motion is non-symmetrical about the equilibrium position.

To illustrate by a numerical example, if we take $a = 10$ ft., $m = 0.1$ lb., $M = 2.9$ lb., $a\dot{\theta}_0 = 100$ ft./sec. and $g = 32.16$ ft./sec^2., then

$$\begin{array}{rlrl}
A &= 33333\ \text{ft.}^4/\text{sec.}^2 & \mu &= 0.0005465 \\
g_1 &= 31.088\ \text{ft.}/\text{sec.}^2 & \delta &= 0.000455 \\
k &= 0.097702\ \text{ft.}^{-1} & c &= 0.239\ \text{ft.} \\
 &= 9.1120\ 8\ \text{sec.}^{-2} & \lambda &= 0.0234 \\
\gamma^2 &= 0.5217\ \text{ft.}^2/\text{sec.}^2 & &
\end{array}$$

Then the period is

$$P = \{(1+\delta)/K\}^{1/2}2\pi = 2.08\ \text{sec}.$$

Solution by interpolation shows that the motion below the equilibrium position requires 1.02 sec. per oscillation and that above 1.06 sec. The total range of oscillation is 0.478 ft., of which 0.235 ft. is below the equilibrium position and 0.243 ft. is above.

SUGGESTED READINGS IN CONNECTION WITH "SUCCESSIVE GENERALIZATIONS IN THE THEORY OF NUMBERS"[1]

By E. T. Bell, California Institute of Technology

R. Dedekind. *Sur la théorie des nombres entiers algébriques*, Bulletin des Sciences Mathématiques (Darboux' Bulletin), (2), vol. 1, pp. 17–41, 69–92, 114–164, 207–248. Also published separately by Gauthier Villars, Paris, 1877.

——— *Vorlesungen über Zahlentheorie*, von P. G. Lejeune Dirichlet, 4th edition, 1894 (reprinted in 1912), Supplement XI.

G. Zolotareff. *Théorie des nombres entiers complexes, avec une application au calcul intégral.* St. Petersbourg, 1873–4. Chapter III for theory of ideals.

——— Journal des Mathématiques (Liouville's Journal), (3), vol. 6, (1880) pp. 51–84, 115–129, 145–167.

[1] This paper by Professor E. T. Bell was published in the February number of this Monthly, vol. 34 (1927), pp. 55–75.

E. Landau. *Einführung in die elementare und analytische Theorie der algebräischen Zahlen und der Ideale*, 1918, Teubner. Especially pp. 1–54.

E. Lasker. Mathematische Annalen, 60, 1905, pp. 20–115, 607–8.

P. Bachmann. *Zahlentheorie. V, Allgemeine Arithmetic der Zahlenkörper.* Leipzig, 1905.

K. Hensel. *Zahlentheorie* (Göshen, 1913).

L. E. Dickson. Bulletin of the American Mathematical Society, vol. 13 (1907), pp. 348–362. Annals of Mathematics, (2), vol. 18 (1917) pp. 161–187.

H. Prüfer. *Neue Begründung der algebräischen Zahlentheorie*, Mathematische Annalen, vol. 94 (1925), pp. 198–243.

A. Speiser. *Allgemeine Zahlentheorie.* Vierteljahrschrift der Naturforschenden Gesellschaft zu Zürich, vol. 71 (1926), pp. 1–48.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. THE COMPOSITION OF ANGULAR VELOCITIES.

By F. E. KESTER, University of Kansas

Perhaps the most convenient and elegant treatment of the theorem for the composition and resolution of angular velocities for class use consists in showing that such quantities are completely represented by vectors, invoking then the general principle of vector composition and resolution. There is, however, excuse for reasonable doubt, it has always seemed to me, that a body can follow strictly the dictates of two or more simultaneous angular velocities without some distortion of form—doubt, in other words, that a *rigid* body can be subjected simultaneously to more than one angular velocity.

Such doubt as this is not quieted by the usual treatment found in text books on mechanics designed for intermediate courses, say for students of a senior college. Books of this grade, when they deign to treat the theorem, usually limit the considerations to such points of the body as lie in the plane of the axes (concurrent) of two given angular velocities.[1] The question as to distortions of the body, if the above mentioned doubt exists at all, regards more seriously those points which are out at more general positions in the body.

The only complete treatments of which I know make use of the Euler transformation of coordinates (Thomson and Tait, Webster, Whittaker, and

[1] The only extension of the usual treatment that I have been able to find is in Gray & Gray, *Treatise on Dynamics* (1911), pp. 448–9.

similar treatises), and these are more formidable than one cares to weave into an undergraduate course in mechanics. Sometimes, moreover, the theorems so given are somewhat misleadingly stated in terms of the composition of very small angular displacements.

The following presentation of this theorem has been used in my courses in mechanics over a number of years. I first stated it entirely in terms of direction cosines and sines; recently however, in class, I inadvertently started with expressions for the component velocities (linear) of a general particle of the body which is subjected to the given angular velocities, in terms of the coordinates of that particle, and I was much surprised, on following this lead through, to find how much simpler the formulation became and how much more evident the conclusion. Thinking that the theorem in its present form may prove useful to others who may have experienced the same desire to give the proof in complete yet simple manner, I am publishing it here for general distribution.

Assume that the body in question is subjected simultaneously to three angular velocities, w_x, w_y, w_z, about three concurrent and mutually perpendicular axes. Choose for consideration any particle P of the body, distant r from O, the point of intersection of the axes; the coordinates of P are x, y, z, (Fig. 1).

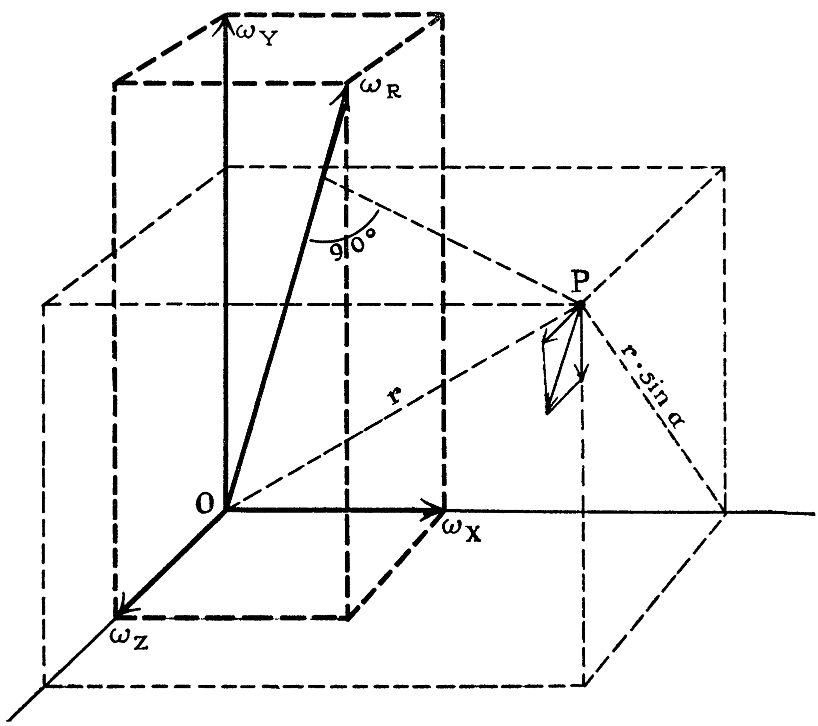


Fig. 1.

If α, β, γ, are the angles which OP makes with the three axes, the linear velocity of P due to w_x is $w_x r \cdot \sin\alpha$, in a plane perpendicular to the x-axis. The x-component of this velocity is of course zero; the y-component is $-w_x r \cdot \sin\alpha(z/r\sin\alpha)$, or $-w_x z$; the z-component is $+w_x y$. We may express the various linear velocity components in tabular form:

Velocity component along	x	y	z
due to w_x	0	$-w_x z$	$+w_x y$
due to w_y	$+w_y z$	0	$-w_y x$
due to w_z	$-w_z x$	$+w_z y$	0

Then the resultant linear velocity of P is given by

$$
\begin{aligned}
\mathrm{v}^2 &= (w_y z - w_z y)^2 + (w_z x - w_x z)^2 + (w_x y - w_y x)^2 \\
&= w_x^2(y^2 + z^2) + w_y^2(z^2 + x^2) + w_z^2(x^2 + y^2) - 2w_x w_y xy - 2w_y w_z yz - 2w_z w_x zx \\
&= (w_x^2 + w_y^2 + w_z^2)\mathrm{r}^2 - w_x^2 x^2 - w_y^2 y^2 - w_z z^2 - 2w_x w_y xy - 2w_y w_z yz - 2w_z w_x zx \\
&= (w_x^2 + w_y^2 + w_z^2) r^2 - (w_x x + w_y y + w_z z)^2 \\
&\equiv (w_x^2 + w_y^2 + w_z^2)\left[r^2 - \left(x\frac{w_x}{(w_x^2 + w_y^2 + w_z^2)^{1/2}} \right.\right. \\
&\qquad \left.\left. + y\frac{w_y}{(w_x^2 + w_y^2 + w_z^2)^{1/2}} + z\frac{w_z}{(w_x^2 + w_y^2 + w_z^2)^{1/2}} \right)^2 \right].
\end{aligned}
$$

The quantity $(x \cdot w_x/(w_x^2+w_y^2+w_z^2)^{1/2} + \cdots)$ may be considered as the projection of r, through its components x, y, z, on to a line whose direction cosines are $w_x/(w_x^2+w_y^2+w_z^2)^{1/2}, \cdots$, so that

$v^2 = (w_x^2 + w_y^2 + w_z^2)(r^2 - r^2\cos^2\theta)$, and $v = r\sin\theta(w_x^2 + w_y^2 + w_z^2)^{1/2}$,

where θ is the angle between r and the above mentioned line.

This expression for v may be interpreted as the result of a single angular velocity of magnitude $(w_x^2 + w_y^2 + w_z^2)^{1/2}$ about an axis the direction cosines of which are $w_x/(w_x^2 + w_y^2 + w_z^2)^{1/2}$; $w_y/(w_x^2 + w_y^2 + w_z^2)^{1/2}$; $w_z/(w_x^2 + w_y^2 - w_z^2)^{1/2}$; and this resultant angular velocity may be represented by the diagonal of the rectangular parallelopipedon whose adjacent edges are w_x, w_y, w_z.

Inasmuch as P is any general particle within the body, it is evident that the whole body moves as though subjected to only one angular velocity, therefore, also that a rigid body may be subjected strictly to two or more angular velocities.

By the converse of the theorem thus far proved we may conclude that any given angular velocity may be resolved into components along mutually perpendicular axes. By such resolution of, say, two given angular velocities ω_1 and ω_2 with axes concurrent and in general directions, and by subsequent composition of the components thus obtained, it becomes easy to prove geometrically that the resultant ω_R of the two angular velocities (with concurrent axes) is given by the adjacent diagonal of the parallelogram plotted with the two given vectors as sides. Finally we may proceed to the more complete theorem of the polygon of angular velocities—which brings us back inductively to the treatment, first mentioned, of vector addition (and subtraction).

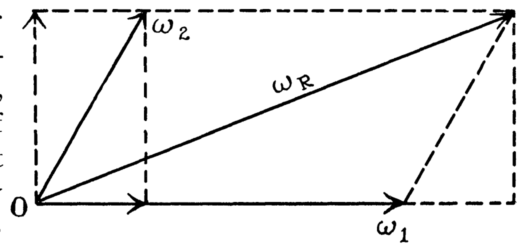


FIG. 2.

II. The Devil's Curve and Abelian Integrals

By Paul R. Rider, Washington University

1. The devil's curve. The curve

$$y^4 - x^4 + ay^2 + bx^2 = 0 \tag{1}$$

is called by French geometers *la courbe du diable*.[1] The origin of this satanic appellation seems to be unknown, but B. H. Brown[2] suggests that "in all probability the curve was summarily christened by some exasperated youth who felt strongly and expressed himself thus forcibly on the subject." Professor Brown gives in his note a bibliography, to which the reader is referred.

The principal interest in the curve seems to be that it affords a good exercise in curve tracing—discussion of symmetry, determination of horizontal and vertical tangents, asymptotes, singular points, etc. Professor Pierpont once called the attention of the writer to the fact that it is also an excellent example to be used in presenting the theory of Riemann surfaces and Abelian integrals.

The form of the curve most frequently met is

$$y^4 - x^4 - 96ay^2 + 100a^2x^2 = 0, \tag{2}$$

a special case of (1). The special case which we wish to consider here is

$$y^4 - x^4 - y^2 + 2x^2 = 0, \tag{3}$$

[1] See Loria, *Spezielle algebraische und transzcendente ebene Kurven* (1902), p. 97, footnote 2 or Zweite Auflage (1910), p. 101, footnote 2. See also Cramer, *Introduction à l'Analyse des Lignes Courbes Algébriques* (1750), p. 19ff. Loria refers to L'Intermédiaire des Mathématiciens, vol. 4 (1897); the page number, which Loria gives incorrectly as 222, should be 104. In this place Brocard lists (without definition) a large number of curves which have received special names; among them is *courbe du diable*.

[2] B. H. Brown, *La courbe du diable*, this Monthly, vol. 33 (1926), pp. 273–274.

results for which have been worked out by the writer on a previous occasion. The general characteristics of curve (3) are the same as those of curve (2). Curve (1) has a node or an isolated point at the origin according as $ab \lessgtr 0$; consequently (2) and (3) have nodes at the origin. Results for (3) are merely summarized; much of the work is too elementary to be given in detail, and for any points of theory concerning which the reader desires further information he may consult Landfriedt, *Theorie der algebraischen Funktionen und ihrer Integrale*, or Appell and Goursat, *Théorie des Fonctions Algébriques et de leurs Intégrales*. Other interesting features of the curve will doubtless be found by a more extensive investigation.

Singular point (node) at (0, 0).

Horizontal tangents at $(0, 1)$, $(0, -1)$.

Vertical tangents at $(\sqrt{2},0)$, $(-\sqrt{2},0)$, $(\frac{1}{2}\sqrt{3}+\frac{1}{2}, \frac{1}{2}\sqrt{2})$, $(\frac{1}{2}\sqrt{3}-\frac{1}{2}, \frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}+\frac{1}{2}, \frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}-\frac{1}{2}, \frac{1}{2}\sqrt{2})$, $(\frac{1}{2}\sqrt{3}+\frac{1}{2}, -\frac{1}{2}\sqrt{2})$, $(\frac{1}{2}\sqrt{3}-\frac{1}{2}, -\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}+\frac{1}{2}, -\frac{1}{2}\sqrt{2})$, $(-\frac{1}{2}\sqrt{3}-\frac{1}{2}, -\frac{1}{2}\sqrt{2})$.

Asymptotes: $y=x, y=-x$ (also imaginary asymptotes $y=ix, y=-ix$).

Plücker's formulas:

$m=$ degree of curve $=4$,

$d'=$ number of possible double points $=(m-1)(m-2)/2=3$,

$d=$ number of actual double points $=1$,

$p=d'-d=$ genus or deficiency $=2$,

$k=$ number of cusps $=0$,

$c=$ class $=m(m-1)-2d-3k=10$,

$i=$ number of inflection points $=3m(m-2)-6d-8k=18$,

$t=$ number of double tangents $=\frac{1}{2}c(c-1)-m-3i=16$.

2. Riemann surface. The values of x for which $F(x,y)\equiv y^4-x^4-y^2+2x^2=0$, has multiple roots in y are $x=0$, $\sqrt{2}$, $-\sqrt{2}$, $\frac{1}{2}+\frac{1}{2}\sqrt{3}$, $-\frac{1}{2}+\frac{1}{2}\sqrt{3}$, $\frac{1}{2}-\frac{1}{2}\sqrt{3}$, $-\frac{1}{2}-\frac{1}{2}\sqrt{3}$, ∞.

Developments about various points:

At $(0,0)$, $y=\pm 2^{1/2}x+\cdots$.

At $(\sqrt{2},0)$, $y=\pm 2^{5/4}i(x-\sqrt{2})^{1/2}+\cdots$.

Similar developments to those at $(\sqrt{2},0)$ hold at $(-\sqrt{2},0)$.

At $(\frac{1}{2}+\frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{2})$, $y=\frac{1}{2}\sqrt{2}\pm i(\frac{3}{2}+\frac{1}{2}\sqrt{3})^{1/2}(x-\frac{1}{2}-\frac{1}{2}\sqrt{3})^{1/2}+\cdots$.

Similar developments to these are found at the other multiple roots (except ∞).

At (∞,∞), $1/y=\pm 1/x+\cdots$, $1/y=\pm i/x+\cdots$.

In obtaining these developments one may illustrate Puiseux's method and call attention to Newton's parallelogram amd De Gua's triangle.

The branch points are therefore $x_1=\sqrt{2}$, $x_2=-\sqrt{2}$, $x_3=\frac{1}{2}+\frac{1}{2}\sqrt{3}$, $x_4=-\frac{1}{2}+\frac{1}{2}\sqrt{3}$, $x_5=\frac{1}{2}-\frac{1}{2}\sqrt{3}$, $x_6=-\frac{1}{2}-\frac{1}{2}\sqrt{3}$.

If we solve equation (3) for y we find that

$$y_1=(\tfrac{1}{2}+r)^{1/2},\quad y_2=(\tfrac{1}{2}-r)^{1/2},\quad y_3=-(\tfrac{1}{2}+r)^{1/2},\quad y_4=-(\tfrac{1}{2}-r)^{1/2},$$

where $\quad r=(x^4-2x^2+\frac{1}{4})^{1/2}=[(x-x_3)(x-x_4)(x-x_5)(x-x_6)]^{1/2}.$

Let us form a Riemann surface by joining by branch cuts x_1 and x_2, x_3 and x_5, x_4 and x_6. Then the branches permute as shown in the following scheme:

Point	Connection of sheets	Order of Cycles
x_1	13 (24)	$0+0+1=1$
x_2	13 (24)	$0+0+1=1$
x_3	(12)(34)	$1+1=2$
x_4	(12)(34)	$1+1=2$
x_5	(12)(34)	$1+1=2$
x_6	(12)(34)	$1+1=2$

Thus the sum of the orders of all cycles is 10, and by means of the well-known formula $p=(\frac{1}{2})(\text{sum of orders of cycles})-(\text{m}-1)$ we find that $p=5-3=2$, which checks our previous result.

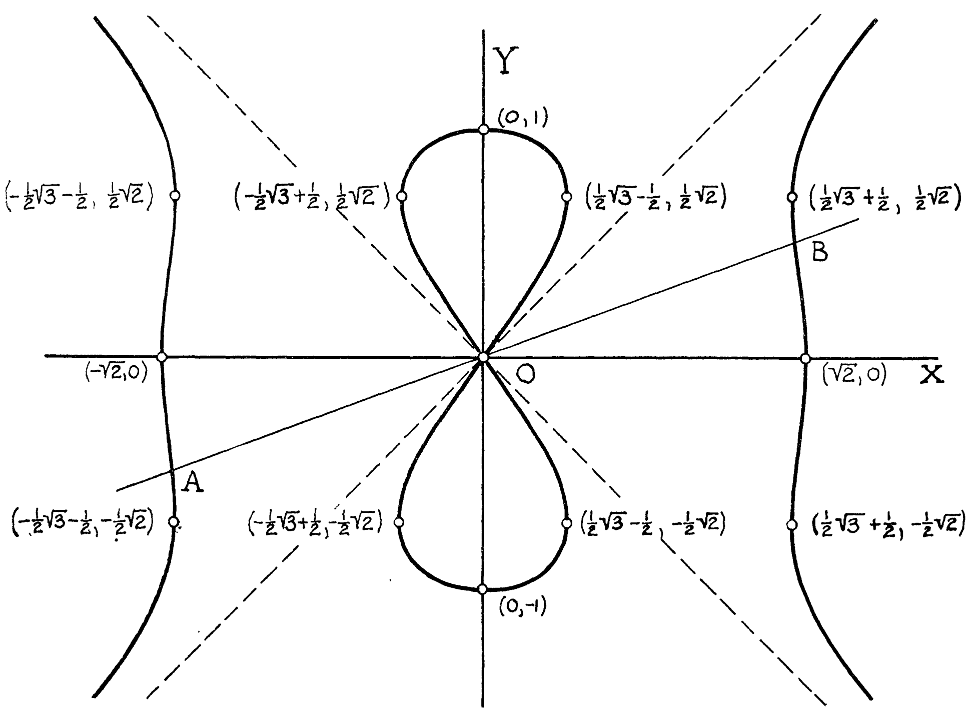


FIGURE 1

3. **Abelian integral of first species.** To form an Abelian integral of the first species belonging to (3) let us take the adjoint polynomial of degree $\text{m}-3=1$, which, since the curve represented by it must pass through the double point $(0,0)$, is of the form cx. The line $y-cx=0$ is the line AB in Figure 1 and the integral

$$I_1 = \int \frac{y - cx}{F_y} dx = \int \frac{(y - cx)dx}{2y(2y^2 - 1)}$$

is an Abelian integral of the first species, being finite at all points of the devil's curve.

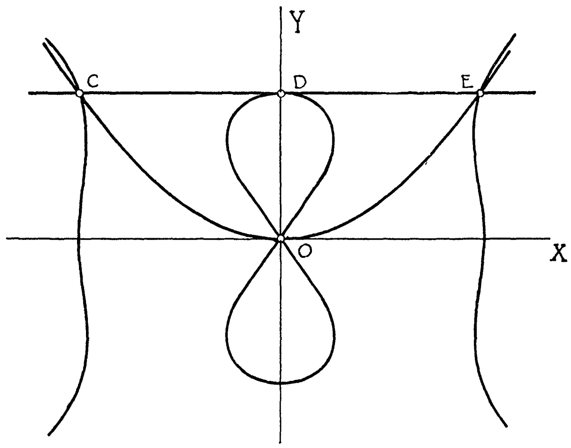


FIGURE 2.

4. **Abelian integral of second species.** Let us take the line $y-1=0$, which is tangent to the devil's curve at $D(0,1)$ and cuts it in the other points $C(-\sqrt{2}, 1)$, $E(\sqrt{2},1)$. (See Figure 2.) The adjoint of degree $m-2=2$ which passes through C and E is $y-(c_1x^2+c_2x)=y-\frac{1}{2}x^2=0$. Then

$$I_2 = \int \frac{y - \frac{1}{2}x^2}{(y-1)F_y} dx = \int \frac{(y - \frac{1}{2}x^2)dx}{(y-1)(4y^3 - 2y)}$$

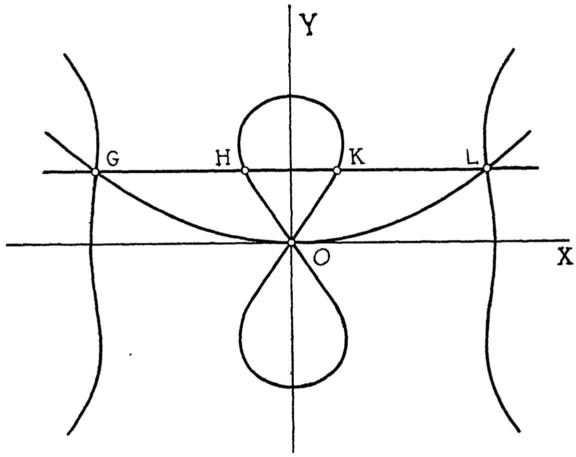


FIGURE 3.

is an Abelian integral of the second species. It has a pole of order 1 at the point D, but is finite elsewhere on the devil's curve.

5. Abelian integral of third species. Take the line $y-\frac{1}{2}=0$, which cuts the devil's curve in the points $G[-(1+\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$, $H[-(1-\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$, $K[(1-\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$, $L[(1+\frac{1}{4}\sqrt{13})^{1/2}, \frac{1}{2}]$ of Figure 3. The adjoint of degree $m-2=2$ which passes through G and L is $y-\frac{1}{3}(8-2\sqrt{13})x^2=0$. Then

$$I_3 = \int \frac{y - \frac{1}{3}(8 - 2\sqrt{13})x^2}{(y - \frac{1}{2})F_y}dx = \int \frac{y - \frac{1}{3}(4 - \sqrt{13})x^2}{(y - \frac{1}{2})(2y^3 - y)}dx$$

is an Abelian integral of the third species, as it can readily be shown that its only singularities are at H and K and that these are logarithmic.

III. Note on the Function $y=a^x$, $a<0$.

By ALAN D. CAMPBELL, University of Arkansas.

No mention has been found by the author anywhere concerning the peculiarities of the function of x defined by $y=a^x$, $a<0$. If we try to plot the graph of this function we find how discontinuous it is. Thus $a^{\pm 1/2}$ is imaginary, $a^{\pm 1/3}$ is real and negative, $a^{\pm 2/3}$ is real and positive, $a^{\pm 2}$ is positive, $a^{\pm 3}$ is negative. We cannot define uniquely the value of $y=a^x$, $a<0$ for $x=\sqrt{2}$; because if we let $x\rightarrow\sqrt{2}$ thru the sequence 1.41, 1.414, $\cdots$ we get some imaginary values for a^x, but if we let $x\rightarrow\sqrt{2}$ thru the sequence 141/99, 1414/999, $\cdots$ we get real values for a^x, some positive and some negative.

Again $a^x/a^x=1$; but $\lim_{x\rightarrow 0}a^x=-1$ if $x\rightarrow 0$ thru such a sequence as $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\cdots$, whereas if $x\rightarrow 0$ thru $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\cdots$, a^x takes on imaginary values; and a^x takes on both positive and negative real values as $x\rightarrow 0$ thru the sequence $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{9}$, $\frac{1}{9}$, $\frac{2}{27}$, $\frac{1}{27}$, $\cdots$; finally a^x takes on both real and imaginary values as $x\rightarrow 0$ thru the sequence $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\cdots$. If we endeavor to define a derivative for $y=a^x$, $a<0$, by the same method as when $a>0$, we find that the derivative does not exist. The same is true of the integral. For a complete discussion of $y=a^x$, $a<0$, recourse should be had to complex variables. In this note we are looking at this function from the viewpoint of real variables. It seems as though $y=a^x$, $a<0$, might be used along with such functions as $y=\sin(1/x)$, $y=x\sin(1/x)$, etc., to introduce students to the singularities of real functions.

I repeat, then, that my own experience of over thirty years in teaching the calculus leads me to prefer substantially the form of Duhamel's theorem you have used, as being suggestive and inspiring to the beginner.

WILLIAM F. OSGOOD, Harvard University

P.S. I do not follow Ettlinger in his criticism of your example on page 25. Your conclusions are justified by your hypotheses.

III. A NOTE BY ASSOCIATE EDITOR W. B. CARVER

EDITOR'S NOTE. It seems important to the Editor that our readers should see precisely the point on which there is a difference of opinion in this matter. In Professor Osgood's paper of 1903, he showed that in the form of Duhamel's theorem, which is stated on page 22 of Professor Woods' book, the conclusion of the theorem, in certain cases which do not ordinarily arise in practice, would not follow from the hypotheses as stated. This point seems not to be in dispute. Professor Ettlinger has criticized the use of such a form of the theorem, preferring the use of some one of the forms which are rigorously correct but which, in the opinion of many teachers, are beyond the grasp of the students for whom the book is written. This kind of question arises frequently in other connections in mathematics, and it is one of profound scientific and pedagogical significance. Should we state a theorem in a text-book with a hypothesis which, in the light of the most rigorous modern analysis, we know to be incomplete; or should we state it in a complete form which the student will probably not be able to understand? Perhaps we are not driven to either of these alternatives. Is it not possible to give the reader only so much of the truth as he is likely to understand without making any statement which is false even from the most rigorous point of view?

W. B. CARVER, Cornell University

REVIEWS

The Work of the College Entrance Examination Board 1901–1925. Boston, Ginn and Company, 1926. 300 pages. Price, $4.00.

The College Entrance Examination Board celebrated the completion of twenty-five years of work by a dinner in New York on November 6, 1925, and by the publication of a "brief record of its origin, achievements, and aims." The aims are discussed in various connections throughout the book but are summarized on the title page as "the solution of educational problems through the cooperation of all vitally concerned."

The speakers at the dinner included Nicholas Murray Butler, Wilson Farrand, Henry S. Pritchett, and Julius Sachs. These men were able to speak with authority on the history of the formation and development of the Board

and abstracts of their remarks occupy the first part of the book. Two recent articles, *A Brief History of the College Entrance Examination Board*[1], by Wilson Farrand, and *The Art of Examination*[2], by A. Lawrence Lowell, are reprinted in full.

The bulk of the volume consists of a digest of the publications of the Board. In addition to the twenty-five annual reports of the Secretary, the Board has published over one hundred numbered documents containing information on specific topics, and the examination questions have been published in various collected forms by Ginn and Company.

The reports of the Secretary contain a large number of statistical tables showing in detail the work carried on by the Board and presenting many interesting statistical studies of educational problems as reflected in the results of the examinations. In 1901 the Board examined 973 candidates. The number has grown steadily until in 1925 there were 19,775. These figures may be taken as a fair indication of the increase in the general activities of the Board and of its widening influence among both the colleges and the preparatory schools.

The Board's first requirements were based largely upon the report, published in 1899, of the Committee on College Entrance Requirements of the National Education Association. Other requirements in mathematics were adopted in 1903 and were retained for two decades. These were formulated by a committee of the American Mathematical Society which published its report in the Bulletin in November, 1903.

The definition of the requirements adopted in April, 1923, was formulated by a commission of eleven members appointed by the Board. Four of the members were also members of the National Committee on Mathematical Requirements created by the Mathematical Association in 1918. The commission submitted two reports, one covering algebra and trigonometry, the other covering geometry. Both of these reports were widely circulated with a view to securing constructive criticisms from teachers of mathematics, copies being addressed to every name on the subscription lists of the *Mathematics Teacher* and of THE AMERICAN MATHEMATICAL MONTHLY. After revision these reports were adopted and published May 15, 1923, as Documents No. 107 and 108, respectively.

The requirements conform in substance to the recommendations contained in the report of the National Committee published in February, 1923, under the title "The Reorganization of Mathematics in Secondary Education." The report of the National Committee has had a profound influence on the teaching of secondary school mathematics and particularly on the preparation of textbooks in algebra. Unfortunately the report is now out of print and the

---

[1] *School and Society*, Vol. XXII, No. 568, Nov. 4, 1925.

[2] *Atlantic Monthly*, Jan., 1926.

committee is no longer engaged in its work of spreading propaganda among the vast body of teachers. At the present time the College Board is the most important agency which, by its annual examinations interpreting and emphasizing the new requirements, is actively engaged in carrying on the work started by the Mathematical Association.

The cooperative attitude of the Board, as illustrated by the history of its requirements in mathematics, is apparent throughout the book. Its attitude toward innovations is explicitly stated as follows (p. 204):

"The College Entrance Examination Board endeavers to maintain an open-minded attitude toward any research relating to the principles which underlie its work and toward any changes that are suggested in its organization. It is the policy of the Board to consider no question as closed and no problem as solved if it is regarded as still unsettled by educators in good standing who have given it careful attention."

Naturally there have been some just adverse criticisms; the difficult work of the examiners has not always been perfect; the influence of the examinations may have led to cramming in some schools. But these troubles appear quite insignificant when one appreciates the effects obtained in uniformity of entrance requirements for the colleges of the Board and the high standard of excellence shown by the schools in meeting them.

Considered as a contribution to the literature of importance to every teacher and administrator, the purpose of this book is well described in the preface by the Secretary, Professor Thomas S. Fiske. "Those interested in education will find in these pages two important messages: one is that the solution of educational problems should be sought in the cooperation of all groups, whether of institutions or individuals, whose interests are vitally concerned; the other is that examinations conducted by an agency external to the school and independent of any purely local or provincial authority are not only the best measures of a pupil's attainment but also an instrument of great value for education."

W. R. LONGLEY.

Riemannian Geometry. By L. P. EISENHART. Princeton, University Press, 1926. 262 pages.

This little book makes one more real contribution to the mathematical literature of America. Since so much has been done on this subject by Americans, especially by the author and his followers, it is indeed fitting that Professor Eisenhart should be the first to present the whole subject in English. It is written in a delightful style and should prove an inspiration to many young geometers. The author has promised a later book on Non-Riemannian geometry. Let us hope that it will not be too long in coming and that it will prove as delightful as the present one.

Several books on Riemannian geometry have appeared and in them the element of arc is defined by a positive definite quadratic form but in the present book Riemannian geometry is defined as any geometry with a metric, ds^2, not necessarily a definite form.

The element of arc is then written $ds^2 = eg_{ij}dx^idx^j$, where e is plus one or minus one and will be so chosen that ds^2 is always positive, or in other words so that ds is always real. The introduction of the multiplier e is a new departure. Einstein defined the element of arc as $ds^2 = g_{ij}dx^idx^j$, even in the case of an indefinite form, and of course ds is imaginary in certain regions of the space. Eisenhart defines the length of a curve only in regions of the space in which $g_{ij}dx^idx^j$ does not change sign.

If the quadratic form is definite, a unit vector is defined by the relation

$$\lambda^i\lambda_j = \epsilon_j^i \text{ where } \epsilon_j^i = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

while if the quadratic form is indefinite, a unit vector is defined by $\lambda^i\lambda_j = e\epsilon_j^i$ where $e=1$ if $g_{ij}\lambda^i\lambda^j$ is positive and $e=-1$, if $g_{ij}\lambda^i\lambda^j$ is negative.

The book is divided into six chapters. Chapter I, "Tensor Analysis," develops the notions of covariant and contravariant systems or tensors, using the usual notation with slight modifications. Chapter II, "Introduction of a Metric," treats the general topics of orthogonality, parallelism curvature, metric properties of a V_n immersed in a V_m, associate directions, etc. In this chapter the author presents a considerable amount of his own work, especially on spaces which contain fields of parallel vectors. Chapter III, "Orthogonal Ennuples," treats the congruences of curves, Ricci coefficients of rotation, Ricci principal directions, N-tuply orthogonal systems of hypersurfaces and allied subjects. Chapters IV and V, "Geometry of Subspaces," and "Subspaces of a Flat Space" are generalizations of topics usually discussed in ordinary differential geometry. The last chapter is an excellent presentation of groups of motions.

There are good lists of exercises in each chapter which will prove welcome to any teacher who wishes to use the book for a text book, and to students who wish to read the book by themselves. A good bibliography is given at the end.

C. L. E. MOORE

Plane Trigonometry. By LLOYD L. SMAIL. New York, McGraw-Hill Book Co., 1926. xii+203+43 pages. Price complete, \$2.25; text only, \$1.50, tables only, \$1.00.

The demands of the modern college are so varied that it is exceedingly difficult for one to write a text, even in trigonometry, for general use. The

author of this new text seems to have solved the difficulty by dividing his text into two parts. The first part is made up of the fundamentals that would be given in any course in trigonometry and these are presented in a brief, interesting way. The second part "reviews and extends the topics of Part I, but from a more advanced standpoint, and gives alternative proofs and methods of treatment".

The text has two possible beginnings and many possible endings; that is, it is suited to the needs of the instructor giving a brief course or to one giving a more extended course, and also it is suited to the instructor who wishes to begin by defining the ratios first for acute angles or to the one who wishes to begin with the general definitions. The author outlines a secondary arrangement for the needs of the latter group. The instructor who wishes to divide his students into sections in accordance with their ability will find Part I suitable for average students and Part II will furnish sufficient extra material to take care of the needs of the exceptional students.

The text is well written and the topics are presented in a clear and attractive style which has given to an old subject a new appearance. On the inside of the back cover is a pocket in which is inclosed a brief set of four-place tables which will be very convenient for examinations.

A special set of tables has been prepared for the text. These tables are different from those usually found. For instance, in Table IV, the mantissa for each four-place number up to 2000 and for each three-place number up to 999 is given opposite the number. This arrangement, together with tabular differences and tables of proportional parts, makes interpolation easy, but it remains to be seen by trial whether the new arrangement is an improvement over the old or not.

The problems in both parts are good and answers are given to about half of the exercises and so, again, the author has been able to solve satisfactorily a problem in text writing.

On pages 4 and 5 of the Tables, "Table III" is used several times when it is obvious that "Table IV" is meant.

R. P. STEPHENS

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) the titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 48, no. 4, October 1926: "Regular maps on an anchor ring" by H. R. Brahana, 225–240; "Determination of the type of tricrunodal quartic by means of its invariants" by L. T. Moore, 241–252; "Subgroups of index p contained in a group of order p^m" by G. A. Miller, 253–256; "Mapping by means of linear systems of curves invariant under Cremona involutions"

by T. L. Bennett, 257–276; "Algebraic and transcendental equations connected with the form of stream lines" by H. Bateman, 277–296.

Annals of Mathematics, volume 28, no. 1, December 1926: "On the problem of coloring maps in four colors, I" by C. N. Reynolds, Jr., 1–15; "Some theorems on deducibility" by C. H. Langford, 16–40; "Note on certain associated systems of linear equalities and inequalities" by L. L. Dines, 41–42; "Some further theorems concerning the formation of chains" by A. Arwin, 43–52; "Note on the extensions of groups to obtain n-th roots" by M. H. Ingraham, 53–58; "Periodic solutions of linear differential equations" by W. B. Fite, 59–64; "On the inversion of the order of integration of a two-fold iterated integral" by H. J. Ettlinger, 65–68; "The summability of a single and multiple Fourier series" by H. W. Bailey, 69–91; "The arithmetic of a general algebra" by O. C. Hazlett, 92–102.

Mathematische Zeitschrift, volume 25, no. 4, December 1926: "The identities of affinely connected manifolds" by T. Y. Thomas, 714–722; "A projective theory of affinely connected manifolds" by T. Y. Thomas, 723–733.

Proceedings of the National Academy of Sciences, U. S. A., volume 12, no. 12, December 1926: "Transformations of manifolds with a boundary" by S. Lefschetz, 737–738; "A possible way to discuss the fundamental principles of relativity" by P. Slavenas, 739–744; "Concerning paths that do not separate a given continuous curve" by R. L. Moore, 745–752; "On simply transitive primitive groups" by W. A. Manning, 753–755; "Quadratic forms which represent all integers" by L. E. Dickson, 756; "Congruences of parallelism of a field of vectors" by L. P. Eisenhart, 757–760; "Concerning certain types of continuous curves" by G. T. Whyburn, 761–766; "Summary of results and proofs concerning Fermat's last theorem" (second note) by H. S. Vandiver, 767–762.

Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften, May-June 1926: "Uber den Grad der Approximation einer analytischen Funktion" by J. L. Walsh, 223–230.

Transactions of the American Mathematical Society, volume 28, no. 4, October 1926: "A boundary value problem for a system of ordinary differential equations of the first order" by G. A. Bliss, 561–584; "On the theory of integral equations with discontinuous kernels" by R. E. Langer, 585–639 "The figuratrix in the calculus of variations" by P. R. Rider, 640–655; "On the existence of fields in Boolean algebras" by B. A. Bernstein, 654–657 "Asymmetric displacement of a vector" by J. M. Thomas, 658–670; "Tensors determined by a hypersurface in Riemann space" by H. Levy, 670–694; "A comparison of the series of Fourier and Birkhoff" by M. H. Stone, 695–762; "Analytic approximations to topological transformations" by P. Franklin and N. Wiener, 762–785.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Denison Mathematics Club, Denison University, Granville, Ohio.

The following programs were given at the meetings of the Club during the year 1925–1926:

October 6, 1925. "My year in Constantinople" by Dr. F. B. Wiley.
October 20. "Inversions" by Miss Anna Peckham.
December 8. Annual Christmas party.
January 5, 1926. "Large numbers" by Dr. F. B. Wiley.
March 2. "Mathematics and music" by George Stiblitz '26.
March 16. "Logarithms of negative numbers" by Dr. F. B. Wiley.
April 6. Reception—formal opening of the new Mathematics Floor.
April 20. "Alternating current circuits" by Professor Coons.

May 4. "Mathematics and physics" by Professor Howe.
May 18. "Functions of angles" by Miss Mattie Tippit.
June 4. Annual mathematics banquet. Speaker, Dr. Henry Blumberg of the Ohio State University.

The officers for the year 1925–1926 were: President, George Stibitz; vice-president, Emily King; secretary, Josephine Deeds; treasurer, Leland Powell.

The officers for the year 1926–1927 are: President, Leland Powell; vice-president, Thelma Weimer; secretary, Ruth Garrett; treasurer, Thomas Parks.

The Denison Mathematics Club is the oldest and one of the largest departmental societies on the campus. Its purpose is to bring to the students of mathematics an opportunity of acquiring a knowledge of those topics rarely discussed in the classroom.

(Report by Miss Ruth Garrett, secretary)

Pi Mu Epsilon, University of Missouri, Columbia, Missouri.

The officers cf the Missouri chapter of Pi Mu Epsilon for the year 1925–1926 were: Director, Richard Shewmaker '26; vice-director, Vernon Tiller '27; secretary, Nola Lee Anderson, graduate student; treasurer, George Cramer, '26.

The three highest Phi Beta Kappa students were members of Pi Mu Epsilon. Each year the organization gives a prize to the sophomore or junior who makes the highest grade in a special examination on the elementary calculus.

The program for 1925–1926 was the following:

October 13, 1925. "The place of mathematics" by Dr. W. D. A. Westfall.
October 28. "Rigor in mathematics" by Dr. Louis Ingold.
November 10. "Groups" by Dr. E. F. Allen.
November 24. "Applications of groups" by Dr. G. E. Wahlin.
December 8. "The probability integral" by Mr. George Cramer. "Approximation of the witch by Taylor's series" by Miss Edna Robinson.
February 9, 1926. "Vectors" by Mr. F. O. Duncan.
March 9. "Vectors" continued by Mr. F. O. Duncan.
March 23. "Mathematics in Missouri in 1824" by Miss Kathryn Wyant.
April 13. "Vectors as used in electrical engineering" by Professor Weinbach.
April 27. "Symmetrical functions" by Mr. C. G. Yeager.
May 11. "The history and development of the calculus" by Dr. Betz.

(Report by Miss Katherine Wyant)

The Mathematics Club of Brown University, Providence, Rhode Island.

The program for 1926–27 is announced as follows:

November 11, 1926. "How one once reckoned" by Sadiean Kaull Gladding, '27. "Diophantine equations" by Earl Halliday Bradley, '28.
December 14. "Foucault's pendulum experiment" by Miriam Estelle Ware, '27. "Jerome Cardan" by Alfred William Pett, Jr., '28.
January 11, 1927. "Imaginaries in geometry" by William Caspar Graustein, Associate Professor of Mathematics, Harvard University.
February 22. Mathematical paradoxes" by Isabelle Virginia Rowell, '28. "The slide rule" by Homer Pine Smith, '29.
March 22. "Certain relations between polynomials and integers" by Mark Hoyt Ingraham, Assistant Professor of Mathematics, Brown University.
April 26. "The bicentenary of Newton's death" by Julia Ayer Oldham, '28. "The harmonic analyzer" by Oscar Americus Carnevale, '28.
May. Picnic

Committee on Program: Professor Adams; Nellie Chase Morton, '27; Isabelle Virginia Rowell, 28; Aubrey Henderson Smith, Gr.; Robert Sumner Asbury, '28.

Committee on Arrangements: Mr. Lockwood; Mildred Venetia Mott, '27; Ethel McKechnie, '28; Arthur Rockwell Tebbutt, '27; Alfred William Pett, Jr., '28.

(Report by Professor R. C. Archibald)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NORTH CAROLINA, Chapel Hill, North Carolina.

The officers for the session 1925—26 were: M. A. Hill, president; V. A. Hoyle, vice-president; W. V. Parker, secretary-treasurer.

The program for the year consisted of the following:

October 26, 1925. "The area under the probability curve" by Dr. J. B. Tinker.

November 24. "Some problems in the modern geometry of the triangle" by Dr. A. W. Hobbs.

January 26, 1926. "Curvature of the plane curve in terms of differentials" by Dr. J. W. Lasley, Jr.
"A proof by means of complex variables of the following theorem. *If the joins of the vertices of two triangles are concurrent, the meets of the corresponding sides are collinear*" by Dr. A. Henderson.

February 23. "A diophantine problem" by E. T. Browne. "Solutions of the pair of equations $x^2+y=7$, $x+y^2=11$," by L. E. Bush.

(Report by Professor Lasley)

THE EUCLIDEAN CIRCLE OF THE UNIVERSITY OF INDIANA, Bloomington, Indiana.

The program of the Euclidean Circle of the University of Indiana for the year 1925–1926 was the following:

October 5, 1925. "Development of mathematics in Indiana University" by Professor S. C. Davisson. "Mathematics among Chinese students" by Miss Anna Clark.

October 19. "The life and works of Sir Isaac Newton" by Mr. G. C. Campbell. "Laplace, the second Newton" by Miss Helen Pearson.

November 2. "The history of π" by Mr. L. C. Shugart and "A report on the mathematics section of the State Teachers' Association" by Professor Cora Hennel.

November 16. "Lucretius and mathematics" by Miss Alice Abell. "Probability" by Mr. W. J. Kirkham.

December 14. Social meeting.

January 18, 1926. "The relation between mathematics and poetry" by Miss Edith Bauer.

February 15. "Some modern notions of space and time" by Professor H. T. Davis.

March 1. "The summation of series by the use of Bernoulli's numbers" by Miss Irene Price. "Methods of summing series" by Dr. H. A. Zinszer.

March 22. Social meeting.

April 19. "Diffraction patterns" by Professor M. E. Hufford.

May 3. "Mathematical principles of the theory of wealth" by Mr. V. V. Latshaw.

May 17. Annual picnic.

(Report by Miss Arkys Roberts)

THE NEWTONIAN SOCIETY, State College of Washington, Pullman, Washington.

The officers for the year 1925–1926 were: Marion Upton, president; Florence Johnson, secretary-treasurer; Dorothy Webster, reporter.

The following programs were given:

October 8, 1925. "Four-dimensional theory" by Professor E. C. Colpitts.

October 22. "Infinitesimals" by Professor C. A. Isaacs.

November 13. "Life insurance policies" by H. H. Irwin.

December 10. "Einstein's theory" by Professor R. O. Hutchinson.

January 14, 1926. "Archimedes' cattle problem" by Uarda Davis. "The first Alexandrian school" by Audrey Graber.

January 27. "The source of stellar energies" by C. D. Calogeris.
February 25. "The physical basis of the musical scale" by Arthur Bond.
March 11. "Non-Euclidean geometry" by Alice Grant.
March 25. "Mathematical puzzles" by Marion Upton and Fern Bolick.

(Report by Professor C. A. Isaacs)

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Illinois.

The following programs were given by the Club during the college year 1925–1926:

October 29, 1925. "Modern developments in mathematics" by Professor D. R. Curtiss. Election of officers for the first semester: Joe Potchen, Gr., president; Wm. Bauer, '27, vice-president; Louise Stookey, '26, secretary; Rachel Hawks, '27, treasurer; Professor D. R. Curtiss, faculty adviser.
November 12. "Origin of analytic geometry" by Louise Stookey, '26.
December 3. "π" by Professor F. E. Wood.
December 17. "Mathematics of hydraulics" by Melville Larson, Gr.
January 20, 1926. "Mathematical puzzles" by Rachel Hawks, '27.
February 24. "A demonstration of the slide rule" by Miss Brown of the Keuffel and Esser Company.
March 17. "Orbital motion of the Earth" by Malcolm Bauer, '27. Election of officers for the second semester: Melville Larson, Gr., president; Birney Dysart, '27, vice-president; Janet Bonar, '27, secretary; Phyllis Hayford, '26, treasurer; Professor F. E. Wood, faculty adviser.
April 14. "Magic squares" by Lois Girdner, '28.
April 28. "The three-point problem" by Professor Berger; "Explanation of the abacus" by Mr. Kim.
May 12. "Differentials and integrals of non-integer index" by Joe Potchen, Gr.
May 26. "Harmonic wave analysis" by Birney Dysart, '27.
June 16. Picnic at the Forest Reserve.

(Report by Miss Janet M. Bonar, secretary)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Nebraska.

The membership of the Club for the year 1925–1926 numbered one hundred. The following programs were given:

October 8, 1925. Officers for the first semester were elected and the constitution and by-laws read. "Calculating machines" by Professor W. C. Brenke. Demonstrations were given on a number of different machines.
November 12. "The Eleven Check" by Miss Evelyn Wallwey. Dean Engberg spoke on "Mathematical recreations" giving the formulae for working a number of card tricks, and demonstrating how they worked out.
December 10. Mr. Lester Shoemaker told the club "How to draw a straight line" and illustrated his talk with a Peaucellier cell, which he had constructed, and which he later donated to the Mathematics Club. "A proposed duodecimal number system" by Professor A. L. Candy.
January 14, 1926. Mrs. Florence B. Young discussed the "Problem of Apollonius" giving the constructions and proofs of the nine cases leading to the problem. Mr. Sherer talked on "Mathematics in map making" and illustrated his lecture with lantern slides.
February 11. A social meeting of the club was held and the officers for the second semester were elected.
March 19. "Pascal's theorem" by Miss Irma Wiedeman. Professor M. G. Gaba spoke on the game of "Nim" and after demonstrating how the game is played and won, he explained the theory of the game which depends on using a binary system of numbers.
April 15. Professor Deming of the Department of Chemistry gave an illustrated lecture on "Graphic representation of data."
May 13. "The Importance of the Arabs in the history of science" by Mr. William Cejnar. Professor Runge next spoke on "Some graphic methods of computation."
May 19. A social meeting and picnic was held in Antelope Park.

(Report by Miss F. B. Young, secretary)

THE JUNIOR MATHEMATICS CLUB OF THE UNIVERSITY OF CHICAGO, Chicago, Illinois.

The meetings of the Junior Mathematics Club alternate bimonthly with those of the Graduate Mathematics Club. Meetings for the year 1925–1926 were held as follows:

October 28, 1925. "Pythagoras and modern mathematics" by Professor A. C. Lunn.

November 24. "Maps and map making" by Mr. L. S. Johnston.

January 20, 1926. "Some elementary theorems from the theory of algebraic numbers" by Mr. R. H. Marquis.

February 17. "Some elementary theorems from point set theory" by Mr. Raymond Garver.

March 3. "Numerical solution of algebraic equations" by Mr. J. Williamson.

March 20. Social meeting at the home of Professor H. D. Slaught, in which the faculty and students of the department were guests of the club.

April 21. "The Weierstrass theory of irrational numbers" by Mr. R. H. Bardell.

May 5. "A new expression for the remainder in Taylor's formula" by Mr. L. M. Blumenthal.

(Report by Miss Beenken)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, Eugene, Oregon.

The officers for the year 1925–1926 were: Eula Benson, president; Evan Lapham, vice-president; Helen White, secretary; Hubert Yearian, treasurer; Helen Shinn, historian.

The officers for the year 1926–1927 are: Helen Shinn, president, Hubert Yearian, vice-president; Edna English, secretary; Edmund Veazie, treasurer; Gladys McCornack, historian.

The programs were the following:

October 25, 1925. Miss Wave Leslie gave an account of her experiences in China where she had taught in the Canton Christian College.

November 19. Mr. Willis, graduate assistant, now deceased, spoke on "Rational plane curves as geometrical loci."

January 26, 1926. Helen Shinn explained the use of the slide rule and gave a short history of it. Lawrence Laveridge discussed Einstein's theory of gravitation.

April 20. Mr. Rojansky, instructor, explained the use of the surge tank.

Other meetings were held and were given over wholly to business and social affairs.

(Report by Miss Edna English, secretary)

IRRATIONAL CLUB OF THE UNIVERSITY OF WYOMING, Laramie, Wyoming.

The following meetings of the Club were held:

October 9, 1925. Organization meeting at the home of Mr. O. H. Rechard. Business and social program.

October 29. "Mathematical prodigies" by Miss Lillian Portenier.

November 12. "Non-euclidean geometry" by Robert Burns.

November 24. Continued discussion of non-euclidean geometry, by Clark Biesmier.

December 10. "Determination of π" by Philip Pepoon. "Duplication of the cube" by Mark Taylor.

January 14, 1926. "Elementary mathematics in astronomy" by Dr. Gossard.

February 11. "Mathematical fallacies" by Elizabeth Johnson, Frances Colt, and Stephen Anderson.

February 18. Social meeting in Hoyt Hall.

March 12. "Mathematics for commerce students" by Carl Johnson.

March 26. "Applications of the slide rule" by Ben Bellamy, City Engineer.

April 8. "Archimedes" by George Goemmer. "Newton" by Mark Taylor.

April 28. "The nine point circle" by Marion Preator.

(Report by Miss Frances Colt)

THE TULANE MATHEMATICS CLUB, New Orleans, Lousiana.

The program for the year 1925–1926 was the following:

November 18, 1925. Election of officers for the year: Professor H. E. Buchanan, Tulane University,

president; Professor A. M. Howe, Newcomb College, secretary. Paper on "The mortality table and probability of life" by Mr. E. F. Holtzman, assistant actuary of the Pan-American Life Insurance Company.

March 12, 1926. Joint meeting with the Louisiana-Mississippi section of the Mathematical Association of America, and the New Orleans branch of the American Academy of Science.

May 6, 1926. Paper on "Continuous annuities by means of Gamma Functions" by Mr. William Larkin Duren Jr., Tulane University.

(Report by Professor A. M. Howe)

PI MU EPSILON, University of Illinois, Champaign, Illinois.

In addition to two social meetings and the regular bi-monthly business meetings, the following program meetings were held:

November 12, 1926. "Theory of relativity" by Professor R. D. Carmichael.

December 13. "The history of mathematics" by Professor G. A. Miller.

March 11, 1926. "The algebra of logic" by Professor J. B. Shaw.

April 22. "Water particles expanded into steam" by Professor G. A. Goodenough, Department of Engineering.

May 20. "Application of mathematics to astronomy" by Professor R. H. Baker, Department of Astronomy. The following officers were elected for the coming year: Janet Weston, director; J. C. Springer, vice-director; Lydia Hackman, secretary; R. G. Ehman, treasurer; C. F. Robbins, librarian.

(Report by Miss Lydia Hackman, secretary)

THE MATHEMATICS CLUB OF COLUMBIA COLLEGE, New York City.

The Club held two meetings during the academic year 1925–1926, the first in December, the second in January. At the first meeting, Mr. Leonard Sindeband, '27, spoke on the game of "Nim". At the close of this meeting, L. Sindeband '27, A. I. Dumey, '26, and Edward Hymes, '28, were elected members of the Executive Committee. At the second meeting, Mr. A. I. Dumey, '26, spoke on "Inversions".

Dr. J. P. Ballantine and Mr. A. E. Meder acted as faculty advisers.

(Report by Professor M. H. Stone)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3251. Proposed by J. V. Uspensky, Carleton College.

Show that the system $\phi(x, y) = x + y + \sum A_{i,k} x^i y^k = 1$,

$$x\phi_x' - ay\phi_y' = 0, \text{ where } A_{i,k} > 0,\ a > 0,\ 2 \leqq i + k \leqq n,$$

has one and only one solution in positive numbers.

3252. Proposed by the late Laenas G. Weld.

Find the equation of the plane curve along which a point, moving with a velocity in constant ratio to its ordinate, will pass from (x_1,y_1) to (x_2, y_2) in the least possible time.

3253. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through the vertices of a given tetrahedron, planes are drawn parallel to the opposite faces. Prove that the faces of the new tetrahedron thus obtained have for their centroids the corresponding vertices of the given tetrahedron.

Note: The new tetrahedron may, by analogy with the case in the plane, be called the "anticomplementary tetrahedron" of the given tetrahedron.

3254. Proposed by C. M. Sparrow, Rouss Physical Laboratory, University of Virginia.

ABC is a triangle with angles α, β, γ. Pairs of lines are drawn through the vertices, symmetrical with respect to the bisectors and making angles $\lambda\alpha$, $\lambda\beta$, $\lambda\gamma$ with the sides (λ any real multiple). The two lines adjacent to any side determine a point. Prove that the three points so determined form a triangle which is perspective to the original triangle.

3255. Proposed by R. H. Sciobereti, Berkeley, California.

Find among all the curves of given length and passing through two given points that which has the lowest center of gravity.

3256. Proposed by C. N. Schmall, New York City.

Two spheres, with radii R and r, are inscribed in a right circular cone so that the greater touches the smaller and the base of the cone. If v is the volume of the cone, show that $v=2\pi R^5/3r(R-r)$.

3257. Proposed by Albert A. Bennett, Lehigh University.

Prove that if V is any real function, of three variables, x, y, z, satisfying the condition that $(\partial^2 V/\partial x^2)+(\partial^2 V/\partial y^2)+(\partial^2 V/\partial z^2)=0$ and expressible as a term-wise differentiable power series, and if the magnitude of the vector $(\partial V/\partial x,\ \partial V/\partial y,\ \partial V/\partial z)$ is constant, say equal to unity, then the direction of this vector is also constant. Conversely, if the direction is constant, the magnitude is constant. Note that the restriction to real functions is essential for the first part of the problem but extraneous for the second part.

This is stated without proof as apparently obvious in an old physics book.

SOLUTIONS

3179 [3175; 1926, 159]. Proposed by Otto Dunkel, Washington University.

Is the following statement correct?

The intersections of two curves whose equations are given in polar coördinates (ρ, θ) are obtained by solving the two equations simultaneously for ρ and θ.

Solution by M. S. Knebelman, Princeton University.

The above statement is not correct; for any point (ρ, θ) has also the coördinates $[(-1)^n\rho,\ n\pi+\theta]$, where n is an integer. Hence to find the points of intersection of two curves whose equations are given in polar coördinates, replace (ρ, θ) by $[(-1)^n\rho,\ n\pi+\theta]$ in one of them, leaving the other unaltered, and solve simultaneously using all possible integral values of n.

3181 [3177; 1926, 159]. Proposed by Frank Irwin, University of California.

Prove the identity

$$\frac{1}{\sin(y+z)}\left[\sin y+\frac{\sin x \sin z}{\sin(x+y+z)}\right]=\frac{1}{\sin(x+z)}\left[\sin x+\frac{\sin y \sin z}{\sin(x+y+z)}\right].$$

SOLUTION BY A. PELLETIER, Montreal, Canada and BY THE PROPOSER.

If we reduce on the left to a common denominator, the numerator, $\sin(x+y+z)\sin y+\sin x\sin z$,

$$\equiv \tfrac{1}{2}[\cos(x+z)-\cos(x+2y+z)]+\tfrac{1}{2}[\cos(x-z)-\cos(x+z)]$$
$$\equiv \tfrac{1}{2}[\cos(x-z)-\cos(x+2y+z)]\equiv \sin(x+y)\sin(y+z).$$

The left member, therefore, reduces to $\sin(x+y)/\sin(x+y+z)$. From the fact that this is symmetric in x and y the identity follows.

Also solved by W. L. AYRES, THEODORE BENNETT, J. A. BULLARD, A. G. CLARK, B. W. CURRIE, ALICE A. GRANT, MICHAEL GOLDBERG, M. S. KNEBELMAN, HARRY LANGMAN, R. H. SCIOBERETI, and H. S. UHLER.

3182 [3178; 1926, 159]. Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

Find the smallest integral values of x and y which satisfy the equation

$$x^2-67y^2=1\ ,$$

SOLUTION BY H. S. UHLER, Yale University.

The quotients comprised in the first complete period obtained by approximating the binomial surd $\sqrt{67}$ by a continued fraction are 8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16. The tenth convergent is $48842\div5967$ and since it falls in an even place the solution of the question is $x=48842$ with $y=5967$.

Remark. The theory of "indeterminate problems of the second degree," the answer to this problem, and a table of solutions of the equation $p^2-Nq^2=1$ for every value of N from 2 to 102, may be found in Peter Barlow's *Theory of Numbers*, London, 1811.

Also solved by S. A. COREY, MICHAEL GOLDBERG, A. L. MCCARTY, A. PELLETIER, and F. L. WILMER.

3187[3175; 1926, 228]. Proposed by J. B. Reynolds, Lehigh University.

Prove that if the perpendiculars from each of three vertices of a tetrahedron upon the opposite faces meet in a point, the fourth will also pass through that point.

I. SOLUTION BY S. B. LITTAUER, Hunter College

Let $ABCD$ be the given tetrahedron with the perpendiculars to the opposite faces, BX, CZ, DY, meeting in O, and let X, Y, Z be their feet in the faces. The plane BOD is perpendicular to the faces BAC and DAC, and it cuts them in the lines BH and DH, each perpendicular to AC. In a similar manner, the plane BOC gives CG and BG, each perpendicular to AD; and then the plane DOC gives CJ and DJ, each perpendicular to AB. The lines CG and DH are altitudes of the face ACD and they meet in X. Hence the plane BOA cuts this face in the third altitude AXF. Since BX is perpendicular to ACD and XF is perpendicular to CD, BF must be perpendicular to CD. Thus the plane BOA is perpendicular to BCD at F. In a similar manner we show that AOD is perpendicular to BCD. Hence their line of intersection, AO, is perpendicular to BCD. This proves the proposition.

II. SOLUTION BY T. C. ESTY, Amherst College

With any origin let α, β, γ, δ be the position vectors of the four vertices A, B, C, D, and let ρ be the vector of P, the intersection of the perpendiculars from A, B, C. It will be shown that DP is perpendicular to the face ABC by showing that DP is perpendicular to any two sides of the triangle ABC. Since AP is perpendicular to the face BCD, it must be perpendicular to DC and BD. Hence

(1) $(\rho-\alpha)\cdot(\delta-\gamma)=0,$ (2) $(\rho-\alpha)\cdot(\beta-\delta)=0.$

In a similar manner we obtain the equations

(3) $(\rho-\beta)\cdot(\delta-\alpha)=0,$ (4) $(\rho-\gamma)\cdot(\alpha-\delta)=0.$

The two pairs of equations (1), (4) and (2), (3) give by addition

$$(\rho-\delta)\cdot(\alpha-\gamma)=0, \qquad (\rho-\delta)\cdot(\beta-\alpha)=0.$$

This completes the proof.

If we add (3) and (4), we obtain $(\beta-\gamma)\cdot(\alpha-\delta)=0$, and this shows that the opposite edges BC and AD are perpendicular. In the same way it may be shown that each of the remaining two pairs of edges are perpendicular.

NOTE BY THE EDITORS: A slight modification of either of the above two proofs shows that: In any trihedral angle the three planes, each of which passes through an edge perpendicular to the opposite face, meet in a straight line through the vertex. This result may be used to establish the theorem of the problem. It is also useful in a proof of 3188 [3176, 1926, 228].

Also solved by NATHAN ALTSHILLER-COURT, MICHAEL GOLDBERG, HARRY LANGMAN, A. PELLETIER, and the PROPOSER.

3188[3176; 1926, 228]. Proposed by J. B. Reynolds, Lehigh University.

Through the vertex, O, of a tetrahedron, $OABC$, are passed three planes perpendicular respectively to OA, OB, and OC. Let BC (extended if necessary) cut the first plane in F, CA cut the second in G, and AB cut the third in H. Prove FGH a straight line.

SOLUTION BY HARRY LANGMAN, Brooklyn, N. Y.

Let O-RST be any trihedral angle, and let the plane RST be perpendicular to OR. Draw $SV\perp RT$ and $TW\perp RS$, cutting SV in Z. Draw RZU, cutting ST in U. Draw OU, OV, OW. The line SV is perpendicular to the plane ORT. Hence, the plane OSV is perpendicular to the plane ORT. Similarly $OTW\perp ORS$. Now $RU\perp ST$ and $RU\perp OR$. We have

$$SU^2+OU^2=SU^2+RU^2+OR^2=RS^2+OR^2=OS^2.$$

$\therefore$ $OU\perp SU$. $\therefore$ Plane $ORU\perp SU$, hence plane $ORU\perp$plane OST. Hence the planes through the edges of a trihedral angle perpendicular to the opposite faces have a common line of intersection. It follows that the altitudes of a spherical triangle have a common point of intersection—an orthocenter.

The three perpendicular planes form a second trihedral angle O-$A'B'C'$, such that OC' is perpendicular to OAB and OC is perpendicular to $OA'B'$. Hence, the plane of OCC' is perpendicular to both OAB and $OA'B'$. The same thing is true of OBB' and OAA'. These three planes meet in a straight line OZ. The planes OBC and $OB'C'$ meet in a straight line OF which must be perpendicular to both OA and OA'. Hence OF is perpendicular to OZ which lies in the plane OAA'. In the same way it is shown that OG and OH are perpendicular to OZ, and hence these three lines lie in the same plane perpendicular to OZ. This plane cuts the plane of ABC in a straight line which contains the points F, G, H.

Also solved by NATHAN ALTSHILLER-COURT, THEODORE BENNETT, T. C. ESTY, A. PELLETIER, MICHAEL GOLDBERG, and the PROPOSER.

3190[3178; 1926, 229]. Proposed by B. C. Wong, Berkeley, California.

On OP, the radius vector of any point on the cardioid $\rho=2a(1-\cos\theta)$, as diagonal a rectangle $OMPN$ is constructed with the sides OM and ON bisecting the angles between the radius vector and the axis of the cardioid. Find the loci of M and N and give their constructions without the use of the cardioid.

SOLUTION BY MICHAEL GOLDBERG, Washington, D. C.

The equation of M is $\rho'=2a(1-\cos 2\theta')\cos\theta'=4a\sin^2\theta'\cos\theta'$.

The equation of N is $\rho''=-2a[1-\cos(2\theta''-\pi)]\cos\theta''=-4a\cos^3\theta''$.

These equations suggest the following simple geometrical constructions for points of the loci. Lay off $OA=4a$ on one side of the angle θ. Draw AB perpendicular to the other side, BC perpendicular to OA, CD perpendicular to AB, and CE perpendicular to OB. Then $CD=4a\sin^2\theta\cos\theta=\rho'$, and $OE=4a\cos^3\theta=\rho''$. These values of ρ' and ρ'' are associated with the angle θ.

3191[3179, 1926, 229]. Proposed by Einar Hille, Princeton University.

The differential equation $y'=y^{2/3}$ has two independent solutions which pass through the origin, namely $y\equiv 0$ and $y=\frac{1}{27}x^3$. Prove that these solutions can be obtained by the method of successive approximations using as first approximation in the former case $y_0=0$, in the latter case $y_0=x^{a_0}$ where a_0 is any positive constant (x may be assumed to be positive).

SOLUTION BY THE PROPOSER.

The successive approximations are determined by the set of equations

$$y_n(x)=\int_0^x[y_{n-1}(t)]^{2/3}dt,\quad n=1,2,\cdots$$

Substituting $y_0(x)=0$, we get $y_1(x)\equiv 0$, hence $y_n(x)\equiv 0$ for every value of n, and $\lim y_n(x)\equiv 0$ which is the first solution.

On the other hand, substituting $y_0(x)=x^{a_0}$, $a_0>0$, we get $y_1(x)=(\frac{2}{3}a_0+1)^{-1}x^{2/3a_0+1}$ and by complete induction $y_n(x)=b_n{}^{-1}x^{a_n}$, in which $a_n=ra_{n-1}+1$ and $b_n=a_1{}^{\beta_1}\,a_2{}^{\beta_2}\,a_3{}^{\beta_3}\cdots a_{n-1}^{\beta_{n-1}}\,a_n^{\beta_n}$, $(n\geqq 1)$, where $\beta_k=r^{n-k}$ and $r=2/3$.

It remains to prove that $a_n\rightarrow 3$ and $b_n\rightarrow 27$. We start with a_n. If $a_0=3$, we have $a_1=3$ and consequently $a_n=3$ for every n. Now assume $a_0\neq 3$; then $a_1\neq 3$ and lies between a_0 and 3. Since $a_n-a_{n-1}=\frac{2}{3}(a_{n-1}-a_{n-2})$ for every n, we conclude that the sequence a_n is monotonic, increasing or decreasing according as $a_0<3$ or $a_0>3$. In the former case, 3 is an upper bound of a_n, in the latter case a lower bound. Hence, $\lim a_n$ exists; since $\lim a_n=\frac{2}{3}\lim a_{n-1}+1$, we have $\lim a_n=3$.

We have $b_n=a_n\,b_{n-1}{}^{2/3}$ or, putting $\log a_n=c_n$, $\log b_n=d_n$, $d_n=\frac{2}{3}d_{n-1}+c_n$. Hence $d_n-d_{n-1}=\frac{2}{3}(d_{n-1}-d_{n-2})+(c_n-c_{n-1})$. Now if $a_n\leqq 3$, $c_n-c_{n-1}\geqq 0$. Thus the d_n sequence is monotonic increasing. But in such a case

$$d_n<\tfrac{2}{3}d_n+c_n<\tfrac{2}{3}d_n+\log\ 3\ \text{ or }\ d_n<3\ \log\ 3.$$

It follows that $\lim d_n$ exists and $\lim d_n=\frac{2}{3}\lim d_{n-1}+\lim c_n$ or $\lim d_n=3\log 3$. Now suppose that $a_0>3$. If the sequence d_n is monotonic increasing the same reasoning applies as in the case $a_0\leqq 3$ with the exception of the estimate of the upper bound where we get $d_n<3\,c_0$ instead of 3 log 3. The limit of d_n is of course the same. A similar reasoning applies if the sequence d_n is monotonic increasing from a certain point onwards. On the other hand, if the sequence d_n is not increasing from a certain point on, there must exist an n for which $d_n<d_{n-1}$, say $n=k$. But $c_{k+1}-c_k<0$; hence, also $d_{k+1}-d_k<0$ and we find successively that $d_n-d_{n-1}<0$ for $n\geqq k$. Thus the sequence is monotonic decreasing from a certain point on; since $d_n>0$ for every n, $\lim d_n$ exists, its value of course being 3 log 3.

Hence, $\lim_{n\to\infty} y_n(x)=\lim_{n\to\infty} b_n^{-1}x^{a_n}=\frac{1}{27}x^3$, if $a_0>0$.

3192[3180; 1926, 229]. Proposed by J. Rosenbaum, Milford, Connecticut.

Given the angles of an inscribed polygon and the radius of circumscribed circle to construct the polygon.

DISCUSSION BY NATHAN ALTSHILLER-COURT, University of Oklahoma.

What is wanted is to inscribe in a given circle (O) a polygon having given angles. It is assumed that the order in which these angles are to appear is also given. An elementary solution of this problem depends upon the following:

LEMMA. If two broken lines $ABC\cdots$ and $A'B'C'\cdots$ with the same number of sides, N, are inscribed in the same circle so that their sides are respectively parallel, then their closing lines are parallel if N is odd, of equal length if N is even.

The proof is immediate, if we observe that $AA'=BB'=CC'=\cdots$

Suppose now that we want to inscribe a polygon with an odd number of sides, say, a pentagon. Take the line $A'B'$ arbitrarily as a chord of (O), and then draw the chords $B'C'$, $C'D'$, $D'E'$ so that these four chords will make with each other in succession the given angles B', C', D'. Draw the circle (O') concentric with (O) and tangent to $A'E'$. Now inscribe in (O) an angle of the given size A so that one of its sides, AE, is tangent to (O'), and the other, AB, is parallel to $A'B'$. Then draw BC parallel to $B'C'$, CD parallel to $C'D'$. The parallel to $D'E'$ through D will pass through E, since $AE=A'E'$, by construction.

Observe that the size of the angle DEA is determined by the construction; hence, only four of the five angles of the pentagon may be given in the conditions of the problem. This could be seen a priori.

NOTE: If the four given angles have the sum of 360°, the pentagon is degenerate.

If the polygon has an even number of sides, say, six, we may proceed with the construction as above. But the direction of the closing line AF being fixed, according to the lemma, both the angle F and the angle A will be determined by the construction, so that only four of the six angles may be given. The line AF may then be taken arbitrarily, and the problem has an infinite number of solutions, for a given direction of the line $A F$.

NOTE: The proposed problem may also be stated as follows: In a given circle, to inscribe a polygon so that its sides shall have given directions. Problems of a closely related nature are discussed by E. Catalan in *Théorèmes et problèmes de géométrie élémentaire*, (Dunod, Paris, 1879), pp. 220–225. The problem may be generalized by interpreting *direction* as *point at infinity*, then taking these points at a finite distance, and not necessarily collinear. A very elegant solution, by the method of inversion, of the problem thus stated is to be found in J. Petersen's, *Methods and theories for the solution of problems of geometrical construction*, (Stechert, New York).

The problem may further be generalized by substituting a conic for the given circle. This problem is solved by Poncelet in his famous *Traité des propriétés projectives des figures*, (Paris, 1822), pp. 349–357. Poncelet gives a number of bibliographical references concerning the status of this problem up to his time. Another solution, also based on projective considerations, although different from those of Poncelet was given by M. Chasles in his *Traité des sections coniques* (Gauthier-Villars, Paris, 1865), page 159.[1]

Also solved by MICHAEL GOLDBERG and HARRY LANGMAN.

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

[1] See also M. Simon, *Uber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert* (Teubner Leipzig), pp. 105 etc.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

On the inside of the back cover of this issue of the Monthly is printed a list of "Suggestions to Authors." The editors request that, in the future, authors prepare their manuscripts with those suggestions in mind.

The attention of chairmen of departments is called to the fact that a number of candidates are already registered in the files of the "Information Bureau for Appointments" under the direction of Professor Kuhn. (See page 221.) It is hoped that this bureau will be of real service to departments of mathematics.

The date of the eleventh summer meeting of the Association is Monday and Tuesday, September 5–6, 1927. There will be sessions of the Association in the afternoon of each of these days. The Society will also hold two sessions, namely on Thursday afternoon and Friday morning. The colloquium lectures will be given on Tuesday, Wednesday, Thursday and Saturday mornings and Thursday evening. Wednesday and Friday afternoons are reserved for lake excursions. The joint dinner will be on Wednesday evening. The complete program will be mailed to all members as early as possible in August. Meanwhile we may all mark our calendars for Madison—September fifth to tenth.

Attention should be called again to the basis of cooperation between the Association and the *Annals of Mathematics*. For the past ten years the Association has assisted in subsidizing this journal and in return the Annals has increased the size of its volume by approximately one hundred pages consisting largely of expository papers and has also reduced its subscription price from three dollars to one dollar and fifty cents to members of the Association. Two hundred and eighty members are now taking advantage of this reduced rate privilege. Doubtless many others will be glad to do so when again reminded of the opportunity. Now is an especially opportune time since the Annals is to be still further enlarged through additional subsidy provided by the National Research Council.

The Program and Arrangements committee for the Madison and Nashville meetings are already busily engaged in laying their plans. This is the second occasion when the combined meetings and colloquium have been held at Madison and doubtless there will be a large attendance both on account of the richness of the programs to be offered and because of the many attractions which Madison and its environs provide for those who wish to spend a week in September in such beautiful surroundings. The Nashville meeting, however,

which will be the Annual Meeting of the Society and the Association will be the first to be held so far South. The chief desire of the officers of both organizations in choosing this location was to afford an opportunity for every member of these mathematical bodies in the South and Southwest to attend one of the great national meetings and to share in the inspiration and enthusiasm which such gatherings afford. In order to fully meet these aims it is necessary for large numbers from the North and East to be in attendance but it is especially important that every member from the southern territory should begin to plan now to spend the holiday week in Nashville and to pass the word along to those who are not as yet members but will welcome the invitation to become members and to share in the fellowship which this occasion will afford. The opportunity is unique in another respect in that there will be reduced railroad rates to Nashville from all parts of the country due to the large numbers who will be in attendance upon the meetings of the American Association for the Advancement of Science.

The award of fellowships at the University of Chicago for 1927–28, which has just been announced, contains the names of eight persons who have been appointed in the department of mathematics as follows: ABRAHAM ADRIAN ALBERT, S.B., University of Chicago; MAY MARGARET BEENKEN, Ed.B., University of California, A.M., University of Chicago; BURTON WADSWORTH JONES, A.B., Grinnell College, A.M., Harvard University; EDWARD JAMES MCSHANE, B.E. and S.B., Tulane University; DAVID CLARENCE MORROW, A.B., University of Manitoba, A.M., University of Toronto; GORDON PALL, A.B., University of Manitoba; CHARLES ANDREW RUPP, A.B. and A.M., Harvard University; ROBERT CLARENCE SHOOK, A.B., Miami University.

The National Council of Teachers of Mathematics issued last year its First Year Book on "A General Survey of Progress in the Past Twenty-five Years." It was a book of 200 pages containing nine articles among which was a reprint of E. H. Moore's presidential address on "The Foundations of Mathematics." The Second Year Book has just been published on "Curriculum Problems in Teaching Mathematics" under the editorship of W. D. Reeve of Teacher's College, Columbia University. It contains ten articles covering 297 pages and is divided into three parts dealing with (1) Arithmetic, (2) Junior High School Mathematics, (3) Senior High School Mathematics. Either of these volumes may be purchased of Mr. C. A. Austin, Oak Park High School, Oak Park, Illinois, or of the Bureau of Publications, Teachers College, Columbia University, New York City.

Professor G. D. BIRKHOFF of Harvard University has been elected an honorary member of the Edinburgh Mathematical Society.

Professor J. L. COOLIDGE of Harvard University has been appointed Exchange Professor to France for 1926–27, his term of service to fall in the second half year.

Dr. M. H. STONE of Columbia University has been appointed instructor and member of the Faculty at Harvard University.

Dr. H. W. BRINKMANN of Harvard University has been promoted to an assistant professorship of mathematics.

Associate Professor O. D. KELLOGG of Harvard University has been appointed to a full professorship of mathematics.

Dr. M. S. DEMOS has been appointed instructor at Harvard University.

The following additional courses in mathematics are announced for the summer of 1927.

Brown University. June 20 to July 30. By Professor R. E. LANGER: Theory of integral equations. By Professor M. H. INGRAHAM: Finite groups and the Galois theory of equations.

Cornell University. July 5 to August 13. By Professor J. I. HUTCHINSON: Modern higher algebra. By Professor W. A. HURWITZ: Advanced calculus. By Professor W. B. CARVER: Analytic projective geometry. By Professor D. C. GILLESPIE: Projective geometry. By Professor C. F. CRAIG: Elementary differential equations. Reading and research courses are offered by Professor J. I. HUTCHINSON, Professor VIRGIL SNYDER, Professor F. R. SHARPE, Professor W. A. HURWITZ, Professor W. B. CARVER, Professor D. C. GILLESPIE, and Professor C. F. CRAIG.

University of Missouri. In addition to the usual elementary courses, the following advanced courses are offered: By Professor INGOLD, Modern geometry; Selected topics from geometry. By Professor WAHLIN: Advanced algebra; Selected topics from algebra. By Professor WESTFALL: Calculus; Selected topics from analysis.

University of Wyoming, first term, June 13 to July 20. By Professor O. H. RECHARD: College algebra; Projective geometry; Differential equations. By Professor C. F. BARR: Calculus; Intermediate algebra. Second term, July 21 to August 26. By Professor RECHARD: Calculus; Seminar. By Professor BARR: Solid geometry; Trigonometry; College geometry.

Professor FRANK H. LOUD, for fifty years professor of mathematics and astronomy at Colorado College (emeritus professor since 1907) died March 2, 1927, at St. Petersburg, Florida. He was seventy-five years old.

Sir GEORGE GREENHILL died February 17, 1927.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.
BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.
Twelfth Annual Meeting, Nashville, Tenn., December, 1927.
The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.
INDIANA, De Pauw University, April 29-30.
IOWA, University of Iowa, May 6-7.
KANSAS, Topeka, Kan., February 5.
KENTUCKY, Lexington, May 7.
LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, April.
MINNESOTA, St. Peter, Minn., May 21.
MISSOURI, St. Louis, Mo., November 25-26.
NEBRASKA, Lincoln, May 14.
OHIO, Columbus, Ohio, April 8.
PHILADELPHIA, Philadelphia, Pa., November.
ROCKY MOUNTAIN, Colorado College, April 22-23.
SOUTHEASTERN, Columbia, S. C., April 15-16.
SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.
TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formula. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: /̸1. Similarly, 0 means "zero"; to indicate "cap O", backspace and overprint period; thus: ʘ .
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\,\beta\,\gamma \cdots$ (all) $\mathrm{A\ B}\ \Gamma \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\,\beta\,\gamma \cdots}$ (all except ι and o).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma \cdots}$ (all except ι and o);

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \varrho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q\ A\ B\ C\ D\ E\ F\ G\ H\ J\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$

* Boldface German—$\boldsymbol{\mathfrak{d\ A\ B\ D}}$

Script (special font) $\mathcal{A\,B\,C} \cdots$(all). No lower case manufactured.

* Hebrew—א א ב ב troublesome to handle.

★ Dashed Italics—$\bar{A}\ \bar{a}\ \bar{B}\ \bar{b}\ \bar{C}\ \bar{c}\ \bar{E}\ \bar{e}\ \bar{F}\ \bar{f}\ \bar{G}\ \bar{g}\ \bar{H}\ \bar{I}\ \bar{\imath}\ \bar{K}\ \bar{k}\ \bar{M}\ \bar{m}\ \bar{N}\ \bar{n}\ \bar{P}\ \bar{p}\ \bar{q}$
* $\bar{R}\ \bar{r}\ \bar{s}\ \bar{t}\ \bar{u}\ \bar{v}\ \bar{X}\ \bar{x}\ \bar{Y}\ \bar{y}\ \bar{Z}\ \bar{z}$

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{p}\ \bar{\omega}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

---

* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 5, MAY

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE ASSOCIATION AND ITS SECTIONS

By H. E. SLAUGHT, University of Chicago

It must be a source of great satisfaction to all those who have closely observed the development of the Mathematical Association of America during the first decade of its history to see the steady and substantial development of its Sections. It will be recalled that Ohio and Missouri were contestants for the honor of securing the first charter for a section and that Ohio won by the margin of a few minutes, both petitions being presented within an hour after the final adoption of the constitution at the organization meeting of the Association in Columbus, Ohio, in December, 1915.

Since that time the number of sections has grown to seventeen, the latest being the Philadelphia Section intended to serve about one hundred members residing in contiguous portions of Pennsylvania, New Jersey and Delaware. All of the sections hold yearly meetings and some of them twice yearly with an average attendance of about twenty-five members, the highest numbers recently recorded being 51, 32, 31, 29, ... and the lowest 11, 16, 17, 21 . . . . .

Aside from members the attendance is usually swelled by about one-half, consisting of visitors from allied departments and students of mathematics in the institution acting as host. The presence of this latter group is especially important in that it forms a bond of union between the Association and the clientéle from which future recruits for mathematics are to be drawn. Moreover the president or dean of the host institution usually welcomes the visitors and attends the dinner thus adding significance to the occasion and giving prominence to mathematics among all the institutions represented. Indeed, the host institution, or some organization in the town not infrequently gives a complimentary dinner or luncheon to the Section—and the local papers give corresponding publicity. For example, at the recent meeting of the Louisiana-Mississippi Section, the Shreveport Chamber of Commerce gave a complimentary dinner to over ninety persons consisting of members of the Section and of secondary teachers who had met to organize a Branch of the National Council. The Shreveport *Journal* gave nearly three columns of publicity to the various phases of the meeting.

During the ten years, the sections, as fast as organized, have "carried on" *continuously*, with only *two* exceptions. In each of these cases an interruption of one year occurred owing to failure of officers to function efficiently. It is believed now that in both of these cases the proper stimulus has been applied and the result will be continuity of action from now on. A study of the programs of the section meetings as reported in the MONTHLY is interesting and significant.

An average of about seven papers have been presented at the meetings during 1925 and 1926, the lowest number being five and the highest nineteen. In some cases discussion has centered around local conditions such as state requirements in the high schools and admission to the colleges and state universities. In other cases valuable expository papers have been presented on various topics. In still other cases new processes or new results have been exhibited. An inspection of these programs affords an illuminating cross-section of the activities of the sections. And not the least important feature of these meetings is a by-product in the form of mutual acquaintanceship among colleagues and mutual exchange of views among those engaged in similar work.

If we measure the influence of the Association by the attendance of its members at meetings and the presentation of papers, we see at once the importance of the sections. At the two national meetings the total attendance has been around 250 (it was over 300 in 1926 because of the record attendance at Philadelphia), while the total attendance at sectional meetings has been over 400. At the two national meetings about 16 papers (total) are usually presented while at the sectional meetings over 100 papers are presented yearly. If, in addition, we consider the cumulative and combined influences resulting from seventeen sectional meeting places and host institutions as against two for the national meetings, we see again the importance of the sections. Moreover, there is little prospect that the attendance at national meetings will become much larger than at present owing to the great distances to such centers from the more remote parts of the country, whereas it is not impossible that the total attendance at section meetings may be greatly increased—even doubled or trebled—as the interest continues to grow,—especially in view of the fact that stone roads are rapidly spreading to all parts of the country so that more and more members may reach their sectional meeting places in their own conveyances or in public busses.

Naturally, the question arises: What can we do to increase the scope and usefulness of the sections of the Association? One answer to this question is to assist them in improving their present activities wherever such assistance is needed. Some of the sections are much more favorably located than others. Some have a far larger quota of members than others. Some have a much larger geographical distribution than others. Evidently the Association should do all in its power to encourage the sections which are laboring under disadvantages. For instance, it is a great stimulus to those sections to have some outstanding representative of the Association attend their meetings. This has been done occasionally—sometimes at the expense of the section, sometimes at the expense of the Association and sometimes dividing the expense between them. Professor Coolidge, when president, proposed that the Association should send a delegate to every section meeting, but the matter was held in

abeyance because the funds of the Association were not sufficient to finance such a project. The question was again discussed at the Philadelphia meeting and a special committee was appointed to consider it in connection with the related question of giving expository lectures at section meetings. As a start in this direction it was agreed that it would be very stimulating to the sections if each author of a Carus Monograph could deliver one or more lectures on his subject at each of the section meetings, or at least at a number of such meetings especially of those sections which are most disadvantageously located. It is hopéd that the Carus Fund may eventually be adequate to finance this procedure and that such a beginning may grow into a more extended series of expository lectures each year before the sections.

Another answer to the question of increasing the usefulness of the sections is to give them some important objective toward which to direct their energies. Such an objective was also suggested by Professor Coolidge when he negotiated the affiliation agreement between the Association and the New England Association of Teachers of Mathematics, an organization concerned chiefly with the secondary field. By this action our Association reaffirmed its vital interest in secondary education, having already emphatically proclaimed that fact when, at the very beginning of its career, it sponsored the National Committee on Mathematical Requirements and actually appointed the chairman, Professor J. W. Young, and the original nucleus of six members of the committee with power to enlarge their number. The report of that committee, everywhere recognized as an Association product, after being distributed to more than 25,000 teachers is still in active demand and has probably produced a more far-reaching effect on secondary teaching than any document of its kind in this generation. With the completion of the work of this committee, the responsibility for carrying on its spirit and influence seemed to fall upon the National Council of Teachers of Mathematics, a federation of secondary organizations, one of which is the above mentioned New England Association.

The foregoing statement of facts throws into clear light the two-fold opportunity and responsibility of the Association. On the one hand it stands squarely in the collegiate field for better teaching, for better mathematical understanding, for wider dissemination of mathematical knowledge and interest through the medium of exposition, especially as embodied in the idea of the Carus Monographs, and for the beginnings of research so essential to the highest success in teaching and so important a stimulus to later scientific activity. In all of these respects the Association has held to its ideals throughout the past decade and by its close cooperation with the Annals of Mathematics and with the American Mathematical Society, as well as in many other ways, it has, by common consent, contributed strongly to the general forward movement of mathematical interests in this country. On the other hand the Associ-

ation, as already explained, has used its strong influence for the betterment of mathematical conditions in the secondary field, not by usurping authority nor by supplanting any of the organizations in that field, but by acting as adviser and friend to their representatives and sitting with them to work out in tolerant cooperation the best solution of the difficult problems which confronted that National Committee. It was, indeed, an inspiring sight to observe the sessions of that body composed, as it was, of university, college and secondary teachers, as they together weighed and tested every idea or plan that was presented.

And now, in the same fraternal spirit, it has occurred to some of us that an opportunity is open for cooperation between the sections of the Association and the Branches of the National Council. It would seem that meetings of these two organizations held at the same time and place, the sessions extending over a Friday and Saturday, with a joint dinner on Friday evening, would result in definite advantages to both groups. In particular it would give each group an opportunity to become acquainted with difficulties confronting the other and to exchange suggestions and remedies. But most important, perhaps, would be the opportunity for mutual acquaintanceship between the two groups and hence for mutual understanding and sympathy. So obvious seem the possibilities of such a scheme that the Trustees at Philadelphia voted to approve a tentative plan proposed for the Louisiana-Mississippi Section as follows: (1) To urge all secondary teachers of mathematics in these two states to meet with the Section at Shreveport on March 4–5 for the purpose of forming a Branch of the National Council. (2) To invite any secondary teachers who might wish to join the Association to do so without the customary two dollar initiation fee, provided they are already members of the National Council and are thus using the two dollars for subscriptions to the *Mathematics Teacher*, the official journal of the Council. This agreement is the same as that already existing with the New England Association, except that we have no section of our Association in New England and hence can meet with them only when the national meetings are held in New England territory.

Great credit is due to Professor S. T. Sanders of Louisiana State University, Chairman of the Louisiana-Mississippi Section of the Association, for the effective way in which he directed the publicity campaign for the meeting. The attendance of secondary teachers was good but not at all commensurate with the effort put forth in their behalf. Nevertheless, considering the fact that the National Council had heretofore been little known to most of them, it was pleasant to observe the serious and earnest spirit of those present and the effective way in which they proceeded to perfect their organization. They were doubtless inspired by the presence of Miss Marie Gugle, assistant superintendent of schools, Columbus, Ohio, president of the National Council, and Professor W. D. Reeve of Teacher's College, Columbia University, joint editor of the

Mathematics Teacher, both of whom took prominent parts in the program. The Association was also represented by Professor H. E. Slaught as official delegate appointed by the Trustees. It seems certain that the influence of such a meeting must result in the widening and deepening of mathematical interest in those two states. Professor Sanders was reelected chairman of the Section and he proposes to continue the good work during the coming year.

At least one other section has been thinking of a similar cooperation with the National Council and still others may wish to consider its desirability. Also there are other secondary organizations in locations where there are no sections of the Association, some of which might wish to affiliate with us as in the case of the New England Association. Again there are portions of the country where new sections of the Association might well be established. All of these possibilities seem destined to be tried out as the Association grows in power and influence. Finally, the sections have a great responsibility for maintaining and steadily increasing the membership of the Association. A good slogan for any section would be: "One hundred percent membership," meaning that every person engaged in teaching mathematics in any collegiate institution within the territory of the section should be a member of the Association. This slogan might well apply also to institutional memberships. The total membership (individual and institutional) is now well over 2,000 but this slogan fully realized in all the sections would swell the number to 3,000 or more, and correspondingly increase the potential activities.

THE FIFTH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fifth regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of California at Los Angeles, on Saturday, March 12, 1927. Professor H. C. Willett presided.

There were fifty present, inculding the following thirty-four members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, H. Bateman, Clifford Bell, E. T. Bell, R. W. Bolton, W. D. Cairns, J. R. Campbell, M. Collier, P. H. Daus, J. B. Ernsberger, H. H. Gaver, H. E. Glazier, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, G. R. Livingston, W. E. Mason, A. McClellan, B. Podolsky, L. E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, M. Skarstedt, D. V. Steed, F. C. Touton, H. C. Van Buskirk, H. C. Willett, Clyde Wolfe, E. R. Worthington.

The meeting began with a luncheon at the Newman Club. Professor W. D. Cairns, Secretary-Treasurer of the Association, was the guest of the Section, and after the luncheon, told us something of the aims and plans of the Association.

The following officers were elected for the coming year: Professor W. P. RUSSELL, chairman, Professor E. T. BELL, vice chairman, Professors O. W. ALBERT and L. D. AMES, program committee.

The program consisted of the following papers. Short abstracts appear below.

1. "Symmetric functions in the algebra of logic," by Mr. MORGAN WARD, California Institute of Technology.

2. "On the concepts of area and volume," by Professor E. R. HEDRICK, University of California at Los Angeles.

3. "A matrix notation for m-dimensional geometry," by Professor L. D. AMES, University of Southern California.

4. "Unsolved problems in the theory of periodic functions," by Professor E. T. BELL, California Institute of Technology.

5. "A set of construction problems related to the problem of Apollonius," by Professor P. H. DAUS, University of California at Los Angeles.

6. "Functions of closest approximation on an infinite range," (by invitation), by Professor W. D. CAIRNS, Oberlin College.

1. Mr. Ward defined the logical symbols $A+B$ and $A\times B$ with reference to classes and explained the fundamental operations in such an algebra. He proceeded to discuss the relations between the elementary symmetric functions of the roots of an equation in this algebra, indicating how serial order can be defined. He showed the relations between the symmetric functions of the roots of two equations and indicated how they led to certain problems in Diophantine analysis.

2. Professor Hedrick discussed a set of axioms concerning area, which force one to the Riemannian definition of area. They are: (1) The area of a rectangle is the product of the base and its altitude; (2) The area of a figure formed by a finite number of rectangles is the sum of their areas; (3) Of two regions, of which one is interior to the other, the inner one has not the greater area. He indicated how these axioms assume too much, and how they may be replaced by assumptions which enable one to apply these concepts to some of our modern definitions of space, as for example, a Fréchet space.

3. Professor Ames considered the matrix of m points $(x_i, y_i, z_i, \cdots)$. If the rank of the matrix is $1, 2, 3, \cdots$, then the matrix represents a point, line, plane, etc. He indicated how certain problems could be conveniently solved using this notation.

4. Professor Bell discussed several problems in the theory of multiply-periodic functions, which are unsolved in the sense that the so-called solutions refer to problems in another branch of mathematics for which we do not have solutions. Among these problems are the following: (1) How many solutions does the Diophantine equation $m=x_1^2+\cdots+x_r^2$ have? If $r<8$ and odd, the answer

is a complicated transcendental function, which is a mathematical but not a practicable solution. (2) Weierstrass's first problem, that is, to expand $sn(u, K^2)$ in a power series in u. The coefficients are polynomials in K, which are considered as known, when their determination has been referred to a problem in linear difference equations. He also discussed the evolution of certain theta functions, and the problem of expanding doubly-periodic functions in two variables, indicating how the so-called solutions were not really solutions.

5. This paper will appear in the MONTHLY.

6. Professor Cairns extended to the interval $(-\infty, \infty)$, a theorem by Professor Dunham Jackson given in Vol. 22 of the Transactions of the American Mathematical Society. If $f(x)$ is continuous for finite x and vanishes at $\pm\infty$ like $|x|^n e^{-x^2/2}$, n a definite positive number, there exists a set of constants c_i, for which we may approximate to $f(x)$ by $\phi(x)$, a linear combination of continuous and linearly independent functions $p_i(x)$ and limited at $\pm\infty$ like $f(x)$, the approximation being such that the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\left[\frac{f(x)-\varphi(x)}{e^{-x^2/4}}\right]^m dx$$

attains a minimum, whenever $m>1$.

P. H. DAUS, *Secretary*

THE TWENTIETH REGULAR MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The twentieth regular meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held on Saturday, December 4, 1926 at the United States Naval Academy, Annapolis, Md., the morning session opening at 11 A.M. and the afternoon session at 2 P.M. Those attending the meeting were guests of the Annapolis members at luncheon. Chairman J. A. Bullard presided at both sessions.

There were 60 present, including the following 41 members of the Association. O. S. Adams, R. N. Ashmun, H. G. Avers, W. W. Bigelow, G. A. Bingley, L. M. Blumenthal, C. C. Bramble, J. A. Bullard, P. Capron, J. R. Clements, J. A. Duerksen, J. B. Eppes, P. J. Federico, G. L. Fentress, M. Goldberg, W. M. Hamilton, G. W. Hansen, H. P. Kaufman, A. E. Landry, C. L. Leiper, Florence P. Lewis, E. S. Mayer, F. Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, E. C. Philips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, J. A. Schad, J. T. Spann, T. H. Taliaferro, A. A. Tomeldon, Marion M. Torrey, J. Tyler, J. E. Willis.

The program follows, accompanied by abstracts of the papers:

1. "The theorems of Menelaus and Ceva and their extension," by Dr. P. WERNICKE, U. S. Patent Office, by invitation.

2. "Invariants under rotation" by Professor F. MORLEY, Johns Hopkins University.

3. "Numerical integration" by Assistant Professor L. M. KELLS, United States Naval Academy.

4. "Modern navigation" by Commander A. M. R. ALLEN, United States Naval Academy, by invitation of the section.

5. "The Byrd polar flight" by H. G. AVERS, United States Coast and Geodetic Survey.

6. "A problem in maxima and minima" by Professor PAUL CAPRON, United States Naval Academy.

7. "A monogram designed to simplify certain geodetic calculations" by H. S. RAPPLEYE, United States Coast and Geodetic Survey, by invitation.

8. "Functions formed by continuous substitution" by Professor JOHN TYLER, United States Naval Academy.

1. Two triangles in a plane determine by the joins of their corresponding vertices an "imperspective central triangle" and by the intersections of corresponding sides an "imperspective axial triangle." Calling the product of the ratios in which one triangle is divided by the joins of vertices, its "menelaus," and the product of the ratios in which its sides are divided by the intersections with those of the other triangle, its "ceva," the menelaus of either triangle is proved equal to the negative ceva of the other. In the case of perspective, the menelai become unity (theorem of Menelaus) and the cevas, negative unity (theorem of Ceva). A similar theorem is proved for two quadrilaterals, or "tetragrams" in three-dimensional space, which define tetragons by their vertices, and tetrahedra by the planes drawn through consecutive sides.

2. The expression $x_1+\epsilon x_2+\epsilon^2 x_3+\cdots+\epsilon^{n-1}x_n$, where $\epsilon^n=1$, is called a Lagrange resolvent of the equation whose roots are x_i. See Pascal, *Repertorium*, 2nd edition, vol. 1, p. 307. Regarding the roots as points of a plane, so ordered as to form a polygon and with a definite starting point x_1, the expressions become attached to the polygon. Under the translations $x_i=y_i+b$ where $\epsilon\neq 1$, the expression is an invariant. We then have $n-1$ Lagrange invariants (under translation):

$$v_1 = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \cdots ; \; v_2 = x_1 + \epsilon^2 x_2 + \epsilon^4 x_3 + \cdots ;$$

where $\epsilon=\exp 2i\pi/n$. If we add to these $v_0=x_1+x_2+x_3+\cdots$ which is not an invariant, we can readily solve for the x_i in terms of the v_i and hence show that the above invariants are a complete system. To illustrate the theory the norm and area of a hexagon were expressed explicitly in terms of Lagrange invariants.

3. The paper on numerical integration illustrated one method of procedure by finding several points on that solution of $dy/dx=x+y$ which passes through point (0, 1), derived a trial formula and an improvement formula, showed how

the method could be applied to the trajectory of an object shot through the air, and discussed briefly several different methods of procedure.

4. Commander Allen discussed the contributions of various nations to the problems of modern navigation. The methods of teaching navigation at the United States Naval Academy were presented. A display of literature on the subject and models to illustrate problems were on exhibit. Upon completion of the paper, the members were at leisure to examine the exhibit and obtain information on desired points.

5. Mr. Avers spoke on the scientific preparation for the Byrd polar flight, the methods of taking observations during the flight and the computation of the observations to determine when Byrd was exactly over the pole. He also spoke concerning the committee who were appointed to re-check Byrd's data.

6. Gores are cut from the corners of a square sheet of tin and a strip folded up along each edge so that a container with sloping sides is formed. The problem required the maximum capacity to be obtained. The interest in the problem lay in the necessity for unusual care in its treatment and in the fact that the capacity found was exactly 7/6 as great as that of a container formed when the gores become squares and the container is rectangular (1.16667 to five decimals). Professor Capron also developed that the breadth of the strip is almost exactly a mean proportional between the half edge of the sheet and the length cut off from the edge at each corner, so that a correspondingly simplified problem in one variable gives a maximum capacity within 0.001 of 1% of the capacity desired, and an approximate solution of the original problem depending only upon a quadratic surd ($\sqrt{3}$) is within 0.003 of 1%.

Characteristics were also mentioned of the corresponding problem for a rectangular sheet.

7. Mr. Rappleye discussed a nomogram designed to give the reduction of horizontal directions to sea level. Copies of the nomogram were given to the members.

8. Professor Tyler discussed a class of functions formed by continuous substitution. Thus if $f(t)$ be a function of t and if we replace t by $f(t)$, the result is denoted by $f^2(t)$. If the argument t be replaced by t, $x-1$ times, we would have the expression $f^x(t)$. He then defined $f^x(t)$ for all values of x, real or complex by the relation $f^x(t)=r^{-1}[x+r(t)]$ where $r(t)$ is a function of t and $r^{-1}(t)$ its inverse function. The relation between f and r is given by the relation $f(t)=r^{-1}[1+r(t)]$. From this definition the index law of $f^x[f^y(t)]=f^{x+y}(t)$ follows and $f^0(t)=t$, also $f^{-x}(t)$ is the inverse function of $f^x(t)$. He also showed that if x is the period of $r^{-1}(t)$, then $f^x(t)$ will repeat under substitution for every integral multiple of x. Various illustrations of this class of functions were given.

J. R. MUSSELMAN, *Secretary*

THE DUTY OF EXPOSITION WITH SPECIAL REFERENCE TO THE CAUCHY-HEAVISIDE EXPANSION THEOREM[1]

BY FRANCIS D. MURNAGHAN, Johns Hopkins University

As the speaker representing the Mathematical Association of America at this session I propose this morning to call your attention to one of the more important duties of a mathematician, namely, the duty of explaining as clearly as possible mathematical truths and discoveries both to his fellow mathematicians and to students of mathematics in general. The American Mathematical Society is especially devoted to the encouragement of research and the secretary of that society has called to your attention during this meeting the remarkable increase in the number of papers presented annually to the Society during the past five years. This increase has been such as to make the problem of publishing the results of mathematical research in America a very acute and difficult one. At the same time it has become more desirable than ever before that the papers published in our American journals should be as clear and as easily readible as possible. The Mathematical Association of America is especially concerned with the teaching and exposition of mathematical truth and it is pretty generally agreed that it is highly desirable that care should be taken to make this teaching and exposition to students as good as possible. I fear, however, that some of my friends who are particularly interested in research agree to this in a rather condescending manner; their tone implying that, whilst nothing should be done to discourage anyone beginning the study of mathematics, such care is not so necessary nor even so desirable when writing for fellow mathematicians. The underlying idea is that a competent mathematician usually prefers to glance at a paper, see the results arrived at, and then derive these results in his own individual manner. I believe that this opinion is not justified by the facts and in support of this belief I shall mention two instances which have recently come to my attention where mathematicians of great competence failed, through a lack of detail or of clarity in available expositions of known results, to arrive at immediate and important corollaries of these results. I shall be happy if my talk this morning tends to make the editors of our mathematical journals insist more definitely on clearness of exposition when considering papers submitted for publication. One can surmise that the writers of at least some of our papers set down their results with the referee of the paper more in mind than the prospective readers. They neglect, therefore, to state or emphasize points which they think will be familiar to the referee. Of course this is unfortunate since there is usually a one-to-one

[1] Read at the joint session of Section A of the American Association for the Advancement of Science, the American Mathematical Society and the Mathematical Association of America, Philadelphia, Dec. 30, 1926.

correspondence between the writer of a paper and his referee and it is the proper function of an editor of a mathematical journal to make the correspondence between the writers of the papers appearing in that journal and their readers as multiform, in a one-sided way, as possible.

It is a good rule of life to have a vocation and an avocation. My own particular vocation is the study and teaching of the applications of mathematics. The study of pure geometry serves as an avocation and I shall, therefore, rather naturally choose my first illustration from the realm of pure geometry reserving for my closing illustration a rather important result in applied mathematics.

One of the problems of pure geometry possessing an interesting history is the Torricelli or Fermat problem of finding a point in a plane having the property that the sum of the three distances from it to three given points in the plane is a minimum.[1] Every American student of mathematics is familiar with the discussion of this problem given in the Goursat-Hedrick *Mathematical Analysis*. However, as Darboux once said, an elegant demonstration touching the heart of the matter and nothing else is often of more interest to a geometer than the result itself and such a solution of the problem in question was given by Steiner.[2] He pointed out that, since the sum of the perpendiculars from any point inside an equilateral triangle upon the three sides is constant, all one has to do is to find a point P such that the perpendiculars from the three given points A, B, C to PA, PB, PC, respectively, form an equilateral triangle; in the technical language of the geometer the anti-pedal triangle of P with respect to the triangle ABC is to be equilateral. In fact, if P' is any other point, each of the distances $P'A$, $P'B$, $P'C$ is greater than, or at least equal to, the corresponding perpendicular distance $P'A'$, $P'B'$, $P'C'$ from P' to a side of the equilateral triangle whose sides pass through A, B and C. Hence $P'A+P'B+P'C \geqq P'A' + P'B'+P'C'$, the equality sign holding only when P' coincides with P. But $P'A'+P'B'+P'C'=PA+PB+PC$ so that $P'A+P'B+P'C>PA+PB+PC$ if P' is distinct from P. The construction of P follows at once on observing that each of the angles BPC, CPA, APB is 120°. This is all delightful but now comes the point where Steiner failed as an expositor. He remarks in a note that the result can easily be extended to space. What was the consequence of this economy of paper? The famous geometer Sturm, whose four volume treatise on synthetic geometry[3] is familiar to every student of that subject,

---

[1] For historical references see the article by M. Zacharias in the Encyclopädie der mathematischen Wissenschaften, vol. III–1, p. 1129. See also this Monthly, vol. 24 (1917), pp. 42–44, 243–244 and vol. 30 (1923), pp. 127–131.

[2] J. Steiner, Werke, vol. 2, p. 93 and p. 729.

[3] R. Sturm, *Die Lehre von den Geometrischen Verwandtschaften* (Teubner, 1908–1909).

considered in 1884[1] the problem of finding a point in space having the property that the sum of the four distances PA, PB, PC, PD to four points A, B, C, D in space is a minimum. He treated the problem again on page 67 of his extremely interesting book on *Maxima und Minima in der Elementaren Geometrie* (Teubner, 1910), and finally returned to it in 1913 in a paper in Crelle's journal[2] in which he says as his final remark: "Aber konstruierbar ist er nicht, weil er Schnittpunkt von drei Flächen höherer Ordnung ist." Nevertheless Steiner's remark that the method given for the plane problem is aplicable to the space problem is justifiable if the proper viewpoint is taken. Instead of focussing attention on the fact that the three angles BPC, CPA, APB are each equal to 120° one may observe that points P and Q whose distances to the sides of the triangle ABC are inversely proportional have the property that the pedal triangle of one is similar to the antipedal triangle of the other. The antipedal triangle of P being equilateral, the pedal triangle of Q must be equilateral and so Q is given as one of the two intersections of the three Apollonian circles of the triangle ABC. (There are two solutions of the Torricelli-Fermat problem since the radical occurring in the distance formula forces an algebraic treatment to consider the problem of stationary values of $PA \pm PB \pm PC$.) The points P and Q are known as isogonal conjugates or as focal conjugates since they are the foci of a conic inscribed in the triangle ABC. The argument holds precisely as above in space. Points P and Q whose distances to the faces of the tetrahedron $ABCD$ are inversely proportional have the property that the pedal tetrahedron of one with respect to $ABCD$ is similar to the antipedal tetrahedron of the other. If the areas of the faces of the antipedal tetrahedron of P are equal the sum of the distances PA, PB, PC, PD is a minimum (assuming P inside the antipedal tetrahedron). Q is then such that its pedal tetrahedron is equifacial and on expressing the fact that the opposite edges of an equifacial tetrahedron are equal we find three quadric surfaces of the type $y_2^2 + y_3^2 + 2c_{23}y_2y_3 = y_1^2 + y_4^2 + 2c_{14}y_1y_4$ through Q. Here (y_1, y_2, y_3, y_4) are the perpendicular distances of the point Q from the faces of $ABCD$ and c_{12} etc. denote the cosines of the dihedral angles of the tetrahedron $ABCD$. There are, then, 8 points $P = (x_1, x_2, x_3, x_4)$, found by setting $x_1y_1 = x_2y_2 = x_3y_3 = x_4y_4$, making $PA \pm PB \pm PC \pm PD$ stationary. Sturm failed to arrive at this result through paying too much attention to the fact that the trihedral angles $PBCD$, $PCDA$, $PDAB$, $PABC$ are congruent. The 8 points P may well be known as the Neuberg points of the tetrahedron $ABCD$ as they were arrived at by him, in an incidental manner, in 1909.[3]

---

[1] Journal für Mathematik, vol. 97 (1884), pp. 49–61.

[2] Journal für Mathematik, vol. 143 (1913), pp. 241–249.

[3] Annales de la Société Scientifique de Bruxelles, vol. 33 (1909), pp. 320–333.

This illustration in support of my thesis has for me the charm of elegance and, if one may say so, of uselessness. Here is a beautiful result unspoiled by any suspicion of a commercial or practical application; just like a beautiful lake surrounded by woods before the engineer comes to dam it or the lumberjack to convert beauty into usefulness. But now I come to an example which will no doubt appeal to many as more forceful since the practical applications of the theorem in question are numerous and of fundamental importance. I refer to the Heaviside expansion theorem which is well known to every research engineer engaged in a study of the problems of telegraphy and telephony. Some years ago the American Mathematical Society had the pleasure of hearing a lecture on the Heaviside operational calculus by Mr. J. R. Carson. This lecture appeared in the Bulletin of the Society for 1926 and those interested in the theorem and its applications are referred to this paper for details. Heaviside stated his theorem in 1886 but his exposition was certainly far from clear and later writers endeavored to give satisfactory proofs. In particular I may refer to a paper by Bromwich in the Proceedings of the London Mathematical Society for 1916.[1] Having occasion to use the theorem during the past year in a problem in electrical engineering, I was rather surprised to find that it is but a very slight modification of a classical result of Cauchy's concerning the determination of a certain particular integral of a linear, non-homogeneous, differential equation with constant coefficients. Heaviside was a genius, original in his methods and rather given to railing at "orthodox" mathematicians for failing to recognize his results or even to be interested in them. I hope to make it clear that a sufficient reply by a mathematician, if one sufficiently addicted to controversy could have been found, would have been the sentence of three words "Read your Cauchy!" Much better than this, however, would have been a clear and simple exposition of Cauchy's theorem. Even today the elementary text books on differential equations seem to ignore Cauchy's result and it is not, then, entirely surprising that Heaviside should have been in ignorance of it or of its connection with his own problem. I was, however, more surprised that Bromwich (for whom, as a member of the famous Galway galaxy, I have naturally a great respect) should have failed to notice this connection.

What, then, is the Heaviside Expansion Theorem and what is the result of Cauchy from which it is at once derivable? In his studies on the propagation of electric vibrations Heaviside was led to a consideration of a system of n ordinary linear differential equations of the type

$$\sum_{\alpha=1}^{n} g_{r\alpha} x_\alpha = f_r(t) ; \qquad r = 1, 2, \cdots, n.$$

[1] For a list of references see a paper by M. S. Vallarta, Transactions of the American Institute of Electrical Engineers, vol. 45 (1926), pp. 383–387.

Here the coefficients g_{rs} are usually of the type $g_{rs}=a_{rs}D^2+b_{rs}D+c_{rs}$, where the a_{rs}, b_{rs}, c_{rs} are all constants and D denotes differentiation with respect to the single independent variable t. The f's are given functions of t and it is desired to find functions $(x_1, x_2, \cdots x_n)$ of t which satisfy the differential equations just given and vanish together with their derivatives with respect to t when $t=0$. Following the usual procedure we eliminate all the unknown functions but one, x_1, say, by multiplying the equations by the cofactors G_{r1} of the elements of the first column of the determinant of the coefficients and then adding the equations together. This gives an equation of the form $F(D)x_1 =\sum_{\alpha=1}^{n} G_{\alpha 1}f_\alpha(t)$ where $F(D)$ is the determinant of the coefficients g_{rs} and is, in general, a polynomial of order $2n$ in D. We may, without lack of generality, assume the coefficient of the highest power of D, namely D^{2n} to be unity. On account of the linear character of the differential equations and the special nature of the initial conditions, it is sufficient to consider separately the various equations obtained by taking, one at a time, the terms on the right hand side and we can, at the end, add the solutions of these equations. One of the equations to which we are led in this way is $F(D)x_1=G_{11}f_1(t)$. In order to find a solution of this equation it is convenient to solve the auxiliary equation $F(D)u_1 =f_1(t)$ with the initial conditions to the effect that u and all its derivatives up to the $(2n-1)$st, inclusive, vanish when $t=0$. Then, on account of the constant character of the coefficients of the polynomials $F(D)$ and $G_{11}(D)$, it is apparent that the function x_1, defined by $x_1=G_{11}(D)u_1$ will satisfy the equation $F(D)x_1=G_{11}(D)f_1(t)$, for the differential operators F and G_{11} are permutable. Moreover, since G_{11} is a polynomial of degree $(2n-2)$ at most in D, it is true that x_1 will vanish together with its first derivative when $t=0$. The whole problem, therefore, consists in the determination of that particular solution of an equation of the type $F(D)u=f(t)$ which vanishes, together with its derivatives up to the $(2n-1)$st, inclusive, when $t=0$. Now this is the question which was solved by Cauchy (the essentials of his solution going back, indeed, to Euler) and of which the solution may be said to have been classical at least a half-century before Heaviside became interested in his problem. It is, I think, hard to see at just what point in the preceding page of argument anyone familiar with the classical solution would fail to observe the immediate connection with the Heaviside question. But the unfortunate state of affairs is that the elementary texts on the theory of differential equations with which I am familiar do not explain the Cauchy method and the situation must have been the same some forty years ago when Heaviside was seeking eagerly for help from the mathematicians.[1]

[1] Following the delivery of this address I had the pleasure of a conversation with Dr. T. C. Fry of the Bell Telephone Laboratories in the course of which he informed me that after a survey of existing

What is Cauchy's method? In order to find that particular solution of a non-homogeneous linear differential equation of order m which vanishes together with its derivatives up to the $m-1$st, inclusive, when $t=0$, he proceeds as follows. Writing the equation to be solved as $F(D)x=f(t)$, where the coefficient of D^m in $F(D)$ is taken as unity, the general solution of the corresponding homogeneous equation $F(D)u=0$ is of the type $u=C_1u_1+C_2u_2+\cdots+C_mu_m$, the C's being arbitrary constants and the u's a set of m distinct solutions of the homogeneous equation. These constants may be so determined that u and its derivatives up to the $(m-2)$nd, inclusive, vanish when $t=\tau$, the $(m-1)$st derivative taking at that instant the value $f(\tau)$; here τ is a fixed but arbitrary value of t. The solution u of the homogeneous equation which is found in this way involves the parameter τ and, to stress this, we may write it $u(t, \tau)$. Then Cauchy showed that the function x defined by $x=\int_0^t u(t,\tau)d\tau$ is that particular solution of the non-homogeneous equation $F(D)x=f(t)$ which vanishes with all its derivatives up to the $(m-1)$st, inclusive, when $t=0$. In fact since $u(t, \tau)$ and its derivatives up to the $(m-2)$nd, inclusive, vanish when the two arguments t and τ become equal we have

$$Dx = \int_0^t Du(t,\tau)d\tau \; ; \cdots D^{n-1}x = \int_0^t D^{n-1}u(t,\tau)d\tau$$

whilst

$$D^n x = \int_0^t D^n u(t,\tau)d\tau + D^{n-1}u(t,\tau)_{\tau=t} = \int_0^t D^n u(t,\tau)d\tau + f(t),$$

so that

$$F(D)x = \int_0^t F(D)u(t,\tau)d\tau + f(t) = f(t) .$$

In the particular case when the coefficients of $F(D)$ are constants and its zeros simple, the explicit expression for x is easy to find. Writing $F(D)=(D-r_1)(D-r_2)\cdots(D-r_m)$ we may set

$$u_1 = C_1 e^{r_1(t-\tau)} \; ; \cdots \; ; u_m = C_m e^{r_m(t-\tau)}$$

and we have to determine the constants C so as to satisfy the equations

$$C_1 + C_2 + \cdots + C_m = 0 \; ; \qquad r_1C_1 + r_2C_2 + \cdots + r_mC_m = 0 \; ; \text{etc.},$$

the last equation being

$$r_1^{m-1}C_1 + r_2^{m-1}C_2 + \cdots + r_m^{m-1}C_m = f(\tau) .$$

These equations show that the C's are the numerators in the development of $f(\tau)/F(r)$ into its simple fractions; that is, we have the identity $f(\tau)/F(r)$

books they have had to mimeograph a special course in differential equations to meet their needs. Incidentally the deduction of Heaviside's expansion theorem from Cauchy's method, in much the same manner as that indicated here, is given, I am informed, in that course.

$=C_1/(r-r_1)+\cdots+C_m/(r-r_m)$. In fact if we expand each of the simple fractions on the right near $r=\infty$ and express that $r=\infty$ is a zero of order m of the left hand side, the first term in its expansion being $f(\tau)/r^m$, we get the equations just given for the C's. The observation that the points $(r_1, r_2, \cdots r_m)$ are simple poles of the expression on the left gives at once, on expanding near one of these poles, that $C_p=f(\tau)/F'(r_p)$, and so

$$u(t,\tau)=f(\tau)\sum_1^m e^{r_p(t-\tau)}/F'(r_p).$$

In the particular case when $f(t)$ is constant and equal to unity, say, we have

$$x(t)=\int_0^t u(t,\tau)d\tau=\sum_1^m\frac{e^{r_pt}}{r_pF'(r_p)}-\sum_1^m\frac{1}{r_pF'(r_p)}.$$

The constant term on the right is $1/F(0)$ since we have the identical relation

$$1/F(r)=\sum_{p=1}^m C_p/(r-r_p)=\sum_{p=1}^m 1/[(r-r_p)F'(r_p)],$$

so that we have

$$x(t)=\frac{1}{F(0)}+\sum_{p=1}^m\frac{e^{r_pt}}{r_pF'(r_p)}.$$

In order to step from this classical result to the Heaviside expansion theorem, all we have to do is to operate on $x(t)$ with the operator G_{11} and we get

$$h(t)=G_{11}x(t)=\frac{G_{11}(0)}{F(0)}+\sum_{p=1}^m\frac{G_{11}(r_p)e^{r_pt}}{r_p\,F'(r_p)}$$

which is the expansion theorem in question. It may be appropriately remarked here that the contour integral method adopted by Bromwich and Wagner connects with the method explained here through the evaluation of the C's as the residues of the function $1/F(r)$ at its simple poles r_p.[1]

It seems clear from these examples that carefulness in exposition is necessary even when writing for experts. The slightest deviation from the natural course of the argument often suffices to make quite difficult what would otherwise be simple. But this is not the worst result. A situation sometimes develops where a man like Heaviside, by some flash of intuition, obtains a result whose derivation is not very clear even to the discoverer. Yet the result is true and a kind of "operational mathematics" develops. Amongst the followers of the new creed a certain laxity of mathematical morality may be observed. We are

[1] The case where the zeros of $F(r)$ are not all simple is dealt with in an article by the writer in the Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 81–89.

told by them that the method "works"; that they do not know how it works and in extreme cases we are given to understand that they do not particularly care. Those of you who had the pleasure of listening two days ago to Professor Swann's address on the "New Quantum Mechanics" must feel that a state of affairs similar to that which arose during the last decade of the nineteenth century in regard to Heaviside's operational calculus has now arisen in Physics. Let us hope that some one will soon be able to give a logically sound and connected exposition of the new theories which can be readily understood.

THE ROTATIONAL DERIVATIVE AND SOME APPLICATIONS[1]

By M. E. MULLINGS, University of Texas

1. Introduction. The problem of the rotation, and the description of the motion of a rotating body has been considered from various angles by the use of vector analysis in the component form. But the practice of referring problems in vector analysis back to component forms loses many of the advantages to be gained by the vector methods. In 1921 Gans[2] gave a definition, from a purely geometric point of view, of a component of the curl of a vector function and by combining the components in three fundamental directions obtained the curl. He did not state the conditions under which the curl would exist, and in his proof of Stokes' Theorem based on this geometric definition he does not state the conditions, nor does he carry his proof through in detail. The notion of a directional derivative and some of its applications is quite familiar, even to students of advanced calculus. I propose, in this paper, to give a definition of a similar nature, which I shall call the rotational derivative about an axis, and to use it to prove several theorems based on carefully formulated conditions.

I shall use the Greek letter ρ throughout this paper to denote the position vector. The directional derivative of a vector function $\phi(\rho)$ about a direction α_1 may be considered as a derivative with respect to the unit vector α_1, and I shall use the notation $D_{\alpha_1}\phi$ to designate the directional derivative of ϕ.

2. Definitions. (a). By a curve of type C I shall mean a curve given by:

(1) *$\rho = C(s)$ where s is the length of arc.*

(2) *$C(s)$ has a continuously turning tangent except for a linear null set of points.*

---

[1] This paper is a portion of a thesis for the M. A. degree presented to the University of Texas in June, 1926, which I prepared, independently of the work of Gans, under the direction of Dr. H. J. Ettlinger.

[2] *Einführung in die Vektoranalysis mit Anwendungen auf die Mathematische Physik* (Teubner, Leipzig-Berlin, 1921), Ch. II pp. 34–38.

(3) *$C(s)$ has a backward and a forward tangent everywhere.*

(b) A bifacial surface S defined by the equation $\rho = S(u,v)$ is said to be a simple surface if:

(1) *$S(u,v)$ is continuous in u and v, where u is the element of length along the curve $\rho = S(u,\bar{v})$ and v along the curve $\rho = S(\bar{u},v)$, $\bar{u}$ and $\bar{v}$ being fixed values of u and v.*

(2) *$S_{\alpha_1}(u,v) = D_{\alpha_1}S$ and $S_{\beta_1}(u,v) = D_{\beta_1}S$, where α_1 is a unit vector tangent to the curve $\rho = S(u,\bar{v})$ and β_1 is a unit vector tangent to the curve $\rho = S(\bar{u},v)$.*

(3) *S has a normal $N(u,v) = S_{\alpha_1}(u,v) \times S_{\beta_1}(u,v)$ existing everywhere except for a two-dimensional null set and continuous where it exists.*

(4) *At any point P at which the normal does not exist, there is a manner of approach to P such that there is a limiting normal at P, i. e., a vector $\overline{N}$ such that*

$$\lim_{P'=P} N(u,v) = \overline{N}.$$

(c). *Rotational Derivative about an Axis.* In a vector field $\phi(\rho)$, pass a plane with normal direction α_1 through a point P, and let Δa represent the magnitude of the area of a portion of the plane bounded by a closed curve L of type C, and having P as an interior point. *Then if*

$$\lim_{\Delta a = 0} \frac{\int_L \phi(\rho) \cdot d\rho}{\Delta a} \alpha_1$$

exists, where the integral is taken around L in the positive direction and Δa approaches zero in such a manner that the maximum dimension of Δa approaches zero, we will call that limit the rotational derivative ϕ_{α_1} of the function $\phi(\rho)$ about the direction α_1 at the point P.

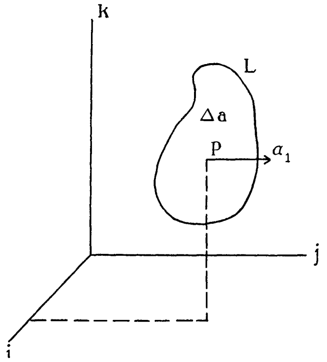


FIG. 1.

This definition will hold equally well if the plane through P is replaced by a simple surface having a normal at P whose direction is α_1.

3. With the above definitions as a basis and making use of the ordinary operations of the vector analysis I shall now proceed to establish three theorems.

THEOREM I. *Hypothesis:*

(1). *$\phi(\rho) = Xi + Yj + Zk$ is a vector function in the neighborhood of a point P_0.*

(2). *Z_y and Y_z exist and are continuous in ρ in the neighborhood of P_0.*

(3). *π is a plane parallel to the jk-plane at a distance p from the point P_0.*

(4). *P is the projection of P_0 on π.*

(5). *L is a closed curve of type C with P in its interior, lying in π, and enclosing an area Δa.*

(6). *The maximum dimension of Δa approaches zero as the distance p approaches zero.*

Conclusion:
$$\lim_{\Delta a=0} \frac{\int_L \phi(\rho)\cdot d\rho}{\Delta a} i = [Z_y(\rho_0) - Y_z(\rho_0)]\, i$$

Proof: Let $d\rho = dx\ i + dy\ j + dz\ k$. Then since the curve L is in a plane parallel to the jk-plane we have that $dx=0$; hence $\int_L \phi(\rho)\cdot d\rho = \int_L (Ydy+Zdz)$ and by Green's theorem[1] for two dimensions we have

$$\int_L (Ydy + Zdz) = \iint_{\Delta a} (Z_y - Y_z) dydz.$$

Now applying the law of the mean to the second member, we have

$$\iint_{\Delta a} (Z_y - Y_z) dydz = (\bar{Z}_y - \bar{Y}_z) \iint_{\Delta a} dydz$$

where $\bar{Z}_y$ and $\bar{Y}_z$ are the values of Z_y and Y_z at some point $\bar{P}$ in Δa. But

$$\iint_{\Delta a} dydz = \Delta a .$$

Therefore

$$\lim_{\Delta a=0} \frac{\int_L \phi(\rho)\cdot d\rho}{\Delta a} i = \lim_{\Delta a=0} \frac{(\bar{Z}_y - \bar{Y}_z)\Delta a}{\Delta a} i = [Z_y(\rho_0) - Y_z(\rho_0)]\, i .$$

Theorem II. *Hypothesis:*

(1). *$\phi(\rho) = Xi + Yj + Zk$ is a vector function in the neighborhood of P_0.*

(2). *X_y, X_z, Y_x, Y_z, Z_x, anl Z_y exist and are continuous in ρ in the neighborhood of P_0.*

(3). *$\phi(\rho)$ has a rotational derivative at P_0 for every α_1.*

Conclusion: There exists a vector curl $\phi = \phi_i + \phi_j + \phi_k$ such that curl $\phi = \nabla \times \phi$ and $\phi_{\alpha_1} = (\alpha_1 \cdot curl\ \phi)\alpha_1$.

---

[1] W. F. Osgood, *Advanced Calculus* (The MacMillan Co., New York, 1925), pp. 222–224.

Proof: Through P_0 pass a plane with normal direction α_1. At a distance p from P_0 along α_1 choose a point P and pass planes through it parallel to the fundamental planes. In the resulting tetrahedron $P\text{-}ABC$ let Δa be the magnitude of the area of the base ABC, and let L be the curve bounding Δa, let Δa_1, Δa_2, and Δa_3 be the magnitudes of the areas of the faces PBC, PCA, and PAB respectively, and L_1, L_2, and L_3 the bounding curves. Then

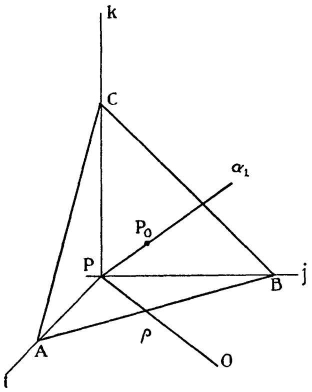


Fig. 2.

$$\phi_{\alpha_1} = \lim_{\Delta a = 0} \frac{\alpha_1 \int_L \phi(\rho)\cdot d\rho}{\Delta a}$$

$$= \alpha_1\left[\lim_{\Delta a_1=0} \frac{\int_{L_1}\phi(\rho)\cdot d\rho}{\Delta a_1}\,\frac{\Delta a_1}{\Delta a}\right.$$

$$\left. + \lim_{\Delta a_2=0} \frac{\int_{L_2}\phi(\rho)\cdot d\rho}{\Delta a_2}\,\frac{\Delta a_2}{\Delta a} + \lim_{\Delta a_3=0} \frac{\int_{L_3}\phi(\rho)\cdot d\rho}{\Delta a_3}\,\frac{\Delta a_3}{\Delta a}\right].$$

But

$$\frac{\Delta a_1}{\Delta a} = \alpha_1\cdot i, \qquad \frac{\Delta a_2}{\Delta a} = \alpha_1\cdot j, \quad \text{and} \quad \frac{\Delta a_3}{\Delta a} = \alpha_1\cdot k,$$

and as p approaches zero, each of the three faces satisfies the conditions of Theorem I and we have

$$\lim_{\Delta a_1=0} \frac{\int_{L_1}\phi(\rho)\cdot d\rho}{\Delta a_1} = Z_y(\rho_0) - Y_z(\rho_0) \text{ etc.}$$

Hence

$$\phi_{\alpha_1} = \alpha_1\left[(Z_y - Y_z)\alpha_1\cdot i + (X_z - Z_x)\alpha_1\cdot j + (Y_x - X_y)\alpha_1\cdot k\right].$$

Now project Δa_1 into $\Delta a_1'$ on a plane through P_0 parallel to the jk-plane. Then by definition (c) we have

$$\phi_i = \lim_{\Delta a_1=0} \frac{\int_{L_1'}\phi(\rho)\cdot d\rho}{\Delta a_1'}\, i$$

and by Theorem I we have $\phi_i = (Z_y - Y_z)i$.
Similarly $\phi_j = (X_z - Z_x)j$, and $\phi_k = (Y_x - X_y)k$.

Substituting the above, we have the second part of our conclusion as follows,

$$\phi_\alpha = [\phi_i\cdot\alpha_1 + \phi_j\cdot\alpha_1 + \phi_k\cdot\alpha_1]\alpha_1 = (\alpha_1\cdot \text{curl}\ \phi)\alpha_1 .$$

To get the first part we have

$$\text{curl}\ \phi = \phi_i + \phi_j + \phi_k = (Z_y - Y_z)i + (X_z - Z_x)j + (Y_x - X_y)k = \nabla\times\phi .$$

THEOREM III. The ease with which Stokes' Theorem follows from the above form of definition of curl ϕ is most suprising. The theorem in full is:

Hypothesis:

(1). *S is a simple surface bounded by a closed curve L of type C.*

(2). *$\phi(\rho)$ is a vector function defined on S and L.*

(3). *$\int_L \phi(\rho)\cdot d\rho$ exists.*

(4). *Curl ϕ exists on S and L and is continuous in ρ everywhere except for a two-dimensional null set on S.*

(5). *Curl ϕ is bounded on S and L.*

Conclusion: $\iint_S \text{Curl}\ \phi\cdot dS = \int_L \phi(\rho)\cdot d\rho$.

PROOF: Divide S into n subdivisions Δa_{in} each bounded by a closed curve L_{in} of type C. Choose any point P_{in} with coordinate ρ_{in} in the subdivision Δa_{in}, and let α_{in} be the normal, or a limiting normal, at P_{in}. Consider

$$h_{in} = \frac{\int_{Lin}\phi(\rho)\cdot d\rho}{\Delta a_{in}}$$

and

$$H_{in} = \text{curl}\ \phi(\rho_{in})\cdot\alpha_{in}) .$$

where the integral is taken around L_{in} in a positive direction. Now $|H_{in}| < M$ by hypothesis, and since

$$\text{curl}\ \phi\cdot\alpha_{in} = \lim_{\Delta a_{in}=0}\frac{\int_{Lin}\phi(\rho)\cdot d\rho}{\Delta a_{in}}$$

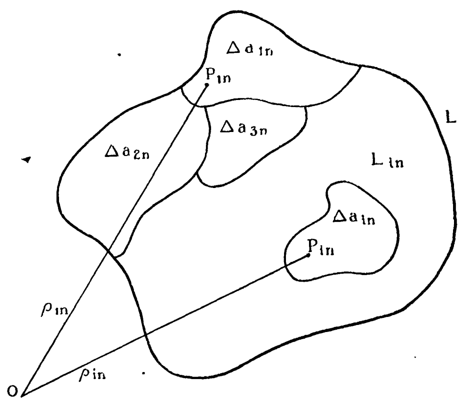


FIG. 3.

we have $|h_{in}| < M$. The difference of two bounded functions is bounded; hence $|h_{in} - H_{in}| < 2M$ for every i and n.

For a fixed point P_0 on S at which curl ϕ is continuous and α_0 exists, let Δa_0 be the subdivision containing P_0 for every value of n. Let L_0 be the curve bounding Δa_0, and let h_0 and H_0 be the values of h_{in} and H_{in} associated with Δa_0. If n increases without limit in such a manner that the maximum dimension of the maximum Δa_{in} approaches zero,

$\lim_{n=\infty} H_0 = \text{curl}\,\phi(\rho_0)\cdot\alpha_0$ and $\lim_{n=\infty} h_0 = \text{curl}\,\phi(\rho_0)\cdot\alpha_0$. Hence $\lim_{n=\infty}(h_0 - H_0) = 0$.

Form the sum $$\sum_1^n h_{in}\Delta a_{in} = \sum_1^n \int_{L_{in}}\phi(\rho)\cdot d\rho\,.$$

We note that the portion of the totality of the curves L_{in} that is not a part of L is traversed twice, once in each direction; hence the net result of integrating over that portion is zero, and the other portion is such that the boundary L is traversed once in the positive direction; hence

$$\sum_1^n \int_{L_{in}} \phi(\rho)\cdot d\rho = \int_L \phi(\rho)\cdot d\rho \text{ and then } \lim_{n=\infty}\sum_1^n h_{in}\Delta a_{in} = \int_L \phi(\rho)\cdot d\rho.$$

If we identify the surface S plus the boundary L as the set E and the subdivisions Δa_{in} as the measure e_{in} of the sub-sets E_{in}, the hypotheses of the Duhamel-Moore theorem[1] are satisfied. Hence

$$\mathit{lim}_{n=\infty} \sum_1^n H_{in}\Delta a_{in} = \int_L \phi(\rho)\cdot d\rho\,.$$

Replacing $\alpha_{in}\Delta a_{in}$ by the vector quantity ΔS_{in},

$$\mathit{lim}_{n=\infty}\sum_1^n H_{in}\Delta a_{in} = \mathit{lim}_{n=\infty}\sum_1^n \text{curl}\,\phi(\rho_{in})\cdot\Delta S_{in} = \iint_S \text{curl}\,\phi\cdot dS\,.$$

Hence

$$\iint_S \text{curl}\,\phi\cdot dS = \int_L \phi(\rho)\cdot d\rho\,.$$

4. It is interesting to note that the above proof of Stokes' theorem is very similar to that given by Gans. However his proof is scarcely more than an outline, as he does not attempt to justify the steps nor to give all the conditions necessary.

It is surprising that in such standard works as Gibbs'[2] text book on vector analysis a statement should be found to the effect that a definition of the nature of the one given herein is of more theoretical than practical interest, in view of the fact that the proof of Stokes' theorem given therein is considerably longer and more difficult to understand than the above. Furthermore, it is very loosely written, depending more on intuition than carefully formulated

[1] R. L. Moore, *On Duhamel's theorem*, Annals of Mathematics, (2), vol. 13(1912), p. 161.

H. J. Ettlinger, *A simple form of Duhamel's theorem* and some applications, this MONTHLY, vol. XXIX (1922), p. 239.

[2] *Vector Analysis*, Yale University Press (New Haven, 1922), p. 194. See pp. 184-193 for proof of Stokes' theorem.

arguments. Osgood[1] and Woods[2] give proofs of Stokes' theorem which are more rigorous than that of Gibbs, Osgood's being more rigorous than that of Woods, but in each the arguments are no more elementary in nature than those in the above proof. Moreover, the kind of bounding curve used is not stated in either of the proofs, and Woods does not state what kind of functions he is using, nor in any way describe the surface used. Osgood is not exact in his statement concerning continuity of his functions, and requires a bilateral surface having a continuous normal everywhere, which condition is less general than the case that I have considered.

A CURIOUS CASE OF THE USE OF MATHEMATICAL INDUCTION IN GEOMETRY

By J. V. USPENSKY, Carleton College

It is known that the ratio sin x/x continually decreases when x increases from 0 to π. Ordinarily the proof is given by means of trigonometry or calculus. The purpose of this note is to show how an equivalent statement can be established by purely geometric means using the process of reasoning by induction, which certainly is a rare case in geometry. Following is the fundamental theorem:

THEOREM I. *If α and β denote the angles at the base AB of a triangle ABC (Fig. 1) opposite to the sides CB and CA, then the following inequalities hold*

$$\frac{CA}{BA} > \frac{\beta}{\alpha+\beta} \text{ and } \frac{CB}{AB} > \frac{\alpha}{\alpha+\beta}.$$

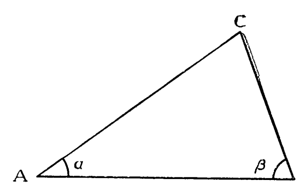


FIG. 1.

PROOF: (a) First let us consider triangles with commensurable angles at the base, so that we can set

$$\alpha = p\delta \text{ and } \beta = q\delta$$

where p and q are relatively prime integers and δ is a certain angle. Supposing that the above inequalities have already been proved in every case when the corresponding sum $p+q$ is *less* than a given integer $N>2$, we shall show that they continue to hold when $p+q=N$. Let ABC be a given triangle (Fig. 2) with the angles at the base $p\delta$ and $q\delta$, where p and q are relatively prime integers and $p+q=N>2$. The sides AC and CB cannot be equal, and we

---

[1] Loc. cit. pp. 237–241.

[2] *Advanced Calculus* (Ginn and Co., Boston, 1926), pp. 197–200.

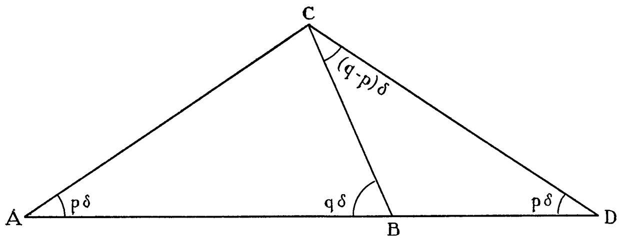


FIG. 2.

can suppose $AC>BC$. Describe from C as center and with CA as radius the circumference which passes through A and cuts the prolonged base AB at the point D. In the triangle ACD,

(1) $$AC=CD \text{ and } 2AC>AD=AB+BD.$$

Considering CD as the base of the triangle CBD we find that the adjoining angles are $p'\delta$, $q'\delta$, where $p'=p$ and $q'=q-p$, so that

$$p'+q'=q<N.$$

As the sum $p'+q'$ is *less* than N we have by supposition

(2) $$BD>\frac{q-p}{q}CD \quad \text{or} \quad BD>\frac{q-p}{q}AC.$$

Adding (1) and (2) we get

$$2AC+BD>\frac{q-p}{q}AC+AB+BD,$$

whence

$$\frac{q+p}{q}AC>AB \text{ or } \frac{AC}{AB}>\frac{q}{p+q}=\frac{\beta}{\alpha+\beta}.$$

From the same triangle CBD we get

$$CB>\frac{p}{q}CA \text{ and } CB>\frac{p}{p+q}AB \text{ or } \frac{CB}{AB}>\frac{\alpha}{\alpha+\beta}.$$

Now if the sum $p+q$ has the least possible value, namely 2, it must be that $p=q=1$; that is the triangle is isosceles, and for such a triangle it is obvious that

$$\frac{AC}{AB}>\frac{1}{2} \text{ and } \frac{BC}{AB}>\frac{1}{2},$$

so that our inequalities hold true when $N=2$. Then they will continue to hold for $N=3, 4, 5, \cdots$, that is in every possible case, provided the angles α and β are commensurable.

(b) Next suppose α and β incommensurable. Divide α into p equal parts, so that $\alpha=p\delta$ and determine an integer q by the condition $q\delta<\beta<(q+1)\delta$. Then construct the triangle ABC' with the angles $p\delta$ and $q\delta$ at its base AB. As these angles are commensurable we have

$$\frac{AC'}{AB} > \frac{q}{p+q} \text{ and } \frac{BC'}{AB} > \frac{p}{p+q}.$$

Now if p increases indefinitely, AC' and BC' approach respectively to AC and BC, and at the same time the ratios $q/(p+q)$ and $p/(p+q)$ converge to the limits $\beta/(\alpha+\beta)$ and $\alpha/(\alpha+\beta)$. Performing the passage to the limit we conclude from the preceding inequalities that

$$\frac{AC}{AB} \geqq \frac{\beta}{\alpha+\beta} \text{ and } \frac{BC}{AB} \geqq \frac{\alpha}{\alpha+\beta}.$$

It is easy to show that the sign = is excluded. Repeat the same construction as in Fig. 1. From the triangle BCD we get

$$BD \geqq \frac{\beta-\alpha}{\beta}AC, \text{ and furthermore } 2AC > AB + BD,$$

whence $$\left(2 - \frac{\beta-\alpha}{\beta}\right)AC > AB, \text{ or } \frac{AC}{AB} > \frac{\beta}{\alpha+\beta}.$$

Combining this inequality with $\dfrac{CB}{AC} \geqq \dfrac{\alpha}{\beta}$, which follows from the same triangle BCD, we get finally $\dfrac{CB}{AB} > \dfrac{\alpha}{\alpha+\beta}$. *Q.E.D*

THEOREM II. *Two arcs AB and AC, neither exceeding a semicircumference, being taken on the same circle and the latter being the greater of the two, the following inequality holds:*

$$\frac{AB}{AC} > \frac{\text{arc}AB}{\text{arc}AC}.$$

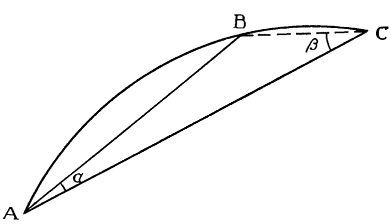


FIG. 3.

PROOF: Applying the preceding theorem to the triangle ABC we have

$$\frac{AB}{AC} > \frac{\beta}{\alpha+\beta}.$$

On the other hand $$\frac{\text{arc}\,AB}{\text{arc}\,BC} = \frac{\beta}{\alpha},$$

whence $$\frac{\text{arc}\,AB}{\text{arc}\,AC} = \frac{\beta}{\alpha+\beta} \text{ and } \frac{AB}{AC} > \frac{\text{arc}\,AB}{\text{arc}\,AC}. \qquad Q.E.D.$$

COROLLARY I. Taking a certain unit of length to measure distances we can express the lengths of the chords AB and AC by the *numbers* s and s'. Taking a certain arc of our circle, e. g. the whole circumference, as a unit we can express the measures of the arcs AB and AC by two *numbers* σ and σ'. We have then the following relation:

$$\frac{s}{s'} > \frac{\sigma}{\sigma'} \quad \text{or} \quad \frac{s}{\sigma} > \frac{s'}{\sigma'}.$$

if $\sigma' > \sigma$, *and this implies that* $\sin x/x$ *diminishes when* x *increases from* 0 *to* $\pi/2$.

THEOREM III. *Denote by P and P' the perimeters of two polygons inscribed in the same circle. If the greatest side of the second is less than the smallest side of the first, then*

$$P' > P.$$

PROOF: Denote by $s_1, s_2, s_3, \cdots, s_n$ the measures of the sides of the first polygon in the *increasing* order of magnitude, and by $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$ the measures of the subtended arcs, the whole circumference being taken as a unit. Denote by $s_1', s_2', s_3', \cdots, s_m'$ the measures of the sides of the second polygon in the *decreasing* order of magnitude, $\sigma_1', \sigma_2', \sigma_3', \cdots, \sigma_m'$ being the measures of the subtended arcs. As

$$\sigma_1 < \sigma_2 < \sigma_3 < \cdots < \sigma_n \quad \text{and} \quad \sigma_1' > \sigma_2' > \sigma_3' > \cdots > \sigma_m',$$

we have by the preceding corollary

$$\frac{s_1}{\sigma_1} > \frac{s_2}{\sigma_2} > \frac{s_3}{\sigma_3} > \cdots > \frac{s_n}{\sigma_n} \text{ and } \frac{s_1'}{\sigma_1'} < \frac{s_2'}{\sigma_2'} < \frac{s_3'}{\sigma_3'} < \cdots < \frac{s_m'}{\sigma_m'}$$

whence it follows that

$$\frac{s_1' + s_2' + \cdots + s_m'}{\sigma_1' + \sigma_2' + \cdots + \sigma_m'} > \frac{s_1'}{\sigma_1'} \text{ and } \frac{s_1 + s_2 + \cdots + s_n}{\sigma_1 + \sigma_2 + \cdots + \sigma_n} < \frac{s_1}{\sigma_1}.$$

By supposition $\sigma_1' < \sigma_1$, whence by the same corollary $\dfrac{s_1}{\sigma_1} < \dfrac{s_1'}{\sigma_1'}$.

That is $$\frac{s_1 + s_2 + \cdots + s_n}{\sigma_1 + \sigma_2 + \cdots + \sigma_n} < \frac{s_1' + s_2' + \cdots + s_m'}{\sigma_1' + \sigma_2' + \cdots + \sigma_m'}.$$

Now $$\sigma_1 + \sigma_2 + \cdots + \sigma_n = 1 \text{ and } \sigma_1' + \sigma_2' + \cdots + \sigma_m' = 1$$

and consequently $s_1 + s_2 + \cdots + s_n < s_1' + s_2' + \cdots + s_m'$.

That is $$P < P'. \qquad Q.E.D.$$

COROLLARY. Perimeters of the regular polygons inscribed in the same circle increase with the increasing number of sides.

NOTE ON THE SMOOTHING OF CURVES

By W. E. MILNE and V. ROJANSKY, University of Oregon

In many statistical investigations it is found that the values of a function obtained from observation are quite irregular, and it is assumed that these irregularities are due to accidental errors or deviations from the unknown "true values" of the function. Sometimes it is desirable to remove as far as possible these fluctuations, and to obtain new values which are more regular and presumably nearer on the average to the "true values" of the function. For this purpose either free-hand graphical smoothing or some type of moving average is recommended in text-books on statistics.

The object of this note is to suggest a method of smoothing which has a plausible justification in theory, and which in practice yields quite satisfactory results.

For the purpose in hand we shall say that a set of values of a function at equally spaced values of the argument is *regular* if their differences of a certain order, say m, are negligible. For simplicity we shall take $m=4$ in the sequel, though the reasoning applies equally well for any even value of m.

Let $y_0, y_1, \cdots, y_n$ be the values obtained for a function $f(x)$ at equally spaced values of x, let $u_0, u_1, \cdots, u_n$ be the corresponding "true values" of $f(x)$, and let $e_0, e_1, \cdots, e_n$ be the corresponding errors, so that

$$y_i = u_i + e_i. \tag{1}$$

We shall assume:

A. That the values u_i are *regular*.

B. That the errors e_i are due purely to chance, so that positive and negative values are equally probable.

Starting from these two assumptions we wish to determine a formula for smoothing the y_i which shall have the following two properties:

I. The formula will leave the y_i unchanged if they are already regular.

II. If they are not regular it will give new values y_i' for which the new errors e_i' satisfy the inequality

$$\sum (e_i')^2 < \sum e_i^2 .$$

Now the first condition will be satisfied by a formula of the type

$$y_i' = y_i + k\Delta^4 y_i \qquad (i=2, 3, \cdots, n-2), \tag{2}$$

in which k is arbitrary and where $\Delta^4 y_i = y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}$. By substitution from (1) equation (2) becomes $y_i' = u_i + e_i + k\Delta^4 e_i$, since by assumption A it follows that $\Delta^4 u_i = 0$. Therefore

$$e_i' = e_i + k\Delta^4 e_i, \text{ and } \Sigma(e_i')^2 = \Sigma e_i^2 + 2k\Sigma e_i \Delta^4 e_i + k^2 \Sigma(\Delta^4 e_i)^2,$$

the summation extending from $i=2$ to $i=n-2$. The value of k which makes $\Sigma(e_i')^2$ a minimum is

$$(3) \qquad k = -\left(\sum e_i\Delta^4 e_i\right)/\left[\sum (\Delta^4 e_i)^2\right],$$

and the minimum itself is

$$(4) \qquad \sum(e_i')^2 = \sum e_i^2 - \frac{(\sum e_i\Delta^4 e_i)^2}{\sum(\Delta^4 e_i)^2}.$$

We have assumed in B that positive and negative errors are equally probable so that in general we may neglect the sums of cross products. We shall further assume that to a sufficient degree of approximation

$$(5) \qquad \sum_{i=1}^{n-3} e_i^2 = \sum_{i=2}^{n-2} e_i^2, \text{ etc.}$$

If now we substitute in (3) the value of $\Delta^4 e_i$, expand, drop sums of cross products, and apply equations of type (5) to sums of squares we find that (3) reduces to

$$(6) \qquad k = -6/70,$$

and similarly (4) becomes

$$(7) \qquad \sum(e_i')^2 = (34\sum e_i^2)/70$$

so that the sum of the squares of the errors is approximately cut in half. The formula (2) may now be written

$$(8) \qquad y_i' = (1/70)[-6y_{i-2} + 24y_{i-1} + 34y_i + 24y_{i+1} - 6y_{i+2}].$$

This is our desired formula. However for actual practice we can obtain a formula which is nearly as accurate and easier to use by taking $k=-1/12$ instead of $-6/70$. Then in place of (8) we have

$$(9) \qquad y_i' = (1/12)[-y_{i-2} + 4y_{i-1} + 6y_i + 4y_{i+1} - y_{i+2}],$$

and $\Sigma(e_i')^2 = 35/72\Sigma e_i^2$ in place of (7), a result which is for practical purposes just as good as (7).

It remains to consider the four values y_0', y_1', y_{n-1}', y_n', which are not provided for by formula (8) or (9). First however we note that many frequency functions occurring in statistical studies taper off toward zero at both ends, and when this is the case it is merely a matter of notation to extend (8) and (9) beyond the ends so as to cover all significant values. When this can not be done we may by an easy modification of the preceding methods obtain formulas of the type

$$y_i' = (1/70)(4y_{i-1} + 54y_i + 24y_{i+1} - 16y_{i+2} + 4y_{i+3}],$$

and

$$y_i' = (1/70)\,[69y_i + 4y_{i+1} - 6y_{i+2} + 4y_{i+3} - y_{i+4}]$$

which take care of the first two values, and the two formulas symmetrical to these which take care of the last two values.

Formula (8) may be derived by another method which throws additional light on its significance. For if by the method of least squares we fit a cubic to the five values $y_{i-2}, y_{i-1}, y_i, y_{i+1}, y_{i+2}$, we find that the value which the cubic gives in place of y_i is precisely the value obtained by (8).

Turning now briefly to the general case where m is any even integer, we notice that the expression for k has in the numerator the mid-coefficient of the binomial expansion of mth degree, and in the denominator has the sum of the squares of all the binomial coefficients of mth degree. Thus taking account of the fact that the mid-coefficient is $m!/[(m/2)!]^2$ and the sum of the squares of the coefficients is equal to the mid-coefficient in the expansion of degree $2m$, that is $(2m)!/(m!)^2$, we have

$$k = -\frac{(m!)^3}{[(m/2)!]^2(2m!)}\,. \tag{10}$$

Using this value of k we have as the general result in place (4)

$$\sum(e')^2 = \sum e^2\left[1 - \left\{\frac{m!}{(m/2)!}\right\}^4 \cdot \frac{1}{(2m)!}\right]. \tag{11}$$

Since the right hand side of (11) increases as m increases and approaches $\sum e^2$ as a limit it is apparent that we should use no larger value of m than is necessary to make the mth differences of the "true values" negligible.

From equation (10) we easily compute k for $m=6$, 8, 10, etc., obtaining $k=-20/924$; $k=-70/12870$; $k=-252/184756$; etc., respectively. But just as in the case of $m=4$, we gain considerably in simplicity and lose little in accuracy by replacing the above fractions by the simpler ones $k=-1/46$, $k=-1/184$, $k=-1/733$, etc., respectively. With these values of k we obtain smoothing formulas for $m=6$, $m=8$, $m=10$, etc., which may be written as follows (for simplicity we suppress the y's, writing only the coefficients):

$$(1/46)\,[1 - 6 + 15 + 26 + 15 - 6 + 1], \tag{12}$$

$$(1/184)\,[-1 + 8 - 28 + 56 + 114 + 56 - 28 + 8 - 1], \tag{13}$$

$$(1/733)\,[1 - 10 + 45 - 120 + 210 + 481 + 210 - 120 + 45 - 10 + 1], \text{ etc.} \tag{14}$$

For the values of k which we have used formula (11) may still be employed with all the accuracy needed, and we find corresponding to (12), (13), and (14) respectively

$$\sum(e')^2 = .567\sum e^2\,; \quad \sum(e')^2 = .619\sum e^2\,; \quad \sum(e')^2 = .656\sum e^2.$$

CARTESIAN EQUATIONS OF CIRCLES CONNECTED WITH A PLANE TRIANGLE

BY PERRY A. CARIS, University of Pennsylvania

1. Introduction. It is the purpose of this paper to show how to obtain, by methods believed to be new, the Cartesian equations of the circum-circle, the nine-point circle, the in-circle and the ex-circles of a plane triangle. Use will be made only of the coefficients and the constant terms in the equations of the three lines upon which the sides of the triangle lie. The following notation will be adopted:

$$l_1: a_1x + b_1y + c_1 = 0$$
$$l_2: a_2x + b_2y + c_2 = 0 \qquad a_ib_k \neq a_kb_i.$$
$$l_3: a_3x + b_3y + c_3 = 0$$

shall denote the three lines. D shall mean the determinant $(a_1b_2c_3)$. The co-factors of a_i, b_i, c_i in D $(i=1, 2, 3)$ shall be denoted by A_i, B_i, C_i. Also $d_t = +\sqrt{(a_t^2+b_t^2)}$ for $t=1, 2, 3$.

2. To find the equation of the circum-circle. Consider the equation

$$\lambda_1 l_2 l_3 + \lambda_2 l_1 l_3 + \lambda_3 l_1 l_2 = 0. \tag{1}$$

Since it is of the second degree in x and y, it is the equation of some conic and this conic passes through the vertices of the triangle, for if l_i and l_k are simultaneously satisfied so is (1). Now (1) will be a circle if the λ's are so chosen that the coefficients of x^2 and y^2 are equal and the coefficient of the xy-term vanishes, that is when

$$\lambda_1(a_2b_3 + a_3b_2) + \lambda_2(a_1b_3 + a_3b_1) + \lambda_3(a_1b_2 + a_2b_1) = 0$$

and $$\lambda_1(a_2a_3 - b_2b_3) + \lambda_2(a_1a_3 - b_1b_3) + \lambda_3(a_1a_2 - b_1b_2) = 0,$$

whence $$\lambda_1 : \lambda_2 : \lambda_3 = (a_2b_3 - a_3b_2)(a_1^2 + b_1^2) : (a_3b_1 - a_1b_3)(a_2^2 + b_2^2) :$$
$$(a_1b_2 - a_2b_1)(a_3^2 + b_3^2) = C_1d_1^2 : C_2d_2^2 : C_3d_3^2,$$

and the required equation can be obtained by substituting in (1).

3. To find the equation of the nine-point circle. The nine-point circle is the circum-circle of the triangle formed by the mid-points of the sides of the given triangle. Now the point of intersection of l_2 and l_3 is easily seen to be $(A_1/C_1, B_1/C_1)$ and that of l_1 and l_3 is $(A_2/C_2, B_2/C_2)$. Hence the mid-point lying on l_3 is

$$[(A_2C_1 + A_1C_2)/(2C_1C_2), (B_2C_1 + B_1C_2)/(2C_1C_2)].$$

The line l_1' joining this point to the mid-point lying on l_2 is parallel to l_1 and therefore of the form $a_1x+b_1y+k=0$. Hence

$$-\ k = (a_1A_2C_1 + a_1A_1C_2 + b_1B_2C_1 + b_1B_1C_2)/(2C_1C_2)$$
$$= [C_2(a_1A_1 + b_1B_1 + c_1C_1) - C_1(c_1C_2 - a_1A_2 - b_1B_2)]/(2C_1C_2)\,.$$

The value of the first parenthesis is D. Also $a_1A_2+b_1B_2+c_1C_2=0$, since the point $(A_2/C_2,\ B_2/C_2)$ is on l_1. Hence $-a_1A_2-b_1B_2=c_1C_2$. That is, $-k=(C_2D-2c_1C_1C_2)/(2C_1C_2)$. Whence $k=c_1-[D/(2C_1)]$ and $l_1'\equiv l_1-[D/(2C_1)]=0$. By symmetry, $l_2'\equiv l_2-[D/(2C_2)]=0$, $l_3'\equiv l_3-[D/(2C_3)]=0$.

Now the λ's of the preceding problem do not depend on the constant terms of l_1, l_2, l_3. Also l_i' differs from l_i only in the constant term. Hence the λ's for the nine-point circle are proportional to the same set of numbers as those of the circum-circle. That is, the equation of the nine-point circle can be obtained by replacing every l in (1) by the corresponding l'.

4. **To find the equations of the in-circle and the ex-circles.** Consider the equation

$$\lambda_1^2 l_1^2 + \lambda_2^2 l_2^2 + \lambda_3^2 l_3^2 - 2\lambda_1\lambda_2 l_1 l_2 - 2\lambda_2\lambda_3 l_2 l_3 - 2\lambda_1\lambda_3 l_1 l_3 = 0. \tag{2}$$

It is plainly a conic touching l_i for $l_i=0$ reduces it to a perfect square. This conic will be a circle if

$$\lambda_1^2 a_1b_1 - \lambda_1\lambda_2(a_1b_2 + a_2b_1) + \lambda_2^2 a_2b_2 - \lambda_1\lambda_3(a_1b_3 + a_3b_1) - \lambda_2\lambda_3(a_2b_3 + a_3b_2) + \lambda_3^2 a_3b_3 = 0 \tag{3}$$

and

$$\lambda_1^2(a_1^2 - b_1^2) + \lambda_1\lambda_2(2b_1b_2 - 2a_1a_2) + \lambda_2^2(a_2^2 - b_2^2) + \lambda_1\lambda_3(2b_1b_3 - 2a_1a_3) + \lambda_2\lambda_3(2b_2b_3 - 2a_2a_3) + \lambda_3^2(a_3^2 - b_3^2) = 0. \tag{4}$$

Multiplying (3) by $2b_1b_2-2a_1a_2$ and (4) by $a_1b_2+a_2b_1$ and adding gives

$$C_3d_1^2\lambda_1^2 - C_3d_2^2\lambda_2^2 + 2C_1d_1^2\lambda_1\lambda_3 - 2C_2d_2^2\lambda_2\lambda_3 + [(C_1^2d_1^2 - C_2^2d_2^2)\lambda_3^2/C_3] = 0,$$

which when multiplied by C_3 can be broken up into two linear factors, giving

$$C_3d_1\lambda_1 + C_3d_2\lambda_2 + (C_1d_1 + C_2d_2)\lambda_3 = 0 \tag{5}$$

$$C_3d_1\lambda_1 - C_3d_2\lambda_2 + (C_1d_1 - C_2d_2)\lambda_3 = 0. \tag{6}$$

To eliminate the $\lambda_1\lambda_3$-term between (3) and (4), interchange the subscripts 2 and 3, obtaining:

$$C_2d_1\lambda_1 + (C_1d_1 + C_3d_3)\lambda_2 + C_2d_3\lambda_3 = 0 \tag{7}$$

$$C_2d_1\lambda_1 + (C_1d_1 - C_3d_3)\lambda_2 - C_2d_3\lambda_3 = 0. \tag{8}$$

From (5) and (7),

$$\lambda_1:\lambda_2:\lambda_3 = -C_1^2d_1^2 - C_1C_2d_1d_2 - C_1C_3d_1d_3:$$
$$C_1C_2d_1^2 + C_2^2d_1d_2 - C_2C_3d_1d_3:C_1C_3d_1^2 - C_2C_3d_1d_2 + C_3^2d_1d_3$$

or

$$\lambda_1:\lambda_2:\lambda_3 = C_1(-C_1d_1 - C_2d_2 - C_3d_3):$$
$$C_2(C_1d_1 + C_2d_2 - C_3d_3):C_3(C_1d_1 - C_2d_2 + C_3d_3).$$

Now for the lines l_1, l_2, l_3, the following identity holds:

$$C_i^2d_i^2 + C_k^2d_k^2 - C_j^2d_j^2 = -2C_iC_k(a_ia_k + b_ib_k),$$

where i, j, k, are all different. The verification of the identity presents no difficulty at all. Multiplying each term of the above ratios for the λ's by $-C_1d_1+C_2d_2+C_3d_3$ and using the above identity gives:

$$\begin{aligned}\lambda_1:\lambda_2:\lambda_3 &= C_1(C_1^2d_1^2 - C_2^2d_2^2 - C_3^2d_3^2 - 2C_2C_3d_2d_3):\\ &\quad C_2(-C_1^2d_1^2 + C_2^2d_2^2 - C_3^2d_3^2 + 2C_1C_3d_1d_3):\\ &\quad C_3(-C_1^2d_1^2 - C_2^2d_2^2 + C_3^2d_3^2 + 2C_1C_2d_1d_2)\\ &= C_1[2C_2C_3(a_2a_3 + b_2b_3) - 2C_2C_3d_2d_3]:\\ &\quad C_2[2C_1C_3(a_1a_3 + b_1b_3) + 2C_1C_3d_1d_3]:\\ &\quad C_3[2C_1C_2(a_1a_2 + b_1b_2) + 2C_1C_2d_1d_2]\end{aligned}$$

$$(9)\qquad = (a_2a_3 + b_2b_3 - d_2d_3):(a_1a_3 + b_1b_3 + d_1d_3):(a_1a_2 + b_1b_2 + d_1d_2).$$

Equation (7) differs from (8) only in the sign of d_3. Hence the solution of (5) and (8) is

$$(10)\quad \lambda_1:\lambda_2:\lambda_3 = (a_2a_3 + b_2b_3 + d_2d_3):(a_1a_3 + b_1b_3 - d_1d_3):(a_1a_2 + b_1b_2 + d_1d_2).$$

Also (6) differs from (5) only in the sign of d_2. Hence the solution of (6) and (7) is

$$(11)\quad \lambda_1:\lambda_2:\lambda_3 = (a_2a_3 + b_2b_3 + d_2d_3):(a_1a_3 + b_1b_3 + d_1d_3):(a_1a_2 + b_1b_2 - d_1d_2).$$

Now if (5) and (7) be each multiplied by -1, then (6) will differ from (5) only in the sign of d_1 and (8) will differ from (7) in the same way. Hence the solution of (6) and (8) can be obtained from that of (5) and (7) by changing the sign of d_1. Hence, in this case

$$(12)\quad \lambda_1:\lambda_2:\lambda_3 = (a_2a_3 + b_2b_3 - d_2d_3):(a_1a_3 + b_1b_3 - d_1d_3):(a_1a_2 + b_1b_2 - d_1d_2).$$

The four sets, (9), (10), (11), (12), give the ratios of the λ's for the required four circles. It remains to show which set is to be chosen to find the equation of a prescribed circle. For this purpose, write (2) in the form

$$(13)\qquad (\lambda_1l_1 + \lambda_2l_2 - \lambda_3l_3)^2 = 4\lambda_1\lambda_2l_1l_2.$$

It may be remarked in passing that this form is to be preferred to (2) in a numerical problem. Since (13) is the equation of a conic in terms of two tangents and their chord of contact, the point of contact of l_1, say, is at the intersection of $l_1=0$ and $\lambda_2 l_2-\lambda_3 l_3=0$, that is at the intersection of $a_1x+b_1y+c_1=0$ and $(\lambda_2 a_2-\lambda_3 a_3)x+(\lambda_2 b_2-\lambda_3 b_3)y+\lambda_2 c_2-\lambda_3 c_3=0$, which is the point

$$\left(\frac{\lambda_2 A_3+\lambda_3 A_2}{\lambda_2 C_3+\lambda_3 C_2},\ \frac{\lambda_2 B_3+\lambda_3 B_2}{\lambda_2 C_3+\lambda_3 C_2}\right). \tag{14}$$

In the problem on the nine-point circle, one vertex lying on l_1 was found to be $(A_2/C_2,\ B_2/C_2)$. The other one lying on l_1 is found by symmetry to be the point $(A_3/C_3,\ B_3/C_3)$. Now the circle (13) will touch l_1 internally, that is at a point *between* $(A_2/C_2,\ B_2/C_2)$ and $(A_3/C_3,\ B_3/C_3)$, if the point (14) divides the segment between these vertices into segments whose ratio is positive,

that is, when
$$\frac{\dfrac{\lambda_2 A_3+\lambda_3 A_2}{\lambda_2 C_3+\lambda_3 C_2}-\dfrac{A_3}{C_3}}{\dfrac{A_2}{C_2}-\dfrac{\lambda_2 A_3+\lambda_3 A_2}{\lambda_3 C_2+\lambda_3 C_2}}=\frac{\lambda_3 C_2}{\lambda_2 C_3}\ \text{is positive.}$$

Similarly, l_2 will be touched internally when $(\lambda_1 C_3)/(\lambda_3 C_1)$ is positive and l_3 when $(\lambda_2 C_1)/(\lambda_1 C_2)$ is positive. Since the product of the three ratios $(\lambda_3 C_2)/(\lambda_2 C_3)$, $(\lambda_1 C_3)/(\lambda_3 C_1)$, $(\lambda_2 C_1)/(\lambda_1 C_2)$, is unity, there are four and just four possibilities of signs:

(15) $+\ +\ +$ (16) $+\ -\ -$

(17) $-\ +\ -$ (18) $-\ -\ +$

Hence that one of the four sets of λ's which produces (15) when combined with the C's as shown above will give the equation of the in-circle. The sets similarly producing (16), (17), (18), will give the equations of the ex-circles touching internally l_1, l_2, l_3 respectively. That the λ's and C's always will produce exactly the above four sets of signs can be seen from the following consideration.

It is an easy matter to show that the absolute value of $a_ia_k+b_ib_k$ is less than d_id_k. Hence, using actual values in (9), (10), (11), (12), instead of ratios, λ_j has the same sign as the d_id_k associated with it. It is then merely a question of testing the sets of λ's with the eight possibilities of sign that the C's present.

THE DUST NUMERALS AMONG THE ANCIENT ARABS

By DAVID EUGENE SMITH, Teachers College, Columbia University
and SALIH MOURAD, Robert College, Constantinople

Among the books composed by Eastern Moslem scholars in the tenth century, several have special interest with respect to elementary computation and the gobar (ghobar) numerals. One was written by 'Alî ibn Aḥmed, Abû'l-Qâsim, al-Antâki (that is, from Antioch), al-Mujtabâ (the chosen), who lived at Bagdad and died in 987.[1] It was called *Kitâb al-takht al-kabîr fî'l ḥisab al-Hindi* ("The great book of the board on Hindu arithmetic"). The word *takht* was translated at first, in Suter's list, as "the method" and later as "the table,"[2] but it is merely an Arabic form of the Persian word *takhta*, meaning "board" or "wood."[3] The same word appears in other treatises[4] by the same writer, and in works by various other writers. One of these is Mohammed ibn 'Abdallâh, Abû Nasr, al-Kalwâdânî, a contemporary of al-Mujtabâ and, like him, living at Bagdad. His work is entitled *Kitâb-al-takht fî'l ḥisâb al-Hindi*, meaning "the book of the board (takht) or the Hindu arithmetic."[5]

Another writer using the term is Sinân ibn al-Fatḥ (c. 941), a native of Harrân and a mathematician of some merit. The title of this work is *Ilm hisâb al-takht* ("The science of arithmetic of the takht").[6] Besides this there may be mentioned the *Kitâb al-takht* ("Book of the board") by al-Razi (ar-Razi) whose date is unknown; the *Kâfiya fî ḥisâb al-takht w'al mîl* ("Compendium of arithmetic with board and stylus"), by Amîn al-Dîn (eddîn) al-Abharî, who died in 1332[7]; and the *Jawâmi' al-ḥisâb bi'l takht w'al-turâb* ("Encyclopedia of arithmetic with board and dust"), written by 'Abdallah al-Zanâtî whose date, like that of al-Razi, is unknown.

The derivation and significance of the word *takht* seem more reasonable than those advanced by Woepcke, Suter, and others of their time. Just as we

---

[1] *Fihrist*, p. 40, and p. 75, nn. 271–276; Arabic text, vol. 1, p. 284; Suter's list, p. 63. See Casiri, *Bibliotheca Arabico-Hispana*, vol. 1, p. 411; Weissenborn, *Zur Geschichte der Einführung der jetztigen Ziffern in Europa*, p. 87. The transliterations and translations in the text and the notes have been checked and the references in the notes extended by Dr. Soloman Gandz, whose writings on Arabic mathematics are familiar to readers of this Monthly.

[2] This was due to the fact that he misread the Arabic letter *kh* for *ḥ*. See p. 70 and his note 232 on the *Fihrist*.

[3] See Vullers, *Lexicon Persico-Latinum*, vol. 1, p. 425; Lane, p. 298.

[4] *Kitâb fi'l ḥisâb 'alâ al-takht bilâ maḥw* ("The book on arithmetic on the board without erasing"). The *Fihrist*, *loc. cit.* Arabic title *Kitâb al-ḥisâb bilâ takht bal bi'l-yad* ("The book on arithmetic with the hand without the board"). This refers to mental arithmetic. See *Fihrist*, p. 70, on the authority of Ibn al-Ḳifṭi (1172–1248): See his *Ta'rikh al-Ḥukamâ*, ed. Lippert, Leipzig 1903, p. 234.

[5] *Fihrist*, pp. 41, 75; Arabic text, vol. 1, p. 284; Suter's list, p. 74, and Ibn al-Ḳiftî Ta'rikh al-Hukamâ p. 288.

[6] The *Fihrist*, pp. 37, 70; Arabic text, p. 281; Suter's list, pp. 51 and 66.

[7] See Ahlwandt's catalogue of Arabic MSS in Berlin, vol. 5, p. 333, No. 5975.

send a pupil to "the board," meaning the blackboard, so the Arabs of the 10th century sent them to their board tablets, as may also be seen in most of the native Arab schools today. In that century, however, when writing materials, as we know them, were scarce, the pupils sprinkled white dust on a black tablet and did their computing with a stylus or pointed rod. Hence it is that later Arab writers speak of "arithmetic on the dust board," of "arithmetic on the board with the pointer," of "*al-ghobar* (dust) arithmetic," and (as in the case of Sinân ibn al-Fatḥ, of "the science of the board."

The use of the word *takht* is more fully explained in the bibliographical lexicon of Haji Khalfa[1] as follows:

"There are many divisions of arithmetic. One of them is the science of arithmetic of the board and the stylus,[2] *al-takht w'al-mîl.* This is the science by which we learn how to perform the arithmetic operations by numerical symbols for the units and also for numbers exceeding the units by the stages.[3] These symbols are attributed to the Hindus. . . . . This science as also called, *al-takht w'al-turâb* ("The board and the dust")."

The form *takht* should be used instead of *taḥt*, although the *Fihrist* and Haji Khalfa adopted the latter. The *ḥ* is the Arabic letter *h* without a dot, whereas *kh* represents the same letter with a dot and is more gutteral than the other. *Takht* is therefore the more correct form and is properly used by both Ahlwardt and Woepcke.

The eastern Arabs kept the name of Hindu[4] arithmetic, while those in western Europe preferred the title *al-gobar*, or *al-ghobâr* (dust) as characterizing their computations.[5] It is not strange that the Arabs in Spain should have continued to use the dust tablet after it was abandoned in the East, for paper was made in Bagdad as early as the ninth century, but the industry was not carried to Spain until three or four centuries later.[6] It is also quite natural that they should have used the word to mean abacus in general, long after the dust feature was forgotten.

There seems, therefore, to be no doubt that, although the western Arabs appropriated the term *gobar* (*ghobar*), the early scholars of Bagdad used the

[1] *Lexicon Bibliographicum et Encyclopaediam*, ed. of the Arabic text with Latin translation by G. Fluegel, in 7 vols., Leipzig, 1835–1858.

[2] See Smith, *History*, vol. 2, pp. 177–178, "radius."

[3] Or "by the degrees," referring to place value.

[4] But see Professor Carra de Vaux's argument that *al-Hindi* did not necessarily mean Hindu.

[5] Dr. Gandz, whose paper on this subject will be published later, takes *ghobar* to be simply an Arabic transliteration of the Greek term *abakion* (abacus),—the Semitic *abaq*; but this simply carries it back to a word meaning dust, as before.

[6] It dates from the beginning of the Christian era in China; at any rate our first specimens of paper go back to the first century. The industry was introduced into Samarkand by Chinese prisoners of war in 751, and a paper factory was established in Bagdad in 794. The first one set up in Spain was in 1154.

dust numerals in the same way as the later computers of Cordova, Salamanca, and Toledo.[1]

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. CONVERGENCE WITHOUT LIMITS

BY A. A. BENNETT, Lehigh University

The following question is sometimes proposed to catch the unwary mathematician. Suppose $f(x)$ and $g(y)$ are two real one-valued bounded functions defined for all real values of their arguments, $f(x)$ being furthermore continuous. Suppose a, b, c, are numbers satisfying the conditions, $\lim_{x\to a} f(x)=b$, $\lim_{y\to b} g(y)=c$; the question is, does it follow that $\lim_{x\to a} g[f(x)]=c$? The correct answer is of course "No," since the definition of $\lim_{y\to b} g(y)=c$, does not make use of $y=b$, and on the other hand the definition of $\lim_{x\to a} f(x)=b$, does not require that $f(x)\neq b$. Thus, for example, if $f(x)$ is taken as a constant, namely b, and $g(y)$ is discontinuous at $y=b$, but has a limit as y approaches b, then the proposed relation cannot hold. For example let $f(x)\equiv b$, and let $g(y)\equiv c[\sin(y-b)]/(y-b)$, for $y\neq b$, but let $g(b)=c-l$. Then $\lim_{y\to b} g(y)=c$, but $g[f(x)]\equiv c-l$, so that $\lim_{x\to a} g[f(x)]\neq c$.

In view of these facts it may not seem so much of a paradox to assert that the statement: "For a given a, a value b exists such that $\lim_{x\to a} f(x)=b$," may be altered without for many purposes essentially impairing its content and yet in such a way as not to assume the existence of either a or b. One might naturally ask as to what if anything would be gained by such a procedure even if it is possible. Another question somewhat more concrete is the following: Is there anything to be gained by sacrificing the use of the continuity of the real number system?

One might raise the question concerning the possibility of making actual use of more than an enumerable set of numbers. So far as any possible explicit

[1] After this article was prepared, it was submitted to Dr. Soloman Gandz, as stated in a previous note, who was thereby led to make an extended study of the question of the abacus in general among the Arabs. His investigations supplement in various instances the treatment of the special topic as here presented, but they carry the subject so much farther that he has been asked to publish the results of his study in a subsequent number of this MONTHLY.

numerical relation is concerned it is clear that the Dedekind-cut axiom on numbers places at our disposal a nondenumerable set of irrational numbers "almost all" of which are never used. However more elementary considerations may be cited which would suggest the desirability in some cases of omitting some limiting points. Consider the theory of the set of points in the Euclidean plane, constructible from two given distinct points by use of ruler and compasses, only points obtained as intersections being counted. After constructing the vertices of a regular hexagon with one of the points as a center by the use of compasses, we can, by employing the ruler alone, secure a complete rational planar net. The further use of compasses brings in further points. For the complete system of points, quadratic equations with positive discriminants have solutions but cubic equations for example are not in general solvable, and a Dedekind cut among the points of a line may not determine a point of the system. For the totality of constructible points, convergence has a meaning, but limits do not in general exist. One might desire to introduce the limiting points. The distinction between constructible and non-constructible points is not thereby abolished. And the limiting points would continue to have a minor role at best in the theory of geometrical constructions.

In certain other illustrations, the limiting points would be worse than useless. Consider for example the following. Let there be given a rational planar net of points. Let Cartesian coordinates be introduced and suppose that each point of the net is designated by an ordered pair of rational numbers (a, b). In the general theory of such planar nets, a notion of normal region and hence of convergence may be introduced as for example by Minkowski in his geometry of numbers, or much more abstractly by Hausdorff and Fréchet in their discussions of general spaces. Any such theory would be developed as applicable to all rational planar nets using the given form of region. Suppose we should particularize and consider a geometrical representation of the numbers in the algebraic field generated by the square root of 2. Let each number of the form $a+b\sqrt{2}$, where a and b are rational, be represented by the point (a, b) in the net. Convergence may be discussed with reference to a notion of absolute value when we define arbitrarily Abs $(a, b)=\sqrt{(a^2+b^2)}$, and we may denote this by $||(a, b)||$. This will satisfy the triangle inequality needed for convergence, although for most algebraic purposes it will not be as useful as the familiar norm as used in the theory of algebraic fields, which latter fails to comply with this condition of inequality. Consider the sequence $(0,1)$, $(0,1.4)$, $(0,1.41)$ etc., where $(0,b_n)$ is such that $\lim_{n\to\infty} b_n=\sqrt{2}$. If a limiting point for this sequence exists, one would expect it to be a point $(0,\sqrt{2})$, but this would denote $0+\sqrt{2}\sqrt{2}=2=(2,0)$. Thus convergence and limit relations are seriously disordered by supposing any such a limit as existing.

The general question as to how one can handle convergence and relative continuity in any domain where compactness is sacrificed presents no great logical difficulty but offers several features of immediate interest. Let us define a sequence (x_n) as convergent if for each positive real number, δ, there is a natural number m, such that $||x_n - x_{n+p}|| < \delta$ for each $n \geqq m$ and each $p > 0$. We shall define such a number m as *belonging* to δ for the given sequence. A function $f(x)$ will be defined as convergent with respect to a convergent sequence (x_n) if and only if for each positive real number, ϵ, there is a positive real number δ, such that for an m belonging to δ as defined by the reference sequence, $||f(x_m) - f(x)|| < \epsilon$, for each x in the domain for which $||x_m - x|| < \delta$. One may note the close resemblance between this definition and the usual condition that $f(x)$ has a limit for x approaching a limit, although here no limits are required. In particular it follows at once that the sequence $(f(x_n))$ converges. However, it is to be noted that the definition is not for a sequence of values of the function. Using this definition one may show that among rational numbers $f(x) \equiv x^3$ converges as x converges toward the cut usually defined as $\sqrt{2}$, although neither limit exists in the domain.

In the usual discussions of general spaces, the notion of compactness which makes possible the introduction in one dimension of a Dedekind cut, and of a Borel theorem is usually regarded as so important that the continued study of convergent sequences has seldom if ever been carried to the extent of considering the notion of the convergence of functions of a variable point of the discontinuous space. Thus for example both Pierpont and Pringsheim who carry out the Cantor theory to an elaborate degree introduce the complete real number system before venturing upon functions of an independent variable. While for many purposes this is undoubtedly wise, yet for certain algebraic problems the more general treatment here proposed has distinct advantages.

II. The Devil's Curve Again

By Aubrey J. Kempner, University of Colorado

The text of P. R. Rider's article "The Devil's curve and Abelian Integrals"[1] and of B. H. Brown's note "La courbe du diable"[2] may give the impression that the curve is not easy to trace. It may be worth while to point out that the most elementary rules for curve tracing bring this curve within the scope of freshman analytic geometry, without any use of calculus. The special case $y^4 - x^4 - y^2 + 2x^2 = 0$ (considered in the first article mentioned above) would be plotted by the following steps:

$$Y^2 - X^2 - Y + 2X = 0, \quad \frac{(X-1)^2}{3/4} - \frac{(Y-\frac{1}{2})^2}{3/4} = 1.$$

[1] This Monthly, vol. 34 (1927) pp. 199.

[2] This Monthly, vol. 33 (1926) pp. 273–274.

Plot this hyperbola in an X-Y system of coordinates. Then take the square root of the X-coordinate ($x^2=X$), and in this new figure (in an x-Y system of coordinates) take the square root of the Y-coordinate ($y^2=Y$). Fig. 1 of Rider's paper is thus immediately obtained. (Compare, for example, this Monthly, vol. 24 (1917), p. 18).

It goes without saying that the curve actually does all that is claimed for it as far as the illustration of many properties (singular points, maxima and minima, asymptotes, etc.) is concerned, but it might have been mentioned how easy it is to plot the curve itself.

III. An Example in Maxima and Minima

By Elijah Swift, University of Vermont

It sometimes puzzles the beginning student to find functions of two variables, x and y, which possess a minimum in x alone and also in y alone without possessing one in both variables. Thus, if we set $x=0$, the expression $x^2-3xy+y^2$ (which then becomes a function of y alone, namely y^2) takes on a smaller value, namely 0, for $y=0$ than for any other value of y and the same thing is true for x when we set $y=0$; but the original expression may take on negative values for pairs of values of x and y arbitrarily near the origin, e. g. for $x=\epsilon$, $y=\epsilon$, and consequently does not possess a minimum at (0,0).

In this connection it may be interesting to exhibit a function of n variables which has the property that it does not possess a minimum in all these, but is such that if we set any one of them equal to zero it will possess a minimum in the remaining $n-1$. Such a function is

$$(2x_1 + x_2)^2 + (2x_1 + x_3)^2 + \cdots + (2x_1 + x_n)^2 - x_1^2 .$$

A moment's inspection shows that if we set any variable, say x_n, equal to zero, the above expression will have a positive value for all values of the remaining variables not all zero, while it vanishes if all the variables vanish. On the other hand if we set $x_1=\epsilon$, $x_2=x_3=\cdots=x_n=-2\epsilon$, the function is negative and consequently does not have a minimum for the values $x_1=x_2\cdots=x_n=0$.

IV. A Note on the Sources of Mathematical Reality

By Alan D. Campbell, University of Arkansas

In J. B. Shaw's *Lectures on the Philosophy of Mathematics*, page 10, we find the statement that the sources of mathematical reality have been ascribed at various times to four different worlds:

1. The world of natural phenomena.
2. The world of universals.
3. The world of mental activity.
4. The creative action of the intellect.

use the "creation of the intellect" theory as a working hypothesis, but when he comes to ponder on the question may decide in favor of the other source; then again he may later be drawn back to bow in reverence before the majesty of the human mind that can fabricate such a marvel as mathematics.

V. A Theorem on Improper Definite Integrals

By H. L. Slobin, University of New Hampshire

Theorem: If $f(x)$ is a function integrable from $-\infty$ to ∞ and if $f(x-x^{-1})$ is also integrable from $-\infty$ to ∞, then

$$\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Proof: Let $f(x)$ be an even function $\phi(x)$ so that $\phi(x)=\phi(-x)$. Let $z=1/x$. Then

$$\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{z=\infty}^{0} \phi\left(\frac{1}{z}-z\right)d\left(\frac{1}{z}\right)$$

$$= \int_{z=\infty}^{z=0} \phi\left(\frac{1}{z}-z\right)d\left(\frac{1}{z}-z\right) + \int_{z=\infty}^{z=0} \phi\left(\frac{1}{z}-z\right)dz.$$

Let $(1/z)-z=u$. Then $u=-\infty$ where $z=\infty$, and $u=+\infty$ when $z=0$. Hence

$$\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi(u)du - \int_0^{\infty} \phi\left(\frac{1}{z}-z\right)dz$$

or

$$(1) \qquad 2\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi(u)du.$$

But

$$2\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi\left(x-\frac{1}{x}\right)dx$$

because $\phi(x)$ is an even function. Therefore (1) becomes

$$(2) \qquad \int_{-\infty}^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi(x)dx.$$

This completes the proof when $f(x)$ is an even function. Now let $f(x)$ be an odd function $\psi(x)$ so that $\psi(x)=-\psi(-x)$. Then

$$(3) \qquad \int_{-\infty}^{\infty} \psi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \psi(x)dx = 0,$$

because the integral of any odd function from $-\infty$ to ∞ is equal to zero.

It remains to prove the theorem for the function $f(x)$ of the theorem. That function can be expressed as the sum of an even function $\phi(x)$ and an odd function $\psi(x)$ as follows:

$$(4) \qquad f(x) = \phi(x) + \psi(x) \text{ where } \phi(x) = \tfrac{1}{2}[f(x) + f(-x)] \text{ and } \psi(x) = \tfrac{1}{2}[f(x) - f(-x)].$$

Then

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} \left[\phi\left(x - \frac{1}{x}\right) + \psi\left(x - \frac{1}{x}\right)\right] dx, \quad by\ (4)$$

$$= \int_{-\infty}^{\infty} [\phi(x) + \psi(x)] dx = \int_{-\infty}^{\infty} f(x) dx, \quad by\ (2),\ (3),\ (4).$$

This completes the proof of the theorem.

QUESTION: It is desired to know if there are any practical applications of the theorem and what, specifically, are the restrictions which it imposes on the function $f(x)$.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

THE SECOND CARUS MONOGRAPH

Analytic Functions of a Complex Variable. By D. R. CURTISS. Chicago, The Open Court Publishing Company, 1926. ix+ 173 pages. Price $2.00.

This is the second volume to appear in the series of *Carus Mathematical Monographs.* The aim of the series may fairly be termed unique in the history of mathematics. While possible usefulness to mathematical and scientific students is kept in mind, the chief purpose is to make accessible to thoughtful and educated people generally some of the most important and fruitful results in the field of mathematics.

Every mathematician has probably felt at some time or another a sense of hopelessness about trying to satisfy the intellectual curiosity of others regarding his own work. There are public lectures and printed articles without number by which this curiosity may be satisfied regarding special topics of interest in almost any scientific or intellectual movement other than mathematics. But the queen of the sciences, the one that is essential for the highest development of almost every other exact science, communicates its ideas and results in a language that must be studied to be understood. In addition to this, the full significance of a mathematical theory or formula can be obtained,

in many cases, only by prolonged thought, after examining in detail all the steps of the reasoning and verifying the algebraic processes that accompany them. All of this requires technical skill to be obtained only by long training. How, then, is it possible to "popularize" any mathematical theory in even the slightest degree? It is obvious that this cannot be done except possibly for a comparatively small number of persons who have some knowledge of the elements of mathematics, i. e., of algebra, geometry, trigonometry, analytic geometry, and calculus. The task is certainly a difficult one at best, and only by the most thoughtful labor on the part of those who are willing to lend their aid can this series of monographs hope to satisfactorily fulfill its larger purpose. It is, therefore, with especial interest that mathematicians will note the efforts in this direction as they appear from time to time in published form.

No more interesting subject could have been chosen for this new volume than that indicated in the title. Its importance has been characterized by Hadamard in a very recent statement: "The main, and central, subject of modern analysis lies in the two connected theories of functions and of differential equations."

The task of putting a theory of such importance and magnitude in a nutshell, where "he who runs may read," is obviously one of great difficulty. It has been performed so well in this monograph that there seems little need for the reviewer to do more than express his sincere admiration for the completed result. Good judgment has been exercised in the delicate task of selecting material so as to give a clear notion of some of the fundamental and characteristic ideas and methods in this theory. The proofs are skillfully handled so as to reduce the algebraic work to a minimum of detail and to make the essential steps as simple, clear-cut, and effective as possible. The author sticks closely to a well-ordered plan of treating only the most vital and intimately connected principles of the theory, resisting the ever present temptation to make side excursions for inspecting the vast and exceedingly interesting and varied details presented by special classes of functions. Only such illustrative material is used, and that of the simplest kind, as is absolutely necessary to clarify fundamental concepts or general methods.

The author very properly gives the ideas, methods, and formulas of Cauchy predominance in his treatment, but the present-day view point in modern analysis pervades the entire book. The influence of Weierstrass comes to the front in the treatment of series and analytic continuation, while the fertility of Riemann's ideas is brought out with great clearness in the consideration of many-valued functions.

The second chapter, on complex numbers, is the only place where the reviewer would wish to see any change. In his opinion it could advantageously be shortened by omission of the graphical representation of multiplication

and division. It is unessential to the book and is no aid whatever to the study of these operations on complex numbers. It should be dropped from our text books and from our teaching as useless lumber. The algebraic theory of these operations by use of polar coordinates is perfectly simple and intelligible and all that is needed.

It would seem especially desirable to make the formula work in the beginning of such a monograph as simple and brief as possible. I am, therefore, inclined to think it would have been better to explain multiplication, division, and extraction of roots by using the exponential form $\epsilon^{i\theta}$ instead of $\cos\theta + i\sin\theta$ referring ahead to pages 53–4 for a proof of the equivalence of these expressions.

An admirable feature of the book consists in giving, at the end of each chapter, full and explicit page references to several standard treatises for additional reading on the various topics of the chapter.

The book is very attractively printed and bound. The typographical errors are extremely few. In the second line below the formulas at the middle of page 45, the words, "as Z approaches z is $f(z)$," should read "as z approaches Z is $f(Z)$." In formula (10), p. 127, the first two subscripts of the coefficients should be -2 and -1.

With so simple and attractive an exposition of the elements of the theory of analytic functions of a complex variable now available, it is earnestly to be hoped that many who are not yet familiar with this beautiful theory will read this little volume and derive from it some idea of one of the great conquests in the field of mathematics.

To make this task as easy and pleasant as possible, it is suggested that Chapter II be omitted entirely on first reading by those who have any knowledge of complex numbers, and that it be used only for reference as occasion may arise. For those who have too little familiarity with complex numbers, a study of the first five pages of Chapter II together with Article 7 would be sufficient for the reading of almost everything that follows in the book. Finally, we might note the author's suggestion in the preface, that, even without a knowledge of the calculus, the general reader may "obtain some idea of the scope and purposes of the theory of functions" from an examination of the book.

Nothing has yet been said about the usefulness of this book to the mathematical student or teacher. From what precedes, it may easily be inferred that no better introduction to the subject could be found. A mastery of the contents of this book ought to make further study of functions of a complex variable easy and rapid. It is believed, also, that the teacher will find it an ideal text for a very brief course, especially for a Summer School class.

J. I. HUTCHINSON

Nicomachus of Gerasa, Introduction to Arithmetic. Translated into English by MARTIN LUTHER D'OOGE, with Studies in Greek Arithmetic by FRANK EGLESTON ROBBINS and LOUIS CHARLES KARPINSKI. New York, The Macmillan Co., 1926. ix+318 pages.

This volume is a beautiful illustration of the advantages of cooperative effort in research. The late Professor D'Ooge performed the translation of the arithmetic of Nicomachus from the Greek, but left the manuscript unfinished. This phase of the work was continued by Mr. Robbins who also wrote the chapter on the development of Greek arithmetic before the time of Nicomachus. The other mathematical parts are by Professor Karpinski who wrote on the sources of Greek mathematics, on Greek notation, and on the content of the Arithmetic of Nicomachus.

A fitting ornament to the publication is a generalization of the theorem of Nicomachus, that cubical numbers are always equal to the sum of successive odd numbers. Here again cooperative effort displayed itself. Karpinski gave the analogous theorem for fifth and the seventh powers, and generally for the $n^{2k+1}th$ power. Then Professor N. Anning gave a further extension, and Mr. E. B. Escott rose still higher in the general view-point, by a theorem which embraces as special cases the generalizations of Karpinski and Anning.

Nicomachus was not a great mathematician, but his *Introduction to Arithmetic* was widely read; it was used by Boethius and indirectly supplied the content of arithmetical study in Europe down to the time of the influx of Hindu-Arabic arithmetic. It is a work of importance to students of the history of mathematics. The men who cooperated in the production of this translation and commentary have rendered a real service to such students.

FLORIAN CAJORI.

Calculus, second edition. By H. W. MARCH and H. C. WOLFF. New York, McGraw-Hill Book Company, 1926. xvii+398 pages. Price $2.50.

This text is equally adapted to technical students and to others pursuing a course in calculus for four hours a week throughout the year. The subject is presented in a simple and direct manner and the more difficult topics are reserved till the later chapters.

The anti-derivative is introduced in chapter III and simple applications are given early so that courses requiring calculus as a prerequisite may be studied at the same time. The power function is treated early with applications to curves and moving bodies. Implicit differentiation properly appears here. With algebraic functions come applications to maxima and minima and rates. Inflections and simple areas come next with work of a variable force, discussion of the parabolic cable, and further applications to mechanics.

The treatment of infinitesimals starts in chapter VII and "in this connection Duhamel's theorem is used as a valuable working principle, though the refinements of statement upon which a rigorous proof can be based have not been given" (Preface). Too much rigor in a first course tends to kill the student's enthusiasm but is there any reason why this theorem could not be stated and explained as in Osgood's *Introduction to the Calculus*, 1922, pp. 301–304? In a first course there is no valid objection to proving that the anti-derivative of a function may be represented as an area and omitting Duhamel's theorem. There is another road open to those who wish to avoid the use of the law of the mean in establishing the fundamental theorem of integral calculus. That is to use the method of proof given in Snyder and Hutchinson's *Calculus*, 1912, pp. 270–273, or in Edwards' *Integral Calculus for Beginners*, 1920, pp. 14–16.

One of the few obscure statements of the book occurs at the beginning of chapter VIII on circular functions. A reference to §55 instead of §56 was made on the same page in making the changes for the revision. The revision is a distinct improvement and includes a valuable rearrangement of chapter XVII on infinite series, additional or new exercises in chapters III to VII and IX. Certain other changes were made in the text, which now includes the new sections 61 on approximate relative error and 73 on weighted mean.

In chapter IX one is attracted by the simplicity and shortness of the method of obtaining the derivative of an exponential function. Herein lie two weaknesses: first, the existence of the limit involved is not even made plausible; secondly, no reason is given to show that the e there obtained is the base of the natural system of logarithms.

Among the topics reserved till later are polar coordinates, concavity, successive integrations, special methods of integration, improper integrals, curvature, evolutes and envelopes. Infinite series are used to evaluate indeterminate forms. DeMoivre's theorem is obtained and Rolle's theorem stated, and finally the law of the mean is proved and extended to Taylor's theorem.

A chapter on solid geometry enables the uninitiated student to take up the application of triple integration without difficulty.

The last two chapters deal with total differentiation, exact differentials and differential equations. It is noted with satisfaction that the integration of an exact differential in two variables is correctly done, but the treatment of initial conditions in the solution of differential equations is not reinforced by sufficient practical exercises. Chapters XVI and XVIII also end without lists of exercises but there are about 1700 exercises given, the answers to many of which may be obtained from the publishers in an answer book. There are about 170 illustrative exercises and 145 figures.

Inaccuracies in printing occur in a few places as in line 2, p. 258; ex. 10, p. 281, upper limit for y; figure 107, subscript for C; and in ex. 5, p. 342, part of the old fashioned factorial sign is omitted. On page 369 the phrase "we seek to find a test" occurs. There are also what might be regarded as omissions, such as Newton's method and other forms of iteration, Simpson's rule, and a rapid series for π. On the whole, however, the book appeals to the reviewer as one of the most teachable books on the calculus and one of the best texts which have so far appeared in English.

C. C. CAMP.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Bulletin of the American Mathematical Society, volume 32, no. 6, November-December 1926: "Recent progress with the Dirichlet problem" by O. D. Kellogg, 601–625; "On irredundant sets of postulates" by A. Church, 626–628; "Generalization of Lagrange's theorem" by L. Weisner, 629-630; "The transformation of a regular group into its conjoint" by G. A. Miller, 631–634; "A characteristic property of minimal surfaces" by J. Douglas, 635–638; "On the convergence of trigonometric approximations for a function of two variables" by E. Carlson, 639–640; "On a tensor of the second rank in function space" by D. Jackson, 641–643; "All integral solutions of $ax^2+bxy+cy^2=w_1w_2\cdots w_n$" by L. E. Dickson, 644–648; "Nuclear points in the theory of abstract sets" by W. Sierpinski, 649–653; "On the integro-differential equation of the Bôcher type in three-space" by G. E. Raynor, 654–658; "Two way continuous curves" by G. T. Whyburn, 659–663; "On some theorems of Bôcher concerning isolated singular points of harmonic functions" by O. D. Kellogg, 644–668; "The transversality relative to a surface of $\int F(x, y, z, y', z')dx=$ minimum" by J. Douglas, 699–673; "On the extension of a method of Briot and Bouquet for the reduction of singular points" by B. O. Koopman, 674–678; "A curious irreducible continuum" by W. A. Wilson, 679–681; "On a fundamental formula in the theory of class-number relations" by E. T. Bell, 682–688; "On the functional equation $f(x+y)=f(x)+f(y)$" by M. Kormes, 689–693; "On quantifiers for general propositions" by C. H. Langford, 694–704.

Proceedings of the National Academy of Sciences, U. S. A., volume 31, No. 1, January 1927: "Groups generated by two operators of order three whose product is of order three" by G. A. Miller, 24–26

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3258. Proposed by H. E. Arnold, Wesleyan University, Middletown, Connecticut.

The perpendiculars dropped from the vertices of a triangle upon the lines joining the feet of the angle bisectors to the orthocenter of the triangle meet the respective sides of the triangle in three collinear points. The line joining these three points is perpendicular to the line joining the orthocenter to the center of the inscribed circle.

3259. Proposed by Norman Anning, University of Michigan.

Solve the cyclic set of simultaneous equations,

$$x_1 - 2x_n = x_n^2 x_1\,; \quad x_{j+1} - 2x_j = x_j^2 x_{j+1}\,,$$

where n is any positive integer greater than unity and $j=1, 2, 3, \cdots, (n-1)$.

3260. Proposed by B. C. Keeler, New York City.

Prove that the following series is convergent.

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots+\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}+\cdots$$

3261. Proposed by J. Rosenbaum, Milford, Connecticut.

On the sides AB, BC, CD, and DA of a quadrilateral $ABCD$, the points P, Q, R, and S are taken so that $AP/AB=BQ/BC=CR/CD=DS/DA$, thus forming the quadrilateral $PQRS$.

Prove that if the two quadrilaterals are similar then they are parallelograms.

3262. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through two given points, real or conjugate imaginary, to draw a circle so that the ends of one of its diameters shall lie on two given lines.

3263. Proposed by Otto Dunkel, Washington University.

Give an elementary proof, without the use of the calculus, of the theorem that the regular tetrahedron has the greatest volume of all tetrahedrons having the same surface.

SOLUTIONS

3193 [3181; 1926, 278] Proposed by C. K. Robbins, Purdue University.

Solve each of the following equations:

(1) $(p-ay)^a=c(p-by)^b$;

(2) $p-ay=ce^{ay/(p-ay)}$;

(3) $[p^2-2apy+(a^2+b^2)y^2]e^{\mu}=c$ where $\mu=(2a/b)\tan^{-1}[(p-ay)/by]$

where $p=dy/dx$ and a, b, c are given constants.

Solution by the Proposer.

The three equations given above correspond to the three cases which arise when one tries to solve the linear homogeneous differential equation of the second order by expressing it as one of the first order and performing one quadrature. Thus the method for solving all three is the same. Consider (3). Taking the logarithms of both sides gives

$$\log\left[p^2-2apy+(a^2+b^2)y^2\right]+\frac{2a}{b}\tan^{-1}\left(\frac{p-ay}{by}\right)=\log c.$$

Differentiating gives

$$\frac{2pp'-2a(p+yp')+(a^2+b^2)2y}{p^2-2apy+(a^2+b^2)y^2}+\frac{2a}{b}\,\frac{\dfrac{by(p'-a)-(p-ay)b}{b^2y^2}}{1+\left(\dfrac{p-ay}{by}\right)^2}=0,$$

where $p'=dp/dy$. Simplifying gives

$$pp' - 2ap + (a^2 + b^2)y = 0, \text{ or } \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0,$$

since

$$p = \frac{dy}{dx} \text{ and } p\frac{dp}{dy} = \frac{d^2y}{dx^2}.$$

The solution of this equation is found by standard methods to be

$$y = e^{ax}(A \cos bx + B \sin bx).$$

If we set $A = R \cos b\psi$, $B = R \sin b\psi$, where R and ψ are constants, and substitute in the differential equation (3), we obtain $R = b^{-1}c^{1/2}e^{-a\psi}$, so that the solution becomes

$$y = b^{-1}c^{1/2}e^{a(x-\psi)} \cos b(x - \psi).$$

A supplementary solution of (3) is

$$y = c^{1/2}(a^2 + b^2)^{-1/2}e^{\rho} \text{ where } \rho = ab^{-1}\tan^{-1}(ab^{-1}).$$

Equations (1) and (2) can be solved in similar fashion. The results are respectively,

$$y = Ae^{ax} - c^{1/a}A^{b/a}(a - b)^{b/a-1}e^{bx} (\text{if } a \neq b), \text{ and } \frac{Ae^{ax}}{a}(ax + \log A - \log c).$$

NOTE: (1) and (2) also have the solutions $y = [c(-b)^b/(-a)^a]^{1/(a-b)}$ and $y = -c/ae$ respectively.

3195 [3183; 1926, 278] Proposed by Nathan Altshiller-Court, University of Oklahoma.

One of two given circles is fixed, the other rolls on a fixed straight line. Find (1) the locus of the trace of the radical axis of the two circles on their line of centers; (2) the envelope of the radical axis; (3) the locus of the limiting points of the two circles.

SOLUTION BY THEODORE BENNETT, University of Illinois.

After making a suitable choice of coordinate axes, we may write the equations of the circles in the form

$$x^2 + (y - a)^2 = r^2, \quad (x - t)^2 + y^2 = R^2,$$

where a, r, R are constants, and t is a variable. The equations of the line of centers and of the radical axis are

$$\frac{x}{t} + \frac{y}{a} = 1, \quad 2tx - 2ay + a^2 - r^2 + R^2 - t^2 = 0.$$

Eliminating t between these equations, we find that the equation of the locus required in (1) is

$$K_1 \equiv (a - y)^2(a^2 - r^2 + R^2 - 2ay) + ax^2(a - 2y) = 0.$$

Since the equation of the radical axis is quadratic in t, the envelope of the radical axis is easily found by equating to zero the discriminant of this quadratic; we find the locus to be

$$K_2 \equiv x^2 - 2ay + a^2 - r^2 + R^2 = 0.$$

The equation of any circle through the intersection of the two given circles may be written

$$C \equiv (x^2 + y^2 - 2tx + t^2 - R^2) + \lambda(x^2 + y^2 - 2ay + a^2 - r^2) = 0.$$

The center of C is

(a) $$\xi = t/(1 + \lambda), \quad \eta = a\lambda/(1 + \lambda).$$

C is a point circle if

(b) $$t^2 + a^2\lambda^2 + [R^2 - t^2 + \lambda(r^2 - a^2)](1 + \lambda) = 0.$$

Eliminating t and λ between equations (a) and (b) we find the locus required in (3) to be

$$K_3 \equiv a\eta\xi^2 - (a - \eta)[a\eta^2 - \eta(a^2 - r^2 + R^2) + aR^2] = 0.$$

The loci K_1, K_2, K_3 are closely related. K_2 is a parabola, while K_1 and K_3 are circular cubics; K_1 has a double point at $(0, a)$, the center of the fixed circle; K_3 passes through this point simply, and is non-singular. The horizontal asymptotes of K_1 and K_3 are $y=a/2$ and $\eta=0$ respectively. K_1 and K_2 cut the y axis at the same point, and have the same tangent line at this point. According as this common tangent line lies below, coincides with, or lies above the line $y=a$, the singular point of K_1 is isolated, is a cusp, or is a double point with real tangents.

NOTE BY OTTO DUNKEL, Washington University.

The envelope may be determined geometrically. Let C be the center of the fixed circle of radius r, l the line of centers of circles of radii R, and D the foot of the perpendicular from C on l. Let $m=OK$ be the radical axis of the fixed circle r and the circle of radius R with center at D, and suppose that m cuts CD in O. Consider one of the variable circles with center at Q on l. The radical axis of D and Q is perpendicular to QD at its middle point M and it cuts m at M'. Hence the radical axis of circles Q and C is the perpendicular through M' to CQ cutting it in H and cutting CD in T. Then H is a point of the first locus. Let the perpendicular to l at Q cut the radical axis $TM'H$ in P. Then P is a point on the second locus. For, if we consider a circle with a center Q' very near Q, the radical axis of C and Q, meets that of C and Q' on the common chord of Q and Q', and this chord is perpendicular to l. In the limit this chord becomes the diameter of Q perpendicular to l along which lies QP cutting m in K. It is now easy to show from the construction that P lies on a parabola whose focus is the intersection of the parallel FM' to CQ through M' with CD, the axis of the parabola. For M' is the middle point of PT; also from the similar triangles FOM' and CDQ we have $FO=CD/2$ since $OM'=DQ/2$. Now $PF=FT=FO+OT=FO+PK$. Hence P lies on a parabola with vertex at O, focus at F, and with a directrix parallel to m and at a distance from m equal to FO.

The locus in (1) is the pedal curve of this parabola with respect to C. The two limit points in (3) are easily constructed and the curves may be studied from the construction.

Also solved by A. G. CLARK.

3197 [3185; 1926, 279] Proposed by B. C. Wong, Berkeley, California.

Prove geometrically that the tangent to the cardioid $\rho=2a(1-\cos\theta)$ and the bisector of the vectorial angle of the point of contact meet on the cissoid $\rho=2a\sin\theta\tan\theta$.

SOLUTION BY A. G. CLARK, Colorado Agricultural College.

Consider the equal circles with centers at A, B, C, where the circles A and B are fixed and tangent at O, while C is the generating circle of the cardioid touching circle A at D. Denote by E the other extremity of the diameter DC and by J the other extremity of the diameter OB. If the arc DF is constructed equal to the arc DO, the locus of F is the cardioid, and the tangent to the cardioid at F is EF. Let the bisector of the vectorial angle FOB cut this tangent at P, the circle B at G, and the tangent to the circle B at J in the point H.

It is immediately evident from a consideration of angles that the three right triangles OPD, GOJ, and GJH are similar, and since $OD=JG$, because the angle OAD equals twice angle JOG, the first and third triangles named are equal. Therefore, $OP=GH$ thus defining the locus of P as the cissoid.

Also solved by MICHAEL GOLDBERG.

3200 [3188; 1926, 338] Proposed by B. F. Finkel, Drury College.

Find the equation of the curve whose radius of curvature at any point of the curve is n times the radius vector to the same point.

PARTIAL SOLUTION BY H. W. BAILEY, University of Illinois.

The curve is evidently a spiral. Substitution shows that the radius of curvature, R, of the logarithmic spiral $\rho=e^{a\theta}$ is a constant times the radius vector at any point on the curve. Using the given condition,

$R=n\rho$, we get $n=(1+a^2)^{1/2}$ as the equation defining a in terms of n. It follows that the required curve is $\rho=e^{\mu}$ where $\mu=(n^2-1)^{1/2}\theta$. It will be noticed at once that the curve is real only when $n\geqq 1$, and that for $n=1$ it reduces to a circle.

NOTE BY THE EDITORS This work shows that the curve $\rho=e^{a\theta}$ is a solution; it does not show that it is the only solution.

3203[3191; 1926, 338]. Proposed by A. A. Bennett, Lehigh University.

Prove that each positive integer less than 2×10^9, has the property of containing an odd prime factor or else of being expressible as a product (with distinct factors) of the form $\Pi_i(p_i-1)$ where each p_i is a prime.

SOLUTION BY MICHAEL GOLDBERG, Washington, D. C.

A number which is not an odd prime, nor divisible by an odd prime, must be a power of 2. Since Fermat's numbers 2^h+1, where $h=2^n$, are prime for $n=0,1,2,3,4$ (but not for $n=5$), 2^n is then $1,2,4,8$, or 16. The numbers 1 to 31 can be formed by adding these numbers, not more than one of each being necessary. Hence, $\Pi_i(p_i-1)$ can become $2^{31}=\text{antilog}(31\times 0.301030)=\text{antilog } 9.331930>2\times 10^9$.

Also solved by F. L. WILMER.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The summer meeting of the Association will be held at the University of Wisconsin on Monday and Tuesday, September 5–6, in conjunction with the summer meeting and colloquium of the American Mathematical Society. The Society sessions will be on Thursday and Friday and the colloquium lectures on Tuesday to Saturday. The joint committees on arrangments are making great preparations for entertaining the visitors and there will be opportunity for those who may so desire to spend a part of August in this beautiful lake region. For information communicate with Professor Arnold Dresden, Madison, Wisconsin.

The third Carus Monograph by Professor H. L. Rietz on "Mathematical Statistics" has just been published by the Open Court Publishing Company for the Mathematical Association of America. These monographs are printed by the University of Chicago Press, and represent the acme of excellence in mechanical make-up. This series is made possible by the generosity of Mrs. Mary Hegeler Carus. They are available to members of the Association (individual and institutional) at the base cost price of one dollar and twenty-five cents each and the entire receipts from such sales go into a special fund of the Association known as the "Carus Publication Fund." Non-members can procure their monographs only through the Open Court Company at the sale price of two dollars. It is hoped that every member of the Association

will possess the entire series. Numbers four and five are in process of preparation.

There will be a new Register of the Association published in the early Autumn. Please report all changes of address for 1927–28 to the Secretary's office. And, quite as important, please try to induce all non-members in your vicinity to apply for membership before Autumn. Application blanks may be had from the Oberlin Office.

The twenty-seventh Western meeting of the American Mathematical Society was held at the University of Chicago, April 15–16, 1927. There were more than fifty papers presented aside from a two-hour symposium lecture by Professor E. W. Chittenden on "Some phases of general topology."

The next international Congress of Mathematicans will be held at Bologna during the first ten days of September, 1928. American mathematicians appreciate the consideration shown them by the Italian mathematicians in choosing a time of year convenient for those who live on this side of the Atlantic. It is hoped that many Americans will attend the Congress. Further details will be announced later.

John Simon Guggenheim Memorial Fellowships in Mathematics have been awarded as follows:

To Dr. PHILIP FRANKLIN, Assistant Professor of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts—for a study of integral equations, orthogonal functions and their relation to almost periodic functions, principally at Göttingen, Germany, and Zurich, Switzerland.

To Dr. HARRY SCHULZ VANDIVER, Associate Professor of Pure Mathematics, University of Texas, Austin, Texas—to engage in original research in connection with Fermat's last theorem, the laws of reciprocity and related topics in the theory of algebraic numbers, and to confer with specialists in the theory, chiefly in England, Switzerland and Germany.

The Massachusetts Institute of Technology has received a grant of $233,000 from the Guggenheim Fund for the Promotion of Aeronautics; the gift will provide a building and equipment.

The University of Pittsburgh has appointed a number of engineers of the Westinghouse Company as part-time lecturers at the University in electrical and mechanical engineering, physics, and engineering mathematics. Among these lecturers is Dr. JOSEPH SLEPIAN.

Professor S. D. WICKSELL of Lund University has been invited to lecture on mathematical statistics at the University of Michigan during the coming academic year.

Mr. S. F. BIBB of the University of North Dakota has been appointed professor of mathematics at the Armour Institute of Technology.

Professor ARNOLD DRESDEN of the University of Wisconsin has been appointed professor of mathematics at Swarthmore College, Swarthmore, Pa. An interesting feature of his work in that college will be in connection with the honors course for the juniors and seniors. Students in that course are not obliged to attend classes, are free to work at tasks assigned to them on which they have conferences with their instructors as often as may seem desirable. No grades or records are kept during these two years. At the end of the senior year they have to take a comprehensive examination covering the work of these two years and conducted both in its oral and written parts by an outsider.

Dr. I. MAIZLISH, Associate Professor of Mathematics and Physics and Head of the Department of Physics at Centenary College, has been promoted to a full professorship. Also he was elected president of the Louisiana Academy of Sciences on March 5th. Dr. H. L. SMITH, of Louisiana State University, is Vice-president; Prof. S. T. Sanders, of L. S. U., is a member of the Executive Council; and Prof. John A. Hardin, of Centenary College, is a member of the membership committee.

Professor E. D. MEACHEM has been made Assistant Dean of the College of Arts and Sciences at the University of Oklahoma.

Dr. A. D. MICHAL, National Research Fellow at Princeton University, has been appointed assistant professor of mathematics at the Ohio State University.

Professor EUGENIE M. MORENUS of Sweet Briar College, who has been awarded the Anna B. Brackett Memorial Fellowship by the American Association of University Women, expects to study during the coming year at Cambridge, England.

Professor J. J. NASSAU, of Case School of Applied Science, has been granted a year's leave of absence, beginning next September, to study at Cambridge University.

Dr. GEORGE E. RAYNOR, Wesleyan University, has been appointed associate professor of mathematics at the University of Oklahoma.

Mr. JOHN STEHN, University of Iowa, has been appointed instructor at the University of Oklahoma, and Mr. Stephan Brixey has been appointed assistant.

The following additional courses in Mathematics are announced for the summer of 1927:

University of Colorado First term, June 20 to July 23; Second term, July 25 to August 26. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, calculus, the following courses will be offered. First term: By Professor LIGHT, Teacher's Course in Mathematics; History of Mathematics; Functions of a complex variable. By Professor KEMPNER: Advanced Teacher's Course in Mathematics; Differential Equations; Theory of numbers. Second term: By Professor LIGHT: Statistics; Theory of Equations; Functions of a complex variable (continued). By Professor KEMPNER: Advanced Teacher's Course; Differential Equations (continued); Theory of numbers (continued).

The following 44 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1926; the university, month in which the degree was conferred, minor subjects (other than mathematics), and title of dissertation are given in each case if available.

EVELYN F. AYLESWORTH, California, May, *The dielectric constant of atomic hydrogen and the Stark effect.*

WEALTHY BABCOCK, Kansas, June, physics, *On the geometry associated with certain determinants with linear elements.*

H. W. BAILEY, Illinois, May, astronomy, *The summability of single and multiple Fourier series.*

R. W. BARNARD, Chicago, December, *The Fredholm theory of linear integral equations in general analysis for quaternionic-valued functions.*

MARTHA H. BARTON, Johns Hopkins, June, physics, *Some applications of the generalized Kronecker symbol.*

FLORENCE BLACK, Kansas, June, Physics, *A reduced system of differential equations for the invariants of ternary forms.*

H. L. BLACK, Illinois, May, physics, *A Cremona group isomorphic with the group of the twenty-seven lines on a cubic surface.*

E. T. BROWNE, Chicago, September, *Involutions that belong to a linear class.*

N. B. CONKWRIGHT, Illinois, May, astronomy, *The summability of Birkhoff series.*

A. E. COOPER, Chicago, June, *A topical history of the theory of quadratic residues.*

A. H. COPELAND, Harvard, *Studies on the gyroscope.*

C. M. CRAMLET, Washington, June, physics and astronomy, *Invariant tensors and their applications to the study of determinants and allied tensor functions.*

H. T. DAVIS, June, Wisconsin, physics, *An existence theorem for the characteristic numbers of a certain boundary value problem.*

M. S. DEMOS, Harvard, *The group characteristics of the general and special quaternary linear homogeneous congruence groups.*

R. D. DONER, Illinois, May, physics, *The determination of the Peirce and Scheffer's algebras of order eight.*

FAY FARNUM, Cornell, June, physics, *Triadic Cremona nets of plane curves.*

ORRIN FRINK, Columbia, December, *The operations of boolean algebras.*

R. J. GARVER, Chicago, September, I: *On Tschirnhaus transformations*; II *Division algebras of order sixteen.*

J. S. GEORGES, Chicago, September, *Associativity conditions for division algebras corresponding to any abelian group.*

R. F. GRAESSER, Illinois, May, statistics and physics, *A certain general type of Neumann expansions and expansions in confluent hyper-geometric functions.*

MARION C. GRAY, Bryn Mawr, June, physics, *Theory of singular ordinary differential equations of the second order.*

L. S. HILL, Yale, June, *Aggregate functions and an application in analysis situs.*

H. M. HOSFORD, Illinois, May, physics, *On the summability of Fourier-rational Bessel and Dini expansions.*

C. M. HUBER, Illinois, May, theoretical physics, *On complete systems of irrational invariants of associated point sets.*

F. E. JOHNSTON, Illinois, May, astronomy, *Transitive substitution groups containing a regular subgroup of lower degree.*

B. F. KIMBALL, Cornell, June, physics, *Geodesics on a toroid.*

F. W. KOKOMOOR, Michigan, June, physics, *The teaching of elementary geometry in the seventeenth century.*

B. O. KOOPMAN, Harvard, *On rejection to infinity and exterior motion in the restricted problem of three bodies.*

B. C. PATTERSON, Johns Hopkins, June, geophysics, *The algebraic and differential invariants of inversive geometry.*

H. R. PHALEN, Chicago, June, *Metric properties of the quadric of Moutard.*

I. R. POUNDER, Chicago, September, *A method of successive approximations for a partial equation of hyperbolic type.*

H. H. PRIDE, New York University, June, *On a certain quintic curve with a triple point.*

D. E. RICHMOND, Cornell, June, physics, *Geodesics on surfaces of genus zero with knobs.*

C. F. ROOS, Rice Institute, June, theoretical economics, *Generalized Lagrange problems in the calculus of variations.*

D. P. SCHWARTZ, Yale, June, *Studies in non-Euclidean geometry.*

M. M. SLOTNIK, Harvard, *Fundamental transformations of surfaces.*

H. L. SMITH, Chicago, September, *The Minkowski linear measure for a simple rectifiable curve.*

P. A. SMITH, Princeton, *Approximation of curves and surfaces by algebraic curves and surfaces.*

MARION E. STARK, Chicago, September, *A self-adjoint boundary value problem associated with a problem of the calculus of variations.*

GUY STEVENSON, Illinois, May, physics, *Expansions of the Neumann type in terms of products of Bessel functions.*

M. H. STONE, Harvard, *Ordinary linear homogeneous differential equations of order two and the related expansion problems.*

TAKASHA TERAMI, California, May, physics, *The solution of the differential equation of a vibrating membrane by successive approximations.*

E. T. VIRATA, Johns Hopkins, June, geophysics, *W-surfaces which have an isometric spherical representation of their lines of curvature.*

P. S. WAGNER, Johns Hopkins, June, geophysics, *An extension of Clifford's chain.*

Mr. A. C. BOSE, Deputy Magistrate, Bengal Provincial Civil Service, died December 11, 1926.

Father RIGGE of Creighton University, Omaha, Nebraska, died in March of this year.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars, to be known as the .... Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: .... Signature ....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

Two such funds have already been started:
THE CARUS PUBLICATION FUND and
THE ARNOLD BUFFUM CHACE FUND.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.
Twelfth Annual Meeting, Nashville, Tenn., December, 1927.
The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 7.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 14.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November 26.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formula. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, 0 means "zero"; to indicate "cap O", backspace and overprint period; thus: O.
15. Avoid a dash over a letter, except for those shown below.
16. **Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.**

Light Face Greek—$\alpha\ \beta\ \gamma \cdots$ (all) A B $\Gamma \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma \cdots}$ (all except ι and o).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma \cdots}$ (all except ι and o);

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \varrho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q\ A\ B\ C\ D\ E\ F\ G\ H\ J\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$

* Boldface German—$\boldsymbol{\mathfrak{d\ A\ B\ D}}$

Script (special font) $\mathscr{A}\ \mathscr{B}\ \mathscr{C} \cdots$ (all). No lower case manufactured.

* Hebrew—א א כ ב troublesome to handle.

★ Dashed Italics—$\bar{A}\ \bar{a}\ \bar{B}\ \bar{b}\ \bar{C}\ \bar{c}\ \bar{E}\ \bar{e}\ \bar{F}\ \bar{f}\ \bar{G}\ \bar{g}\ \bar{H}\ \bar{I}\ \bar{\imath}\ \bar{K}\ \bar{k}\ \bar{M}\ \bar{m}\ \bar{N}\ \bar{n}\ \bar{P}\ \bar{p}\ \bar{q}\ \bar{R}\ \bar{r}\ \bar{s}\ \bar{t}\ \bar{u}\ \bar{v}\ \bar{X}\ \bar{x}\ \bar{Y}\ \bar{y}\ \bar{Z}\ \bar{z}$

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \bar{\Gamma}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

---

* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 6, JUNE-JULY

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

TWELFTH ANNUAL MEETING OF THE OHIO SECTION

The twelfth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 8, 1927, in connection with the meetings of the Ohio College Association. Chairman H. W. Kuhn presided over the afternoon session. Following the dinner in the evening there was a round table discussion under the direction of Professor I. A. Barnett, chairman of the program committee.

Fifty-five persons registered attendance, among whom were the following forty-three members of the Association: R. B. Allen, W. E. Anderson, Grace M. Bareis, I. A. Barnett, H. M. Beatty, H. Blumberg, R. L. Borger, L. Brand, J. B. Brandeberry, C. T. Bumer, R. S. Burington, V. B. Caris, E. H. Clarke, R. Crane, O. L. Dustheimer, C. A. Garabedian, H. Hancock, E. J. Hirschler, E. M. Justin, J. H. Kindle, H. W. Kuhn, C. C. MacDuffee, J. E. Merrill, C. C. Morris, R. L. Newlin, J. R. Overman, J. Pierce, S. E. Rasor, B. H. Redditt, Hortense Rickard, S. A. Rowland, E. C. Rupp, Mary E. Sinclair, S. A. Singer, A. Helen Tappan, J. C. Tinner, M. O. Tripp, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, P. D. Wilkins, J. B. Winslow, C. H. Yeaton.

Several items of business of local interest were transacted, and the following officers were elected for the coming year: Chairman, C. H. YEATON; Secretary-Treasurer, RUFUS CRANE; Member of the Executive Committee, C. T. BUMER; Member of the Program Committee, HENRY BLUMBERG. It is expected that the next meeting will be held at the Ohio State University on April 6, 1928.

The following five papers were presented:

1. "Galois fields and permutation groups" by the Chairman, Professor H. W. KUHN, Ohio State University.

2. "A new type of singular solution" by Professor H. W. SIBERT, University of Cincinatti, by invitation.

3. "The mystic number seven" by Professor HARRIS HANCOCK, University of Cincinnati.

4. "Mathematics as related to other collegiate work" by Professor O. L. DUSTHEIMER, Baldwin-Wallace College.

5. "The calculus of variations, from the standpoint of vector analysis" by Professor LOUIS BRAND, University of Cincinnati.

Abstracts of these papers follow, the numbers corresponding to those in the list of titles.

1. In this paper Professor Kuhn showed how two groups determined by the addition and the multiplication tables of a Galois field of order 2^3 can be

represented as regular permutation groups of degree 8 and 7 respectively which generate a doubly transitive permutation group of degree 8 and order 56. By reversing the process it was shown how this doubly transitive group can be used to set up the addition and the multiplication tables of a Galois field of order 2^3.

2. Forsyth, Goursat, and others, (i) virtually define a singular solution of a differential equation of the first order to be the envelope, and (ii) subsequently state that a relation derived from the p-discriminant and satisfying the differential equation is a singular solution. Professor Sibert shows that for differential equations of degree $n>2$ the p-discriminant can furnish a relation which satisfies the differential equation but which is not an envelope. Hence if (i) is taken as definition, (ii) is not in general true; if (ii) is taken as definition, the author has found an apparently new type of singular solution.

3. This paper is published in this issue of the MONTHLY.

4. Professor Dustheimer presented a summary of replies to a questionnaire from more than 200 colleges. Of these, 65% require not more than 2 units for entrance; 62% require no mathematics for graduation; 35% require physics of students majoring in mathematics; 34% have a mathematics club. Other interesting statistics were given.

5. Professor Brand considered the problem of finding a plane curve joining two fixed points over which the line integral $\int F(\boldsymbol{r}, \theta)ds$ assumes its maximum or minimum value. Here $\boldsymbol{r}$ denotes the position vector of a current point P on the curve and θ the angle of inclination of the tangent to the curve at P with a fixed line. It was shown that Euler's equation has the vector analogue, $\nabla_r F = d/ds(F\boldsymbol{T}+F_\theta \boldsymbol{N})$, where $\boldsymbol{T}$, $\boldsymbol{N}$ represent unit vectors in the positive directions of the tangent and normal of the curve. Methods of integrating this differential equation were developed in several cases. In the region occupied by a field of extremals which define $\boldsymbol{T}=\boldsymbol{a}$, $\boldsymbol{N}=\boldsymbol{b}$, $\theta=\psi$, as point functions, the vector point function, $\boldsymbol{Q}=F(\boldsymbol{r}, \psi)\boldsymbol{a}+F_\psi(\boldsymbol{r}, \psi)\boldsymbol{b}$, proves to be of fundamental importance in establishing both necessary and sufficient conditions for an extreme. Rot $\boldsymbol{Q}=0$ within the field, and the integral $\int \boldsymbol{Q}\cdot \boldsymbol{T}ds$ has the same value over all curves within the field joining two given points. Moreover, $\boldsymbol{Q}$ is expressible as the gradient of a scalar point function W. The curves, $W=\text{const.}$, are the transversals of the field. The transversality and envelope theorems were then established. Finally, the necessary conditions of Legendre and Weierstrass were expressed in terms of the function $\boldsymbol{Q}$.

RUFUS CRANE, *Secretary*

ELEVENTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The eleventh annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado College, Colorado Springs, Colorado, on April 22, 23. There were forty present including the following twenty members of the association: G. H. Albright, A. G. Clark, E. A. Cummings, I. M. DeLong, Philip Fitch, A. J. Kempner, Claribel Kendall, G. H. Light, W. V. Lovitt, S. L. Macdonald, J. Q. McNatt, W. K. Nelson, Letitia Odell, O. H. Rechard, H. L. Rietz, W. J. Risley, H. E. Russell, Mary S. Sabin, C. H. Sisam, C. W. Wray.

The section voted to hold its next annual meeting at the Colorado School of Mines, Golden, Colorado. The following officers were elected: W. J. RISLEY, chairman; G. W. FINLEY, vice-chairman; PHILIP FITCH, secretary; G. H. LIGHT, treasurer.

At the complimentary dinner given by Colorado College on Friday, Dean C. B. Hershey delivered an address of welcome to which Professor W. J. Risley made the response. The section was favored Friday evening with an address "Group Insurance" by Professor H. L. Rietz of the University of Iowa. Professors H. E. Russell and C. H. Sisam read sketches of the lives and work of the late Dean H. A. Howe and the late Professor F. H. Loud respectively.

The following eleven papers were read:

1. "The sectioning of college freshmen in mathematics by means of the Iowa Placement Test" by Professor C. F. BARR.
2. "A grade weigher" by Professor G. H. ALBRIGHT.
3. "A graphic solution of tensions in cables" by Mr. J. Q. MCNATT.
4. "On a type of involutions in space" by Professor C. H. SISAM.
5. "Concerning the Heusler alloys" by Mr. PHILIP FITCH.
6. "An elementary method of solving matricial equations" by Professor O. H. RECHARD.
7. "Applications of elementary divisors" by Mr. D. L. GUNDER (by invitation).
8. "On a geometrical problem from the MONTHLY" by Professor A. J. KEMPNER.
9. "The Carus monograph on statistics" by Professor H. L. Rietz.
10. "Index number bias" by Professor W. V. LOVITT.
11. "A nomograph" by Professor W. K. NELSON.

In the absence of the author, the abstract of Professor Barr's paper was read by Professor Rechard. Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The Iowa Training Examination in Mathematics was given at Purdue University to some 800 entering freshmen engineers, fall of 1925. This paper is a study of the degree of correlation between the grades obtained, and the subsequent classroom records in mathematics, of the persons involved. There was found a high degree of correlation. Furthermore, there appeared clearly defined lines for segregation into inferior, normal, and superior groups. The careful analysis of classroom records for the members of each group showed a remarkably high power of selectivity for the Iowa examination, and commends it as a tool for segregation of entering freshmen in mathematics. The conclusions are supported by the results of a similar study, made the preceding year, and ending simultaneously with the present study. While the paper demonstrates the possibility, it does not argue the advisability of classroom segregation of students.

2. In this paper Professor Albright described the construction and manipulation of a device for determining the average grade of a student involving the weighing of his various marks to correspond to the units of credit carried by his course. The device was of the type of a simple lever. Such an instrument was exhibited which would calculate the average with an error not exceeding three-hundredths of one percent.

3. Mr. McNatt gave a graphic solution of the tensions in a cable used to carry movable loads as in the case of aerial trams.

4. In this paper, the author discusses involutions in space such that the locus of the lines joining corresponding points is a congruence of order one defined by a (l, m) between the points on and the planes through a fixed line. Several particular involutions of this type were discussed at some length.

5. This paper dealt with mathematics as applied to data obtained from measurements on the resistance, permeability and thermo-electric affect of the Heusler alloys.

6. Professor Rechard showed that the solution of the general equation of the same order reduces to the problem of solving n^2 equations in n^2 unknowns, each equation being of degree n. A quadratic equation was used to illustrate the method.

7. Mr. Gunder showed that the principles of elementary divisors was directly applicable to the solution of problems of the dynamics of a particle where the particle is executing small vibrations about an equilibrium configuration. The study of this problem is greatly facilitated by the reduction of the equations of motion from the usual coordinates to normal coordinates.

8. Problem 3171(3167; 1926, 104) states that if, in an ellipse, a straight line is drawn through one Focus F_1, its intersection P_1 with the circumference joined with the other Focus F_2, the intersection P_2 of P_1F_2 with the ellipse again joined with F_1, etc. etc., the limiting position of the straight line will be

the straight line joining F_1F_2. It is shown that this property is in no way characteristic of the ellipse. An analogous property exists, for example, for all closed convex curves.

9. Professor Rietz discussed first the question of making statistical theory more available to the general mathematical reader by means of the third Carus Mathematical Monograph. He then divided the problems considered in the monograph into two general classes. In problems of the first class, the main interest centers around the properties of a random sample drawn from a "population." In problems of the second class, the main interest centers around the question of making and checking the validity of statistical inferences about the population from which the sample is drawn. In dealing with the sample we introduce very early the concepts of relative frequency, arithmetic mean, various other averages, and certain measures of dispersion. In considering the population, we introduce the parallel concepts of probability, mathematical expectation of the value of a variable, and of the powers of the deviations of a variable from its expected value. After discussing these concepts, the paper gives a summary of the material of the monograph dealing with the following three topics of dominant interest in recent progress in statistics: Generalization of frequency curves, correlation theory, and random sampling theory.

10. Definite proofs were made as to the presence or absence of bias for the weighted arithmetic average of relatives for the weights I p_0q_1, II p_0q_1, III p_iq_i, III p_iq_0, IV p_iq_i. Definite proofs were given as to the relative size of the index numbers with weights I, II, III, IV.

11. In this paper a description was given of a nomograph of an approximation to the formula for the relation between effective and nominal interest rates.

For given values of j and p, i could be read accurately to at least four significant figures.

PHILIP FITCH, *Secretary*

THE TEACHING OF MATHEMATICS IN GERMAN SECONDARY SCHOOLS AND THE TRAINING OF TEACHERS FOR THESE SCHOOLS

By CHARLES A. NOBLE, University of California, Berkeley

The statements which follow have to do primarily with conditions in Prussia. In the other German states the departures from the Prussian system are, in general, inconsiderable. The head of the Prussian school system is the Minister of Science, Art, and Popular Culture (Minister für Wissenschaft, Kunst, und Volksbildung). The regulations governing all educational institutions are issued by him; but the immediate control of the schools of any province is entrusted to the Provincial Schoolboard (Provinzial Schulkollegium) of that province.

There are now, as before the war, three main types of secondary school in Germany. These are the Gymnasium, or classical school; the Realgymnasium, or Latin-modern-language school; and the Oberrealschule, or mathematics-natural science school. They are all nine-year schools. The revolution abolished the three-year preparatory school (Vorschule) which was attached formerly to each of the three types, and set up the universal four-year preparatory school (Grundschule), upon which every higher German school rests. Every German child enters the Grundschule at six years of age. The normal ages in the full secondary school are thus from ten to nineteen. Graduation from the secondary school automatically admits the pupil to the university. At that stage the pupil's training is at least two years beyond that of the graduate of an American high school, as to breadth, and still further ahead as to thoroughness.

The official guide for the Prussian secondary school is the ministerial edict, issued in April, 1925, "Richtlinien für die Lehrpläne der höheren Schulen Preussens." According to this edict the aims of the secondary schools appear to be, briefly, as follows: The particular problem of the Gymnasium is, through the study of Greek and Latin, its characterizing subjects, to prepare the pupil for the appreciation of history and the mother tongue and to discipline his mind by a method of work which he can apply in other fields; that of the Realgymnasium is, through the study of English and French, reinforced by Latin, to prepare the pupil to understand German culture as related to Western culture and as influenced by mathematical and scientific thought; that of the Oberrealschule is, through the greater prominence given to mathematics and

natural science, to unfold to the pupil the possibilities that lie in these subjects for humane culture.

The secondary schools for girls have the same aims and problems as those for boys. The six-year Lyzeum is the foundation of all types. Corresponding, respectively, to the Gymnasium, the Realgymnasium, and the Oberrealschule are the gymnasiale Studienanstalt, the Oberlyzeum, and the Oberlyzeum der Oberrealschulrichtung.

The curriculum of each type of school is heavy and is substantially prescribed. It is, however, less inflexible than formerly, because reform has introduced subsidiary types designed to permit a greater variety of choice.

It may be interesting to set out the curriculum in mathematics for, say, the Prussian Gymnasium, the secondary school in which that subject is least heavy. The "Richtlinien" prescribe as follows:

VI. (Age 10)[1]

The four fundamental rules with integers, German weights, measures, and coins. Practice in writing and calculating with decimals. Applications to home and community life. Intuitional treatment of space-forms with especial reference to the needs of instruction in physical geography.

V. (Age 11)

Continued practice with decimals. Divisibility of numbers. Four fundamental rules with fractions. Representation of numbers by means of line segments and areas. Rule of three. Applications to home, community, and state. Continued study of space-forms.

IV. (Age 12)

Arithmetic. Four fundamental rules with decimals; changing common fractions into decimals and conversely. Short methods. Simplest cases of percentage, discount, and interest, and other problems of everyday life, with application of the rule of three. Representation of number sequences by means of line segments and areas. Use of tables, especially calculation of mean values and ratios. Applications to the life of the community and the state.

Geometry. Intuitional development of fundamental notions; sides and angles of triangles. Simplest triangle constructions. Congruence theorems.

Geometrical Drawing and Measurement. Practice in the use of ruler, drawing triangles, and compasses. Drawing of parallels and perpendiculars and of nets of cubes, parallelopipeds, tetrahedrons, and octahedrons. Orthogonal projection of cube and parallelopiped. Construction of the space-diagonals of these

[1] The normal age for admission to "sexta" is ten; it is possible and not unusual, however, for a pupil to finish the Grundschule in three years in which case he enters "sexta" at nine years of age, as formerly.

bodies. Construction of prisms and pyramids and of nets of these solids. Measurement of segments and angles.

UIII (Age 13).

Algebra. Introduction of operations with letters. Four fundamental operations with integral and with fractional numbers. Representation by means of segments and rays. Calculation of tables from formulas. Simple equations of first degree with one unknown, in connection with operations with rational numbers. Introduction of linear function, if possible. Drawing of curves.

Geometry. Completion of the study of triangles. Continued construction of triangles. Theory of quadrilaterals, especially parallelograms and trapezoids. Calculation and comparison of areas (Pythagorean theorem). Extension of geometrical considerations and measurements to space.

Geometrical Drawing and Measurement. Projection of points, line segments, and plane figures upon the plane. Angle of inclination of line and of plane. Intersections of two planes whose traces and inclinations are given. Pyramids. Roofs. Measure of segments and angles.

OIII (Age 14).

Algebra. Equations of first degree with one or more unknowns; simple applications, especially from everyday life. Introduction of graphical representation of empirical functions; representation of linear functions (the linear function as straight line) and their use in solving equations of first degree. The function $y=mx$. Use of millimeter paper.

Geometry. Circles. Further study of space-forms.

Geometrical Drawing and Measurement. Simple masses of shrubbery and borders of garden paths. Measurement of segments, angles, and areas.

UII (Age 15).

Algebra. Powers with positive and negative integral exponents. The function $y=x^n$, n positive or negative, and its graphical representation, especially the graphs of the general parabola and of the rectangular hyperbola. Operations with radicals, and their graphs. Extraction of square root. Exponential and logarithmic functions. Inverse of a function. Four place logarithmic table and the use of logarithms. The slide rule. Quadratic equations with one unknown.

Geometry. Equality of ratios between segments; theory of similarity. Application to circle and right triangle. Length and area of circle. Selections from the history of geometrical problems (e.g. quadrature of the circle). Calculation of simplest figures.

Geometrical Drawing and Measurement. Tri-rectangular axes; the regular bodies in axonometric representation. Curve drawing in connection with algebra. Approximate constructions (circle). More exact measurements (nonius). Surveying in connection with the theory of similarity.

OII (Age 16).

Algebra. Simple integral and rational functions. Simple equations, and systems of equations which can be solved by quadratic equations—numerical and graphical treatment. Arithmetic and geometric series. Infinite geometric series. Compound interest and bonds, with applications from commercial life (political arithmetic). Binomial theorem with positive integral exponent.

Geometry. The trigonometric functions. Simple triangle calculations. Goniometry. Notion of periodic function. Geometrical calculations continued.

Geometrical Drawing and Measurement. Projection of the circle. Constructions for trigonometric problems, including those whose data are not all in one plane. Curve drawing in connection with algebra. Simple exercises in surveying and leveling.

UI (Age 17) and OI (Age 18).

Algebra. Introduction to infinitesimal calculus. Definition of differential quotient, its geometrical and its physical significance; its application to rational and, if possible, to trigonometric functions, especially in the calculation of greatest and least values, points of inflexion, inflexional tangents, etc. Simplest exercises in finding areas and volumes with the aid of integral calculus (e.g. sphere, paraboloid, etc.). Construction of the number system, from the positive integer to the complex number. Simple representations by means of functions of a complex variable.

Geometry. Review of the curvilinear figures thus far studied and the introduction of analytic geometry as far as the general equation of second degree. Supplementary theorems from stereometry (sphere). Straight line and plane in space. Plane and cone. Fundamental notions of spherical trigonometry (sine theorem and cosine theorem). Applications to geodesy and astronomy.

Geometrical Drawing and Measurement. Fundamental problems on the point, line, and plane. Conic sections. Projection of the sphere. Simple astronomical observations with measurements and calculations.

Review from historical and philosophical points of view.

The following tabular program of the Prussian Gymnasium indicates the number of hours devoted each week to each subject during the nine years. It is taken from the Ministerialerlass of October 31, 1924, Berlin. The "hours" are about forty-five minutes in length.

Gymnasium

	VI	V	IV	UIII	OIII	UII	OII	UI	OI	Total
Religion	2	2	2	2	2	2	2	2	2	18
German	5	4	3	3	3	3	4	3	3	31
Latin	7	7	7	6	6	5	5	5	5	53
Greek	–	–	–	6	6	6	6	6	6	36
Modern Language	–	–	3	2	2	2	2	2	2	15
History	–	1	2	2	2	3	3	3	3	19
Physiography	2	2	2	1	1	1	1	1	1	12
Mathematics	4	4	4	3	3	4	3	4	4	33
Natural Science	2	2	2	2	2	2	2	2	2	18
Drawing	2	2	2	2	2	1	1	1	1	14
Singing	2	2								4
Total	26	26	27	29	29	29	29	29	29	253

In addition to the above there is a weekly prescription of: 4 hours for physical exercise during the entire course of nine years; 4 hours for music during the six years IV to OI inclusive; 6 hours for Arbeitsgemeinschaften during the three years OII to OI inclusive.

It will be seen that the weekly program of the German pupil contains at least fifty per cent more hours than that of the American high school pupil.

Another striking difference is that, whereas in American schools plane geometry is studied daily for one year, to the exclusion of all other mathematics, and similarly for algebra, in Germany all the types of mathematics are studied during the same year. Algebra including arithmetic, and geometry including trigonometry are carried on abreast or in close alternation, at the discretion of the teacher, according as they supplement one another.

The "Richtlinien" of 1925 not only define the aims and prescribe the programs of the secondary schools, but they set out a very considerable body of principles and of suggestions as to method. They insure the same goal and standard for all schools of like type and facilitate the movement of pupils from one school to another. They aim to preserve unity in the variety of secondary schools by placing certain fundamental subjects, viz. religion, German, history, and physiography in the center of all secondary school programs and they demand that each school shall accomplish its special task through the interaction of its characteristic subjects with these fundamental subjects. They insist that the training should move toward an organic whole rather than be merely teaching of separate subjects, as formerly; that teaching should be teaching to work, and not the imparting of information; that the teacher should continually watch for those powers of the pupil which can be developed under his guidance; that the teacher's problem is to make the bridge between the acquirement of certain knowledge, without which high mental activity is impossible, and the acquirement of the power of independent work, without which knowledge is sterile.

During three and a half weeks of the summer of 1926, I visited mathematics classes in Göttingen, Berlin, Dresden, Stuttgart, and Hamburg. This brief time sufficed to give certain impressions as to personality of a considerable number of teachers, their methods of teaching, and its effectiveness. These teachers, always well trained, were nearly always men and women of good presence and pleasing personality who had the confidence of their pupils, with whom the relation was rooted in the teacher's desire to be helpful.

I saw no provision in any class for the separation, into different sections, of pupils of differing ability, as is done in English schools. While the better pupils are stimulated by the offer of more difficult tasks, the Arbeitsgemeinschaft provides in the later years the fuller opportunity for gifted pupils. In it pupils who show especial interest in some subject of the curriculum may, under the informal guidance of a teacher, widen their knowledge and test their ability.

Two factors which contribute to the excellence of the training in German schools are that mathematics is taught in close touch with physics, frequently by the same teacher, and, secondly, that geometrical drawing is taught throughout the entire nine years. This assists materially in the cultivation of the space sense. It is noteworthy also that geometry is begun in the first year of the school, when the pupil is ten years old, earlier than is usually thought possible in America. This instruction is, at first, only intuitional—demonstration comes later—and the pupil is at once encouraged to enlarge his notions of three dimensional space, the space in which he has had experience. One excellent teacher declared to me that it was absurd to start a boy with plane figures when solids are what the child first sees. He sometimes found, with his maturer pupils, that their perception with solid figures was immediate in problems where he was puzzled, because he had been trained in the old way which suppresses and warps the child's experience.

In a Hamburg school I found projective geometry being taught to two classes in UI, corresponding roughly, to our freshman year. I was told that this was the only school in Germany where this subject was taught. The University of California is one of the few institutions in America, if not the only one, where projective geometry is offered to freshmen. In both cases the situation arose from the presence of a stimulating teacher who was an enthusiast for that subject.

The teaching in all the schools which I visited was marked by more emphasis upon careful, detailed attention to theoretical development and less upon illustrative numerical problems than is the case in American schools and colleges. This was as noticeable in the Oberrealschule, where future engineers would be enrolled, as in the classical schools.

The recent school reform in Germany arose, according to some, in a desire to overcome the materialism of the German people, which was thought to be due to too much concern with intellectual things, too little with things of feeling. A corrective was sought in a plan which should direct less attention to science and mathematics, more to music and art; which should give prominence to modern language in order that the Germans, through reading foreign literature, could come into touch with other peoples; and which should improve the tone of the youth by fostering an interest in open air recreation, in "Spiel und Sport."

Let us now consider the training of teachers of mathematics for the secondary schools of Prussia.

1. The prospective teacher must have graduated from a nine-year secondary school in Prussia, or have had equivalent training. This is in general fully, in some respects more than the equivalent of the first two years at an American university.

2. The prospective teacher must have passed the state examination (wissenschaftliche Prüfung) set for candidates for a teaching position. This examination must be in two major subjects, one minor subject, and in philosophy. The examination is held by a committee composed of university professors and schoolmen. To qualify for admission to this examination the candidate must have studied at least eight semesters at a university, of which at least six must have been at a German, and at least three at a Prussian university. He must submit evidence that the studies in the majors have included the appropriate laboratory and seminary courses and that during two semesters he had training in physical exercise, both theoretical and practical.

The examination consists of a written and an oral part. In the former the candidate submits two papers upon assigned subjects, one from the field of the chief major, the other from that of the other major or from philosophy, at his option. He is allowed five months to prepare these papers. The candidate is examined orally in each of his subjects.

3. The prospective teacher, after having passed the state examination, must have spent two years in practice preparation, for which he is assigned to a secondary school recognized for that purpose, and during which he is under the guidance of experienced secondary school teachers.

4. The prospective teacher must pass the pedagogy examination (Pädagogische Prüfung), to qualify for which he must submit a favorable report upon his work during the two practice years. This examination embraces a written part consisting of a thesis on a topic from the theory of education, which shall exhibit his own views acquired during his two practice years and be supported by scientific argument; two test class lessons given in the

presence of the examining committee; and an oral part designed to test his grasp of actual teaching problems.

There is no provision at the universities and technical schools for future teachers to acquire, during their scientific training, detailed drill in the experimental work of elementary science and descriptive geometry which is necessary in secondary school teaching. To supply this lack, summer courses are offered at universities and other institutions for the benefit of those actually engaged in teaching. Such courses, for Prussia, are under the auspices of the Staatliche Hauptstelle für Naturwissenschaftlichen Unterricht, in Berlin. In the courses which I visited in chemistry, physics, and descriptive geometry, secondary school teachers were actually performing, under the direction of experienced university and school men, experiments appropriate for secondary schools, much as is done at our university summer courses for teachers. The apparatus was crude and the accommodations in other respects were meager, due to that widespread poverty which, since the war, has been a blighting handicap in German educational undertakings. University and school men cannot afford to buy books or even the scientific magazines so necessary to keep them abreast of progress; libraries cannot make their needful purchases; many school children are unable to buy the prescribed books; laboratories are short of supplies. In spite of such disabilities, perhaps because of them, there is a close-knitting bond uniting all the school agencies and a spirit of cooperation between community and schools which enables them to carry on and to do admirable work.

THE "MYSTIC" NUMERAL 7

By HARRIS HANCOCK, University of Cincinnati

In 1892 a writer in one of the Berlin newspapers had learned that the integers in the number $N_1 = 142857$ repeat themselves in cyclic order when N_1 is multiplied respectively by 1, 3, 2, 6, 4, 5.[1] For example, $142857 \times 3 = 428571$. Noting the fact that $142857 \times 7 = 999999$ and that three 9's result from adding the digits in pairs $(1+8=9;\ 4+5=9;\ 2+7=9)$, this writer, possibly a fundamentalist, connected the above results with the Holy Trinity. Dividing 1 by 49, I wrote down the 42 numerals that occur before the decimal begins to repeat itself (as given below), indicated that this number repeated itself cyclically when multiplied by the 42 integers that are less than and relatively prime to 49 and queried what was its biblical analogon.

Observe first that, when divided by 7, the remainders of 10, 30, 20, 60, 40, 50, are 3, 2, 6, 4, 5, 1, so that $1/7 = .142857$. It may be remarked parenthetically

[1] Professor Robert E. Moritz, in a paper, *On products whose digits are cyclical permutations of the digits of the multiplicand*, has given recently an interesting theorem from which this peculiar property of the number 142857 follows as a very special case. This Monthly, vol. 34 (1927), pp. 33–36.

that these remainders repeat themselves cyclically (mod 7) when multiplied by 1, 3, 2, 6, 4, 5. Due to the fact that a circulating decimal is a cyclic expression, it follows that 1/7, 3/7, 2/7, 6/7, 4/7, 5/7 are equal to circulating decimals with the same digits occurring cyclically. And these circulating decimals are nothing other than the results of multiplying N_1 by 1, 3, 2, 6, 4, 5. Observe that, otherwise interpreted, this simply means that $10^6 \equiv 1$ (mod 7) and that corresponding to a definite power of 10 there is a definite remainder and vice versa. Observe next that $10^{42} \equiv 1$ (mod 49) and that 42 is the lowest power after zero that satisfies this congruence. For, writing $10^k \equiv 1$ (mod 49) it follows that $10^k \equiv 1$ (mod 7) so that k is a multiple of 6, say $k = 6h$. Hence $(10^{6h}) \equiv 1$ (mod 49); or, since $10^6 - 1 = 7t$, where $t \not\equiv 0$ (mod 7), it is seen that

$$(1 + 7t)^h \equiv 1 \pmod{49} \quad \text{or} \quad 1 + 7ht + \frac{h(h-1)}{1 \cdot 2}(7t)^2 + \cdots \equiv 1 \pmod{49}.$$

Accordingly we must have $h \equiv 0$ (mod 7). Similarly, $s = 6 \cdot 7^2$ is the lowest power of 10 which satisfies $10^s \equiv 1$ (mod 7^3) and in general $s = 6 \cdot 7^{r-1}$ is the lowest power of 10 which satisfies $10^s \equiv 1$ (mod 7^r), where r is a positive integer. And corresponding to a definite power there is a unique residue.

To determine the forty-two digits which compose a period of the circulating decimal for $1 \div 7^2$, we may divide by 7 the circulating decimal D given in the following array. The number N_2 is the quotient.

(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)
D=.142857	142857	142857	142857	142857	142857	142857
1	2	3	4	5	6	0
N_2=.020408	163265	306122	448979	591836	734693	877551

The following table (to be described later) is inserted here for convenience.

	3	0	5	4	4	5	1
	4	1	6	5	5	6	2
N_3	5	2	0	6	6	0	3
	6	3	1	0	0	1	4
	0	4	2	1	1	2	5
	1	5	3	2	2	3	6
	2	6	4	3	3	4	0

Returning to the number N_2, notice that in the first interval (I) the final remainder is 1 which is written above the number; the final remainder in the second interval (II) is 2; in the third interval (III) it is 3; $\cdots$; and in the seventh interval (VII) this remainder is 0, showing that the decimal expressing $1 \div 7^2$ begins then to repeat. Next observe that, corresponding to every digit in N_2, there is a second digit such that the sum of the two digits is 9, so that the sum of the digits of N_2 is $21 \cdot 9 = 7 \cdot 3^3$. Similarly in N_r, the sum of the digits is $7^{r-1} \cdot 3^3$.

It is of interest to observe the sequence of the remainders 1, 2, 3, 4, 5, 6, 0, just noticed. To do this note that

$$10 \equiv 3 \ ; \ 10^2 \equiv 2 \ ; \ 10^3 \equiv 6 \ ; \ 10^4 \equiv 4 \ ; \ 10^5 \equiv 5 \ ; \ 10^6 \equiv 1 \ ; \ (\text{modulo } 7).$$

Hence, if $N = \cdots n_6 n_5 n_4 n_3 n_2 n_1 n_0$, where n_0 is the numeral in the units' place, n_1 the numeral in the ten's place, etc., it is seen that

$$\begin{aligned} N &= n_0 + 10n_1 + 10^2 n_2 + 10^3 n_3 + 10^4 n_4 + 10^5 n_5 + 10^6 n_6 + \cdots \\ &\equiv n_0 + 3n_1 + 2n_2 + 6n_3 + 4n_4 + 5n_5 + n_6 + \cdots \\ &\equiv n_0 + 3n_1 + 2n_2 - n_3 - 3n_4 - 2n_5 + n_6 + \cdots \ (\text{mod } 7) \end{aligned}$$

or

$$\begin{aligned} N \equiv (n_0 - n_3 + n_6 - n_9 + \cdots) &+ 3(n_1 - n_4 + n_7 - n_{10} + \cdots) \\ &+ 2(n_2 - n_5 + n_8 - n_{11} + \cdots) \ (\text{mod } 7). \end{aligned}$$

It is thus seen that

$$142857 \equiv 7 - 2 + n_6 + 3(5 - 4) + 2(8 - 1) \equiv 1 + n_6 \ \ (\text{mod } 7).$$

Hence, in the first interval in the array above, where $n_6 = 0$, the remainder is 1; and as $n_6 = 1$ for the second interval, the remainder in that interval is 2, etc.

To determine the number N_3 which corresponds to $1 \div 7^3$, write down the number N_2 consisting of forty-two digits (corresponding to $1 \div 7^2$) seven times, each of these seven intervals being written the one over the other as in the diagram above. Observe that, if N_2 is divided by 7, the remainder after the sixth digit is 3; the remainder after the twelfth digit is 0, etc.; the remainder after the 42nd digit is 1. Continuing with this remainder into the line below, the next remainder is 4, etc. . . . . ; the final remainder on the seventh line being 0, showing that the period is ended. It is again of interest to note the sequence of remainders which follow the one below the other in the above scheme. Of course, when N_3 is multiplied by any multiple of 7 that is less than 7^3, we have seven periods of digits in cyclic order that compose N_2; and if again these are multiplied by a multiple of 7 that is less than 49, we have 49 periods of the digits that cyclically constitute N_1. Further, if N_3 be multiplied by 7^3

we have $6 \cdot 7^2$ consecutive 9's; and if N_2 is multiplied by 7^2, there are 42 consecutive 9's.

Bachman, on p. 381 of vol. 36 (1891) of the Zeitschrift für Mathematik und Physik writes that Sturm had communicated to him the cyclic properties of N_1 when multiplied by 3, 2, 6, 4, 5, 1, and that these results were taken from a French periodical (Zeitschrift). J. C. Burkhardt, in his *Tables des diviseurs pour tous les nombres du premier million* (Paris, 1817), p. 114, states that 10 is a primitive root of 148 of the 365 primes p, where $5<p<2500$. For example $10^{16} \equiv 1 (\text{mod } 17)$, and $1/17 = .0588235294117647$. Here again among the digits that constitute this decimal, for every 0 there is a 9, for every 5 there is a 4, etc. so that the sum of these digits is $8 \cdot 9$; the sum of the digits in the decimal that expresses $1/17^2$ is $8 \cdot 17 \cdot 9$ while the sum of those in $1/17^3 = 8 \cdot 17^2 \cdot 9$, etc. L. E. Dickson, in his *History of the Theory of Numbers*, vol. 1, p. 161, gives a reference to Goodwyne who, Dickson says, noted that when $1/p$ is converted into two half-periods, the sum of corresponding quotients is 9.

INTERPOLATION FORMULAS DEPENDENT UPON THE UNDERLYING FUNCTION

By JOHN F. REILLY, University of Iowa

We are accustomed to the use of interpolation formulas for finding the approximate value of a function $f(x)$ at $x=x'$, even when we can calculate $f(x')$ directly to any desired degree of accuracy. The approximate value found by interpolation is sufficiently accurate for many purposes, and the labor involved in the direct calculation is thereby eliminated.

The Gregory-Newton interpolation formula

$$y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \cdots$$

proceeds without invoking very directly the properties of the underlying function; it merely makes use of a limited number of values of the function. Thus we would interpolate in the same way when the values (1, 2), (2, 4), and (3, 8) have underlying them the exponential function $y=2^x$, as when they have underlying them the algebraic function $y=x^2-x+2$. The interpolation would accurately reproduce the latter function but not the former. It is the purpose of this paper to develop a scheme of interpolation that is dependent upon the underlying function. This is accomplished by developing formulas containing one or more parameters, and then using the underlying function to determine these parameters.

Denote the values of the function y corresponding to the values 0, 1, 2, 3, of the argument x, by y_0, y_1, y_2, y_3 respectively. It is clear that the slopes of the straight lines P_0P_1, P_1P_2, and P_2P_3, are y_1-y_0, y_2-y_1, and y_3-y_2 respectively. To obtain an interpolation formula for use between y_1 and y_2, we will impose the following conditions: that the formula reduce to y_1 when $x=1$, and to y_2 when $x=2$; that the slope of its graph at P_1 be $k(y_1-y_0)+l(y_2-y_1)$, where $k+l=1$, and that the slope at P_2 be $m(y_2-y_1)+n(y_3-y_2)$, where $m+n=1$. These slopes are arbitrarily weighted arithmetic means of the slopes of the straight lines P_0P_1, P_1P_2, and P_2P_3.

Assuming the interpolation formula to be of the form

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 \cdots$$

we find the slope to be $y' = a_1 + 2a_2(x-1) + 3a_3(x-1)^2$.

Imposing the above conditions leads to the following values of the coefficients in terms of y_0 and its leading differences:

$$a_0 = y_0 + \Delta y_0, \qquad a_2 = (2 - n - 2l)\Delta^2 y_0 - n\Delta^3 y_0,$$

$$a_1 = \Delta y_0 + l\Delta^2 y_0, \quad a_3 = (-1 + n + l)\Delta^2 y_0 + n\Delta^3 y_0.$$

Substituting these values of the coefficients, we obtain after reduction

$$(A) \quad y = y_0 + x\Delta y_0 + (x-1)[l(x-2)^2 + n(x-1)(x-2) - (x-1)(x-3)]\Delta^2 y_0 + n(x-1)^2(x-2)\Delta^3 y_0,$$

a formula containing two parameters l and n. We note the three following special cases:

When $l=n$, formula (A) becomes

$$(B) \qquad y = y_0 + x\Delta y_0 + [n(x-1)(x-2)(2x-3) - (x-1)^2(x-3)]\Delta^2 y_0 + n(x-1)^2(x-2)\Delta^3 y_0.$$

When $n=\frac{1}{2}$, formula (B) becomes

$$(C) \quad y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2}\Delta^2 y_0 + \frac{(x-1)^2(x-2)}{2}\Delta^3 y_0.$$

This is Sprague's third differences osculatory formula.[1]

[1] Journal of the Institute of Actuaries, vol. 41 (1907), p. 545. This is what we should expect, since in this case Sprague's partial interpolation curves are parabolas, and when $l=n=\frac{1}{2}$ the assumed slopes at P_1 and P_2 are those of the lines joining P_0P_2 and P_1P_3 respectively. It is well known that in a parabola a chord is parallel to the tangent at the point midway between the ends of the chord.

When $l=\frac{2}{3}$, $n=\frac{1}{3}$, formula (A) becomes[1]

$$(D)\qquad y = y_0 + x\Delta y_0 + \frac{x^2-1}{3}\Delta^2 y_0 + \frac{(x-1)^2(x-2)}{3}\Delta^3 y_0.$$

In the application of formula (A) or formula (B), the parameters will be determined so that the interpolated values will reproduce as nearly as possible the underlying function. To illustrate the use of formula (B), consider the function $(1+x)^r$ where four values corresponding to equidistant values of x are known, and where its value is required for some x in the middle interval. The value of the parameter n in formula (B) that will produce the best interpolated value will depend upon r. In practice one would compute the value of n corresponding to the value of r in which he was interested. If by the best interpolated value we mean the best in the least squares sense, we determine n so that the sum of the squares of the differences between the interpolated values as given by formula (B) and the actual values as given by $(1+x)^r$, for all values of x over the range x_1 to x_2, is a minimum. This requires the determination of n so that the definite integral

$$\int_{x_1}^{x_2} \{(1+x)^r - y_0 - x\Delta y_0 - [n(x-1)(x-2)(2x-3) - (x-1)^2(x-3)]\Delta^2 y_0 - n(x-1)^2(x-2)\Delta^3 y_0)\}^2 dx.$$

shall have a minimum value.

If we write $f_1(x)$ for $(1+x)^r - y_0 - x\Delta y_0 + (x-1)^2(x-3)\ \Delta^2 y_0$, and $f_2(x)$ for $-(x-1)(x-2)(2x-3)\Delta^2 y_0 - (x-1)^2(x-2)\Delta^3 y_0$, this definite integral may be written

$$\int_{x_1}^{x_2} [f_1(x) + nf_2(x)]^2 dx.$$

Sufficient conditions for a minimum are

$$\int_{x_1}^{x_2} [f_1(x) + nf_2(x)]f_2(x)dx = 0, \text{ and } \int_{x_1}^{x_2} [f_2(x)]^2 dx > 0.$$

The inequality is satisfied except in the case where $f_2(x)$ is identically zero over the range x_1 to x_2, in which case formula (B) is linear in x. When $f_2(x) \not\equiv 0$ we find

$$n = \int_{x_1}^{x_2} f_1(x)f_2(x)dx \div \int_{x_1}^{x_2} [f_2(x)]^2 dx.$$

This value of n substituted in formula (B) will give the best reproduction of $(1+x)^r$ over the range x_1 to x_2, under the least squares criterion adopted.

[1] Reilly, Proceedings of the Iowa Academy of Science, vol. 31 (1924), p. 369.

When applying formula (A), which contains two parameters l and n, the method of least squares requires that the definite integral

$$\int_{x_1}^{x_2} \{(1+x)^r - y_0 - x\Delta y_0 - (x-1)[l(x-2)^2 + n(x-1)(x-2) \\ - (x-1)(x-3)]\Delta^2 y_0 - n(x-1)^2(x-2)\Delta^3 y_0\}^2 dx$$

have a minimum value.

If we write $f_1(x)$ for $(1+x)^r - y_0 - x\Delta y_0 + (x-1)^2(x-3)\Delta^2 y_0$, $f_2(x)$ for $-(x-1)(x-2)^2\Delta^2 y_0$, and $f_3(x)$ for $-(x-1)^2(x-2)(\Delta^2 y_0 + \Delta^3 y_0)$, this definite integral may be written

$$\int_{x_1}^{x_2} [f_1(x) + lf_2(x) + nf_3(x)]^2 dx.$$

Sufficient conditions for a minimum are

$$(1) \quad \int_{x_1}^{x_2} [f_1(x) + lf_2(x) + nf_3(x)]f_2(x)dx = 0, \qquad (3) \quad \int_{x_1}^{x_2} [f_2(x)]^2 dx > 0,$$

$$(2) \quad \int_{x_1}^{x_2} [f_1(x) + lf_2(x) + nf_3(x)]f_3(x)dx = 0, \qquad (4) \quad \int_{x_1}^{x_2} [f_3(x)]^2 dx > 0,$$

$$(5) \qquad \left[\int_{x_1}^{x_2} f_2(x)f_3(x)dx\right]^2 < \int_{x_1}^{x_2} [f_2(x)]^2 dx \times \int_{x_1}^{x_2} [f_3(x)]^2 dx.$$

The inequalities (3) and (4) are satisfied except, as before, when $f_2(x)$ and $f_3(x)$ are identically zero over the range x_1 to x_2, and the equations (1) and (2) will determine values of l and n which will make the definite integral a minimum, provided the inequality (5) is also satisfied.

Such values of l and n substituted in formula (A) will give in this case the best reproduction of the function $(1+x)^r$ over the range x_1 to x_2, in the sense of the least squares criterion.

In the development of the above formula we have employed four values of the function, and hence third order differences, but it is obvious that other formulas could be similarly developed containing either a lesser or a greater number of differences. When this direct appeal is made to the underlying function it seems reasonable to believe that there would be a gain in accuracy in the interpolated values with the use of a given number of differences, or that a given degree of accuracy is attainable with a lesser number of differences.

NOTE ON THE UPPER LIMIT TO THE VALUE OF A DETERMINANT

By HARVEY A. SIMMONS, Northwestern University.

The purpose of this note is to obtain by an elementary method a new expression for the upper limit to the value of a determinant whose elements are real numbers—an expression which never exceeds the Hadamard upper limit,[1] $n^{n/2}M^n$, where n is the order of the determinant and M is a fixed, positive constant such that $|a_{ij}| \leqq M$ for every element a_{ij} of the determinant.

Example. If $n=4$ and $M=1$, Hadamard's upper limit is $4^2=16$, which can actually be attained.[2]

We use the notation

$$(1) \qquad D \equiv \begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n1} \cdots & a_{nn} \end{vmatrix}, \quad \Delta \equiv \begin{vmatrix} A_{11} & A_{12} \cdots & A_{1n} \\ \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} \cdots & A_{nn} \end{vmatrix},$$

where each a_{ij} $(i, j=1, 2, \cdots, n)$ is a real number and A_{ij} is its cofactor in D, so that Δ is the adjoint of D, and where

$$(2) \qquad |a_{ij}| \leqq M_i, \qquad \sum_{j=1}^{n} a^2_{ij} = n_i\lambda_i^2,$$

for every i; M_i being the maximum absolute value of any element in the ith row of D, λ_i being a fixed, positive constant, obviously less than or equal to M_i, and n_i denoting the number of non-zero elements in the ith row of D.

Let the elements in the ith row of D vary in such a way that the value of D remains unchanged. On differentiating D with respect to the elements in its ith row, we obtain

$$(3) \qquad \sum_{j=1}^{n} \frac{\partial D}{\partial a_{ij}} \cdot \delta a_{ij} = 0 \qquad \text{for every } i,$$

where the δa_{ij} are variations which are subject only to the condition

$$(4) \qquad \sum_{j=1}^{n} a_{ij}\, \delta a_{ij} = 0, \text{ which is the result of differentiating}$$

the second of equations (2). Now $\partial D/\partial a_{ij} = A_{ij}$ and we have, by (3) and (4),

$$(5) \qquad A_{ij} = \partial D/\partial a_{ij} = k a_{ij}, \text{ where } k \text{ is a constant.}$$

From the fact that $\sum a_{ij}A_{ij}=0$ for each i when j runs from 1 to n, and equation (5), we conclude that

[1] Bulletin des Sciences Mathématiques, (2), vol. 17, p. 240.

[2] Loc. cit., p. 244. The author proves that if n is a power of 2 or if n is 12 or 20, the upper limit can be attained by determinants whose elements are real numbers.

$$(6) \qquad k \sum_{j=1}^{n} a^2_{ij} = D \text{ or } k = D/(n_i\lambda_i^2).$$

Now substituting in (5) the value of k given by (6), we have $A_{ij} = a_{ij}D/n_i\lambda_i^2$. Substituting this value of A_{ij} in (1) and making use of the well-known theorem that $\Delta = D^{n-1}$, we obtain $\Delta = D^{n+1} \div n_1n_2 \cdots n_n\lambda_1^2\lambda_2^2 \cdots \lambda_n^2 = D^{n-1}$. From this equation, either $D=0$ or $D^2 = n_1n_2 \cdots n_n\lambda_1^2\lambda_2^2 \cdots \lambda_n^2$. Hence the absolute value of D never exceeds $E \equiv (n_1n_2 \cdots n_n)^{1/2}\lambda_1\lambda_2 \cdots \lambda_n$. Now $n_i \leqq n$ and $\lambda_i \leqq M_i$ for every i, and M_i is obviously less than, or equal to, the Hadamard M. Hence $E \leqq n^{n/2}M^n$. If D has a zero-element or if there is an i for which $M_i < M$, the inequality sign, $<$, is effective.

SOME FUNCTIONS ANALOGOUS TO TRIGONOMETRIC FUNCTIONS

By L. E. WARD, University of Iowa

The differential equation $(d^2u/dx^2) + u = 0$ has the two linearly independent solutions e^{ix} and e^{-ix}, which are combined linearly to form the trigonometric functions $\sin x$ and $\cos x$. It is the purpose of this note to define analogous functions based on the differential equation

$$(1) \qquad (d^3u/dx^3) + u = 0,$$

and to derive some properties of them, pointing out the corresponding properties of the trigonometric functions.

We choose as a basis these three linearly independent solutions of (1): $e^{\omega_1 x}$, $e^{\omega_2 x}$, and $e^{\omega_3 x}$, where $\omega_1 = -1$, $\omega_2 = e^{\pi i/3}$, and $\omega_3 = e^{-\pi i/3}$ are the three cube roots of negative one. One third the sum of these is a function analogous to $\cos x$; it is

$$(2) \qquad u_1(x) = \tfrac{1}{3}(e^{\omega_1 x} + e^{\omega_2 x} + e^{\omega_3 x}).$$

The functions $u_2(x)$ and $u_3(x)$, written below, are respectively the negative of the first derivative and the second derivative of $u_1(x)$.

$$(3) \qquad u_2(x) = -\tfrac{1}{3}(\omega_1 e^{\omega_1 x} + \omega_2 e^{\omega_2 x} + \omega_3 e^{\omega_3 x}).$$

$$(4) \qquad u_3(x) = \tfrac{1}{3}(\omega_1^2 e^{\omega_1 x} + \omega_2^2 e^{\omega_2 x} + \omega_3^2 e^{\omega_3 x}).$$

Evidently $(d/dx)u_3(x) = -u_1(x)$, and consequently no new functions are obtained by further differentiation. The three functions $u_1(x)$, $u_2(x)$, and $u_3(x)$ are linearly independent solutions of (1), and each is real for real values of x. They are given in terms of letters not involving complex numbers by

$$u_1(x) = \frac{1}{3}\left[e^{-x} + 2e^{x/2}\cos\left(\frac{x(3)^{1/2}}{2}\right)\right];$$

$$u_2(x) = \frac{1}{3}\left[e^{-x} - 2e^{x/2}\cos\left(\frac{\pi}{3} + \frac{x(3)^{1/2}}{2}\right)\right];$$

$$u_3(x) = \frac{1}{3}\left[e^{-x} - 2e^{x/2}\cos\left(-\frac{\pi}{3} + \frac{x(3)^{1/2}}{2}\right)\right].$$

These formulas show that the functions are not bounded as $x \to \infty$, but that they do oscillate. They are not periodic.

The functions satisfy the boundary conditions

$$\begin{array}{llll} & u_1(0) = 1 & u_2(0) = 0 & u_3(0) = 0 \\ (5) & u_1'(0) = 0 & u_2'(0) = 0 & u_3'(0) = -1 \\ & u_1''(0) = 0 & u_2''(0) = 1 & u_3''(0) = 0, \end{array}$$

and these together with the differential equation (1) determine the functions uniquely. The following power series in x may be obtained for the functions from (1) and (5), or from (2), (3), and (4) using the series for the exponential function:

$$u_1(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \cdots; \quad u_2(x) = \frac{x^2}{2!} - \frac{x^5}{5!} + \cdots;$$

$$u_3(x) = -\frac{x}{1!} + \frac{x^4}{4!} - \cdots.$$

They are series which are well adapted to computation if x is not too large numerically.

To derive a relation analogous to $\sin^2 x + \cos^2 x = 1$, consider the function $\phi(x) = u_1^3(x) + u_2^3(x) + u_3^3(x) - 3u_1(x)u_2(x)u_3(x)$. Differentiating with respect to x we find

$$\begin{aligned}\phi'(x) = &-3u_1^2(x)u_2(x) - 3u_2^2(x)u_3(x) - 3u_3^2(x)u_1(x) \\ &+ 3u_2^2(x)u_3(x) + 3u_3^2(x)u_1(x) + 3u_1^2(x)u_2(x) = 0.\end{aligned}$$

Hence $\phi(x)$ is constant and, taking $x=0$, we find the value of this constant to be unity. This gives the identity

$$u_1^3(x) + u_2^3(x) + u_3^3(x) - 3u_1(x)u_2(x)u_3(x) = 1.$$

The three functions possess simple addition theorems, namely

$$u_1(x+y) = u_1(x)u_1(y) + u_2(x)u_3(y) + u_3(x)u_2(y), \tag{6}$$

$$u_2(x+y) = u_2(x)u_1(y) + u_3(x)u_3(y) + u_1(x)u_2(y), \tag{7}$$

$$u_3(x+y) = u_3(x)u_1(y) + u_1(x)u_3(y) + u_2(x)u_2(y). \tag{8}$$

Of these, (7) and (8) are derived from (6) by differentiating with respect to either x or y, while (6) is found from the power series or from the definitions (2), (3), and (4).

Replacing y in (6), (7), and (8) by x, we get the formulas for the double variable,

$$u_1(2x) = u_1^2(x) + 2u_2(x)u_3(x)\,; \qquad u_2(2x) = 2u_1(x)u_2(x) + u_3^2(x)\,;$$
$$u_3(2x) = 2u_1(x)u_3(x) + u_2^2(x).$$

The formulas for the functions of the half variable can be found by solving these three equations for $u_1(x)$, $u_2(x)$, and $u_3(x)$. Although of degree six, the system can be solved in terms of radicals.

The relations $u_1(-\omega_3 x) = u_1(-\omega_2 x) = u_1(x)$ are analogous to $\cos(-x) = \cos x$, and are derived from (2). We have also $u_2(-\omega_3 x) = -\omega_2 u_2(x)$; $u_2(-\omega_2 x) = -\omega_3 u_2(x)$; $u_3(-\omega_3 x) = -\omega_3 u_3(x)$; $u_3(-\omega_2 x) = -\omega_2 u_3(x)$.

Of course, the list may be extended indefinitely, but the above will serve to indicate some of the resemblances of these functions to the trigonometric functions. We may also extend the work to a similar system of functions satisfying the differential equation $u^{(n)} + u = 0$.

RULED SURFACES REFERRED TO THE TRIHEDRAL OF A DIRECTRIX

By MALCOLM FOSTER, Williams College

Introduction. Let C be a curved line upon a ruled surface S, and let l be the ruling through a point M of C. Relative to the trihedral of C at M the coordinates of any point P on l are

$$(1) \qquad x = \alpha t, \qquad y = \beta t, \qquad z = \gamma t,$$

where α, β, γ, are the direction-cosines of l expressed as functions of the arc s of C, and t is the distance along l from M to P.

1. The line of striction. Let l_1 be the ruling through a neighboring point M_1 on C. The direction-cosines of l_1 are $\alpha+\delta\alpha$, $\beta+\delta\beta$, $\gamma+\delta\gamma$, where[1]

$$(2) \qquad \delta\alpha = d\alpha - \frac{\beta}{\rho}ds, \quad \delta\beta = d\beta + \frac{\alpha}{\rho}ds + \frac{\gamma}{\tau}ds, \quad \delta\gamma = d\gamma - \frac{\beta}{\tau}ds.$$

As M is displaced along C to M_1, any point P on l receives the following absolute displacements in the direction of the axes of the moving trihedral:[2]

[1] Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 32.

[2] Eisenhart, p. 32.

$$\text{(3)} \qquad \delta x = dx + ds - \frac{y}{\rho}ds, \quad \delta y = dy + \frac{x}{\rho}ds + \frac{z}{\tau}ds, \quad \delta z = dz - \frac{y}{\tau}ds.$$

The condition that the locus of some point P be the line of striction is that the displacement of P be orthogonal to both l and l_1. Hence

$$\sum(\alpha + \delta\alpha)\frac{\delta x}{ds} = 0, \quad \sum\alpha\frac{\delta x}{ds} = 0.$$

Subtracting these equations and dividing through by ds we get

$$\sum\frac{\delta\alpha}{ds}\,\frac{\delta x}{ds} = 0, \quad \text{which becomes, on using (1), (2), and (3),}$$

$$\left(\alpha' - \frac{\beta}{\rho}\right)\left(\alpha t' + \alpha' t + 1 - \frac{\beta t}{\rho}\right) + \left(\beta' + \frac{\alpha}{\rho} + \frac{\gamma}{\tau}\right)\left(\beta t' + \beta' t + \frac{\alpha t}{\rho} + \frac{\gamma t}{\tau}\right)$$
$$+ \left(\gamma' - \frac{\beta}{\tau}\right)\left(\gamma t' + \gamma' t - \frac{\beta t}{\tau}\right) = 0,$$

where the primes indicate differentiation with respect to s.

Solving for t, we get the distance along l from M to the line of striction to be

$$\text{(4)} \quad t = \frac{\dfrac{\beta}{\rho} - \alpha'}{\sum\alpha'^2 + \dfrac{2}{\rho}(\alpha\beta' - \alpha'\beta) + \dfrac{2}{\tau}(\beta'\gamma - \beta\gamma') + \dfrac{1}{\rho^2}(1 - \gamma^2) + \dfrac{1}{\tau^2}(1 - \alpha^2) + \dfrac{2\alpha\gamma}{\rho\tau}}$$

Since the condition that C be a geodesic is that $\beta = 0$, we have from (4) the well known theorem of Bonnet:

If a curve upon a ruled surface have two of the following properties, it has the third also:

1. *That it be a geodesic.*
2. *That it cut the rulings under constant angle.*
3. *That it be the line of striction.*

2. Developable surfaces. For l to generate a developable surface, the displacement of some point on l must be in the direction of the ruling. Hence for that point,

$$\frac{\delta x}{\alpha} = \frac{\delta y}{\beta} = \frac{\delta z}{\gamma}, \text{ or } \alpha\delta z - \gamma\delta x = 0, \quad \beta\delta z - \gamma\delta y = 0.$$

Using (3), these equations become

$$t\left(\alpha\gamma' - \alpha'\gamma - \frac{\alpha\beta}{\tau} + \frac{\beta\gamma}{\rho}\right) = \gamma, \quad t\left(\beta'\gamma - \beta\gamma' + \frac{\gamma\alpha}{\rho} + \frac{1 - \alpha^2}{\tau}\right) = 0.$$

Hence the condition that S be developable is that

$$(5)\quad t=\frac{\gamma}{(\alpha\gamma'-\alpha'\gamma-\alpha\beta/\tau+\beta\gamma/\rho)},\quad \beta'\gamma-\beta\gamma'+\gamma\alpha/\rho+(1-\alpha^2)/\tau=0,$$

where t is the distance from M to the edge of regression.

3. Condition that the directrix be a line of curvature. The tangent plane to S at any point M of C is determined by the ruling l and the tangent to C. Hence the normal to the surface is perpendicular to l and to the x-axis of the trihedral. The direction-cosines α_1, β_1, γ_1, of the normal are therefore determined from the following three equations:

$$\alpha_1=0,\qquad \alpha\alpha_1+\beta\beta_1+\gamma\gamma_1=0,\qquad \alpha_1^2+\beta_1^2+\gamma_1^2=1.$$

We find

$$(6)\qquad \alpha_1=0,\qquad \beta_1=\gamma(\beta^2+\gamma^2)^{-1/2},\qquad \gamma_1=-\beta(\beta^2+\gamma^2)^{-1/2}.$$

The condition that C be a line of curvature is that the ruled surface generated by the normal be developable. This condition will be given by substituting the values of α_1, β_1, γ_1, from (6) in (5). This gives after some reduction,

$$(7)\qquad t=\rho(\beta^2+\gamma^2)^{1/2}/\gamma,\qquad \gamma\beta'-\beta\gamma'+(1-\alpha^2)\tau^{-1}=0.$$

4. Orthogonal trajectories of the rulings. Let the point M be displaced along C to a neighboring point M_1. The condition that some point P be displaced orthogonally to the ruling is

$\sum\alpha\dfrac{\delta x}{ds}=0$, which readily reduces to $t'+\alpha=0$ on using (1) and (3).

Hence $$t=-\int\alpha ds=\text{const}.$$

The point on C from which the arc s is measured may evidently be chosen so as to make the constant of integration equal to zero. Hence the distance t to a particular orthogonal trajectory may be given by

$t=-\displaystyle\int\alpha ds$, from which we have the following theorem:

If the rulings cut the directrix under a constant angle, the distance along a ruling from the directrix to an orthogonal trajectory is proportional to the arc of C; and if the directrix be a closed curve every orthogonal trajectory intercepts constant distances on each ruling.

5. Locus of the center of geodesic curvature of an orthogonal trajectory of the rulings. Let the directrix C be an orthogonal trajectory of the rulings. It is well known that the tangent planes at M and P, the center of geodesic

curvature of C, are at right angles. Hence the direction-cosines of the normal to the surface at P are (1, 0, 0). We consider the ruled surface generated by these normals. The coordinates of any point on this normal are

$$x = t, \quad y = \rho, \quad z = \rho\gamma/\beta, \tag{8}$$

where t is the distance from P. We have seen that the distance to the line of striction is determined by the equation

$$\sum \frac{\delta\alpha}{ds}\frac{\delta x}{ds} = 0, \quad \text{which readily reduces to} \quad t = -\rho\left(\rho' + \frac{\rho\gamma}{\beta\tau}\right).$$

Since a characteristic property of asymptotic lines is that they are the lines of striction of ruled surfaces formed by the normals along these lines, the last equation tells us that $\rho' + (\rho\gamma/\beta\tau) = 0$ is the condition that the locus of P be an asymptotic line. From this we readily get $\rho = ce^{-\int \gamma ds/\beta\tau}$.
Hence if the curve C be plane, it is also a circle.

We now consider the condition that the locus of P be a line of curvature. This condition is that the normal at P generate a developable surface. Hence for some point on this normal $\delta y = 0$, $\delta z = 0$. From (8) these two equations become

$$t = -\rho\left(\rho' + \frac{\rho\gamma}{\beta\tau}\right), \qquad \rho\beta' - \gamma^2\beta\rho' + \frac{\gamma\beta^2\rho}{\tau} = 0,$$

where t is the distance to the edge of regression.

The condition that the locus of P be the line of striction is that t as given by (4), satisfy the equation $t = \rho/\beta$. This reduces to

$$\frac{\dfrac{\beta}{\rho}}{\left(\dfrac{\beta'}{\gamma} + \dfrac{1}{\tau}\right)^2 + \dfrac{\beta^2}{\rho^2}} = \frac{\rho}{\beta}, \qquad \text{from which} \quad \frac{\beta'}{\gamma} + \frac{1}{\tau} = 0.$$

Since $\alpha = 0$, we may write this in the form $\beta'(1-\beta^2)^{-1/2} + \tau^{-1} = 0$. Hence $\arccos \beta = \int \tau^{-1} ds +$ const., is the desired condition. From this we have the theorem:

If C be a plane curve, the condition that P describe the line of striction is that $\beta = const.$; and if C be a curve of constant torsion, the angle between the principal normal and the ruling is proportional to the arc of C.

6. The line of striction an asymptotic line. Let C be the line of striction then, from (4), $\beta = \alpha'\rho$. If C be also an asymptotic line, $\gamma = 0$, and the relation

$\sum \alpha^2 = 1$ becomes $\alpha^2 + (\alpha')^2\rho^2 = 1$. This may be put in the form $\rho^{-1} = \alpha'(1-\alpha^2)^{-1/2}$, from which we get $\operatorname{arc}\cos\alpha = -\int \rho^{-1}ds + \text{const.}$, from which we have the theorem:

On a ruled surface whose line of striction is an asymptotic line of constant first curvature, the rulings meet the line of striction at an angle which is proportional to the arc.

7. The rulings the characteristics of the tangent planes along C. From (6), the equation of the tangent plane at M is $\gamma y - \beta z = 0$. The characteristic of the tangent plane as M is displaced along C will be defined by this equation and[1]

$$\gamma\left(\frac{x}{\rho} + \frac{z}{\tau}\right) + \frac{\beta y}{\tau} - y\gamma' + z\beta' = 0.$$

If we eliminate y and z between the last two equations, we get

$$\frac{\gamma^2 x}{\rho} - kz = 0, \quad \frac{\beta\gamma x}{\rho} - ky = 0,$$

where we have set $k = \beta\gamma' - \gamma\beta' - \tau^{-1}(1-\alpha^2)$. From (9), $x:y:z = \rho k : \beta\gamma : \gamma^2$. Hence the direction-cosines of the characteristic are

$$(10) \quad \alpha_1 = \frac{\rho k}{[\rho^2k^2 + \gamma^2(1-\alpha^2)]^{\frac{1}{2}}}, \quad \beta_1 = \frac{\beta\gamma}{[\rho^2k^2 + \gamma^2(1-\alpha^2)]^{\frac{1}{2}}}, \quad \gamma_1 = \frac{\gamma^2}{[\rho^2k^2 + \gamma^2(1-\alpha^2)]^{\frac{1}{2}}}.$$

We seek the condition that the rulings be the characteristics. We must have

$$\rho k/[\rho^2k^2 + \gamma^2(1-\alpha^2)]^{1/2} = \alpha \text{ from which we get } (\rho^2k^2 - \alpha^2\gamma^2)(1-\alpha^2) = 0.$$

Hence, excluding the case where $1-\alpha^2 = 0$, which is the condition that S be developable, the above becomes

$$(11) \qquad (\rho k + \alpha\gamma)(\rho k - \alpha\gamma) = 0.$$

When (11) holds, it is readily seen from (10) that $\beta_1 = \beta$ and $\gamma_1 = \gamma$. Hence (11) is the condition that the rulings be the characteristics. If k vanishes we must have $\alpha = 0$.[2] But if k is zero, we note from (7) that C is a line of curvature. This is, of course, what we should expect as the lines of curvature form the only orthogonal conjugate system.

[1] Eisenhart, p. 65.

[2] We exclude the case where $\gamma = 0$, that is, where C is an asymptotic line.

DID THE ARABS KNOW THE ABACUS?

BY SOLOMON GANDZ, The Rabbi Isaac Elchanan Theological Seminary, New York City

The modern system of numeration with place value and zero certainly belongs among the greatest inventions of the human spirit. "Who it was to whom the invention is due, or where he lived, or even in what country, will probably always remain a mystery."[1] So far, however, we know that this system is closely connected with the so-called Hindu-Arabic numerals and that its origin therefore is to be traced to Hindu and Arabic sources.[2]

On the other hand, we must not imagine that these Hindu-Arabic numerals, with their ingenious system of place value and zero, owe their inception to a spontaneous idea suddenly originating in the mind of some great mathematician. For many hundreds of years the system of place value has been practiced in a mechanical and palpable way on the abacus in its various forms. On the columns of the abacus the Hindus most probably developed their famous system of notation with a special name for each power of ten.[3] On the columns of the abacus the Hindu mathematicians probably first learned to connect different orders of the decimal system with the same marks, counters, or symbols. It seems not only possible but very likely "that one of the forms of ancient abacus suggested to some Hindu mathematician the use of a symbol to stand for the vacant line when the counters were removed."[4] Bubnov and Kaye, although denying the Hindu origin of the numerals, still admit "that the forms of the numerals first found in Europe are derived from ancient symbols used on the abacus."[5] The Hindu-Arabic numerals were not improperly characterized as a means to transform the instrumental abacus into a written abacus;[6] and indeed they are nothing else, the columns being missing but being inserted mentally.

We therefore may find it quite natural that India, which is "historically the home of the Abacus," is at the same time the birthplace of the Hindu numerals representing the written abacus.[7] We are, however, naturally sur-

[1] Smith-Karpinski, *The Hindu-Arabic numerals*, p. 41.

[2] Smith-Karpinski, loc. cit., pp. 47 seq., 91 seq., 112 seq., 117 seq.

[3] Alberuni, *India*, English ed., vol. 1, pp. 174 seq.; Woepcke, *Mémoire sur la propagation des chiffres Indiennes*, Paris, 1863, pp. 92 seq., 110 seq.; Smith-Karpinski, loc. cit., pp. 41 seq.

[4] Smith-Karpinski, loc. cit., p. 41.

[5] Smith-Karpinski, loc. cit., p. 65; Kaye in the Journal of the Asiatic Society of Bengal (1907), p. 488.

[6] Woepcke, *Mémoire sur la propagation*, pp. 67 and 128.

[7] The existence of an abacus in India and the close connection of the Hindu numerals with the abacus may be regarded as the common opinion of the historians of mathematics and Indian literature; see Humboldt, *Kosmos*, vol. 2, pp. 263 seq., 454–456; Chasles, quoted by Humboldt, *ibid.*, p. 454, note 18; Woepcke, loc. cit.; Knott, "The Abacus," in The Transactions of the Asiatic Society of Japan, vol. 14, pp. 19–20, says: "The modern cipher system of notation . . . . owes its origin to the indications

prised at the strange fact that the Arabs who were the first to accept from India this system of the written abacus, namely the Hindu numerals, and to improve it, and to teach and spread it all over the world, should have known nothing of its origin, the instrumental abacus. It is therefore the purpose of this paper to examine once more the old question, "Did the Arabs know the abacus?"

This problem has for a long time been a matter of controversy among historians of mathematics and is not yet solved, and a brief sketch of the points at issue will be appropriate as an introduction to the writer's own theory.

Woepcke[1] thinks it very likely that the Arabs knew the column-abacus. He says: "Tous ces faits rendent plus que probable que les Arabes, en arrivant en Espagne, y adoptèrent pareillement les chiffres et .... le tableau à colonnes." The German mathematicians Hankel and Treutlein strongly object to this opinion. Hankel[2] contends: "Jene alten [*ghobâr*] Ziffern erscheinen bei den Lateinern so wesentlich mit dem Abakus verbunden, ... dass man meinen sollte, der Abakus müsse aus derselben Quelle stammen, wie jene Ziffern. Nun aber findet man in der gesamten arabischen Literatur nicht die geringste Spur einer Bekanntschaft mit dem Abakus." This means: *There is not the slightest trace of an acquaintance with the abacus in the whole Arabic*

of the Abacus." "The Abacus finds its earliest historic home in India." Further on, p. 44, Knott continues as follows: "Historically, the home of the Abacus is in India..... Once the decimal stage was reached, its general similarity to the Abacus indications suggested bringing them into still closer correspondence. This advance seems to have taken place amongst the Aryan Indians, who .... very soon discarded the Abacus for the more convenient cipher notation." This opinion is also accepted by Smith, *History of Mathematics*, vol. 2, pp. 158 and 159, and by Smith-Karpinski, loc. cit., p. 65. Of the same opinion are also Bayley, Taylor, Woepcke, Burnell and Rodet, all quoted by Kaye, "*The use of the abacus in ancient India*," in the Journal of the Asiatic Society of Bengal, 1908, pp. 293–298, and also *ibid.*, 1907, pp. 119, 487 seq. See also Fleet, "*The use of the abacus in India*," in the Journal of the Royal Asiatic Society, 1911, who interprets the word *ganitra* occurring in a writing of the first century of the Christian era as meaning an instrument of reckoning. Kaye himself (*ibid.*, 1907, p. 488; 1908, p. 297) says "that the abacus played a very important part in the development of our notation no one will deny," but he questions the fact that it was known to the Hindus on the ground that "if such an instrument as the abacus were in use in ancient India some real evidence of its use would be forthcoming." That such evidence does not exist is also asserted by the Hindu scholar Datta in this Monthly (1926), p. 450. Datta, however, who also refers to Fleet's theory, mentioned above, formulates his opinion as follows: "Until now no mention of the existence of the abacus, direct or indirect, has been traced in any Indian literature so that it may be taken without any fear of contradiction that if an abacus was ever in use among the learned men of India, it was discarded long ago." However, Cantor (*Geschichte der Mathematik*, vol. 1, 4th ed., p. 609) simply denies that there is any proof whatever for the existence of the abacus in that country. That Cantor is consistent, insofar as he also denies that the Arabs knew the abacus, is considered later in this paper.

[1] *Mémoire sur la propagation* (Paris, 1863), p. 58.

[2] *Geschichte der Mathematik* (1874), p. 328.

literature.[1] In the same way Treutlein expresses his opinion[2] "dass die Araber unzweifelhaft kein Columnenrechnen hatten"; that is, that undoubtedly the Arabs had no column-arithmetic. Weissenborn first takes exception to these statements of Hankel and Treutlein, which were made with such a certainty, and admits the possibility of the Arabs having known the abacus. He says:[3] "Da vernahm man die Araber rechneten auf einer mit Staub bedeckten Tafel . . . . und was konnte eine solche Tafel, welche zum Rechnen diente, anders sein als ein Rechenbrett?" And then he continues:[4] "Allein sie rechneten doch auf einer mit Staub bedeckten Tafel, freilich nicht mit Marken, sondern indem sie die Zahlzeichen schrieben. Ob nun unter solchen Umständen diese Staubtafel ein 'Abakus' genannt werden kann, darüber wird man streiten können." Then he quotes the testimony of the English chronicler William of Malmesbury (*c.* 1150), that Gerbert learned the use of the abacus from the Arabs[5] and, still doubting, he concludes in his summary[6] with the admission of two possibilities: Either Gerbert invented the abacus, or he learned it from the Arabs. But it was only an abacus with written statements. "Schriftliches Verfahren," namely the dust abacus with the dust numerals. Later on, Weissenborn with still more emphasis ascribes the knowledge of the abacus to the Arabs, saying:[7] "It is true that we do not find the abacus mentioned among the Arabs, but we must not draw the conclusion that it was not known to them." His reasons are the testimony of William of Malmesbury mentioned above, and a treatise of Al-Kindi (*c.* 850), —"*De numeris per lineas et grana hordeacea multiplicandis.*"[8] Finally Weissenborn refers to the inner,

[1] According to Kaye, in the Journal of the Asiatic Society of Bengal (1907), p. 488, Rodet is supposed to have stated that in Arabic and Persian manuscripts one often comes across arithmetical calculations effected by means of the "tableau à colonnes." Kaye does not quote exactly the passage where Rodet made such a statement. It is to be found in the Journal Asiatique, (7), vol. 16 (1880), p. 463.

[2] *Geschichte unserer Zahlzeichen* (1875), p. 36.

[3] *Gerbert* (Berlin, 1888), p. 230.

[4] *Ibid.*, p. 235.

[5] *Ibid.*, p. 236: "Abacum certe primus a Saracenis rapiens regulas dedit"; see also Smith, *History*, vol. 2, p. 175.

[6] *Gerbert*, pp. 237–239.

[7] *Zur Geschichte der Einführung der jetztigen Ziffern in Europa* (Berlin, 1892), pp. 5 seq.

[8] This Latin translation is given by Casiri, *Bibliotheca Arabico Hispana*, vol. 1, pp. 353 seq. The Arabic title, however, reads: "*Risâlat fî'l khuṭûṭ w'al ḍarb bi'a'dâd al-sha'îr*" (see Casiri, *ibid.*, p. 357; *Fihrist*, Arabic text, p. 256, line 22; *Ibn Abi Useibia*, ed. Müller, vol. 1, p. 210, line 15), and ought to be translated "A treatise on the lines and the multiplication with the number of the barley-grains." This translation is also given by Suter (*Fihrist*, p. 11), and accepted by Cantor (vol. 1, 2nd ed., p. 675; 4th ed., p. 718) in his polemics against Weissenborn's theory. On the other hand this correct translation gives no sense. The writer therefore prefers the Latin translation of Casiri, which clearly proves the knowledge of the line abacus, and is the work of one who was thoroughly familiar with Arabic and who made no mistake in his translation. His rendition of the text indicates that he could find no sense in the extant Arabic title, and believed that it should read "*Risâlat fî ḍarb al'a'dâd bi'l khuṭûṭ w'al-sha'îr,*" to which emendation the writer would also subscribe. For further literature on Al-Kindi see Steinschneider, *Hebräische Uebersetzungen*, pp. 562 seq., and note 174 *ibid.*

historical evidence, saying:[1] Es war mir schon früher nicht glaubhaft, dass die Araber (sie hätten ja sonst wohl unter allen Völkern allein gestanden) vom Kopf- und Fingerrechnen unmittelbar zum schriftlichen Rechnen übergegangen sein sollten, ohne, wenigstens eine kurze Zeit hindurch, ein Rechenbrett zu benutzen." Cantor, however, opposes Weissenborn's theory with great determination, although without mentioning his name, saying:[2] "Von einem Rechenbrette oder etwas, was demselben irgendwie gleicht, ist bei Alchwarismi keine Rede, und ebenso erfolglos wird unser Suchen danach bei allen arabischen Schriftstellern bleiben." He adds: "Von Al-Kindi wird zwar eine Schrift über die Linien und das Multiplizieren mit der Zahl der Gerstenkörner erwähnt, aber daraus auf einen Abacus zu schliessen, dürfte allzukühn sein"; and further on we read:[3] "Von Arabern, bei welchen die Kunst [des Abacus] geblüht haben könnte, ist auch bei Radulph von Laon (*d.* 1131) mit keinem Worte die Rede . . . . Auch von Atelhart von Bath, welcher um 1130 über den Abakus schrieb, ist ein sehr beredtes Schweigen zu melden." Atelhart, although he perfectly mastered the Arabic language, as is shown by his translations, did not mention the Arabs in his treatise *Regulae Abaci.* Sein [Radulph von Laon's] Schweigen [about the Arabs; he ascribes the abacus to the Assyrians] ist als Beweis anzusehen, dass ihm und mit ihm den Zeitgenossen, vor welchen er durch Gelehrsamkeit sich auszeichnete, ein Vorkommen des Abakus bei den Arabern gerade so unbekannt war wie bei uns." Further on we read:[4] "Wesentlich ist auch das Nichtvorkommen des Abacus im Algorithmus des Johannes von Sevilla." And finally he says:[5] "Das Kolumnenrechnen kommt erst bei Ibn al-Bannâ am Ende des dreizehnten Jahrhunderts vor; hier ist abendländischer Einfluss nachgewiesen."

From the English school, however, Professors D. E. Smith and L. C. Karpinski seem fully to accept Weissenborn's theory, and, without further entering into the discussion of the problem, are simply stating[6] that "the significance of the term *ghobâr* is doubtless that the numerals were written on the dust-abacus." Still more clearly Professor Smith gives as his opinion[7] that the dust-board was common among the Arabs in the Middle Ages, and the numeral forms derived from being written on such a tablet were therefore called in the schools of the western Arabs the ghobâr numerals. Not satisfied with this Professor Smith started an investigation relating to the existence of the ghobâr numerals among the eastern Arabs, concerning which he also enter-

[1] *Zur Geschichte,* as above, p. 8.

[2] *Geschichte der Mathematik,* vol. 1, 2nd ed., p. 675; 4th ed., p. 718.

[3] *Ibid.,* vol. 1, 2nd ed., pp. 836, 838; 4th ed., pp. 891, 893.

[4] *Ibid.,* vol. 1, 2nd ed., p. 854; 4th ed., p. 909.

[5] *Ibid.,* vol. 1, 2nd ed., p. 758; 4th ed., p. 807.

[6] Smith-Karpinski, *The Hindu-Arabic Numerals,* p. 65.

[7] *History,* vol. 2, p. 175.

tained a correspondence with his friend Salih Mourad, then a professor in the Turkish Naval College, Constantinople. Finding then that the eastern Arabic scholars Al-Antâkî and Al-Kalwâdânî, living in the tenth century in Bagdad, wrote certain books under the title "The book of the board on Hindu arithmetic," both Smith and Salih Mourad inferred that the eastern Arabs used the dust numerals in the same way as the western computers of Cordova, Salamanca and Toledo. Professor Smith was then kind enough to show to the writer of these lines the incomplete manuscript prepared by them upon the subject, giving him (with Professor Salih Mourad's approval) liberty to use it.

While studying these notes and trying to collect some further material of the same kind the writer was struck by the idea that there is indeed abundant proof and clear evidence for the statement that the eastern as well as the western Arabs knew not merely the dust abacus but the general form as well, and created two Arabic terms for abacus, which have not as yet been recognized as such by students of Arabic philology and mathematics. This material, together with the writer's opinion, is now submitted as follows:

'Alî ibn Aḥmed, Abû'l Qâsim, al-Antâkî (that is from Antioch), al-Mujtabâ (the chosen), who lived at Bagdad and died in 987, wrote several books on arithmetic bearing the name *Kitâb al-takht* "The book of the board."[1] They are (1) The *Kitâb al-takht al-kabîr fî'l ḥisâb al-Hindî*, "The great book of the board on Hindu arithmetic"; (2) *Kitâb fî'l ḥisâb 'alâ'l takht bilâ maḥw*, "The book on arithmetic on the board without erasing"; (3) *Kitâb al-ḥisâb bilâ takht bal bi'l yad*, "The book on arithmetic with the hand without the board."[2] Mohamed ibn Abdallâh, Abû Nasr, al-Kalwâdânî, a contemporary of al-Mujtabâ and, like him, living at Bagdad, also wrote a book entitled *Kitâb al-takht fî'l ḥisâb al-Hindî*, "The book of the board on Hindu arithmetic."[3] Sinâ ibn al-Fatḥ (whose father, Al-Fatḥ, may be identical with Fatḥ ibn Naghiyya (died *c.* 941), a native of Harrân and a mathematician of some merit) is also mentioned as the author of a book *'Ilm ḥisâb al-takht*, "The science of arithmetic of the board."[4] There is also a *Kitâb al-takht* written by al-Râzî, whose date is unknown.[5]

[1] See *Fihrist*, pp. 40 and 75, nn. 271, 276; Arabic text of the *Fihrist*, p. 284; Suter's list, p. 63; Casiri, loc. cit., vol. 1, p. 411 (quoted also by Weissenborn, *Zur Geschichte*, p. 87); Ibn al-Qifṭi, *Tarikh al-ḥukamâ*, ed. Lippert (Leipzig, 1903), p. 234; Woepcke, *La propagation*, pp. 157, 160. In the library of the Jewish Theological Seminary there is a copy of Woepcke, formerly in the Steinschneider library, which contains on the above-mentioned pages numerous marginal notes by Steinschneider referring to further literature upon the subject.

[2] This book is quoted directly by Ibn al-Qifṭi, and by others on his authority.

[3] *Fihrist*, pp. 41, 75; Arabic text, vol. 1, p. 284; Suter's list, p. 74; Ibn al-Qifṭi, loc. cit., p. 288; Woepcke, *loc. cit.*

[4] *Fihrist*, pp. 37, 70; Arabic text, vol. 1, p. 281; Suter's list, pp. 51, 66; Woepcke, loc. cit., p. 157.

[5] *Fihrist*, p. 37; Arabic text, p. 281 at the bottom.

The question now arises as to what is the exact meaning of *Kitâb al-takht*, "Book of the board."[1] It can hardly be assumed that a kind of a board tablet of the Arabic school is meant. There is nothing characteristic on such a board to give the name to the science of arithmetic. On the other hand we see that all the treatises named *Kitâb al-takht* dealt with the Hindu arithmetic. Now we know that the characteristic of the Hindu numerals was the place value and the zero. This was also the characteristic of the abacus. It would therefore be proper to call the Hindu arithmetic the abacus arithmetic. Besides that we know that the abacus gave the name to arithmetic in general, like the calculus (pebble).[2] We know that "the period from the time of Gerbert (*c*. 1000) until after the appearance of Leonardo's monumental work (1202) is called the period of the abacists."[3] But although "even for many years after the appearance, early in the twelfth century, of the books explaining the Hindu art of reckoning, there was strife between the abacists, the advocates of the abacus, and the algorists, those who favored the new Hindu numerals,"[4] the word abacus, however, finally came to mean any kind of elementary arithmetic, and even books dealing with the algorismus and the Hindu-Arabic numerals bore the name *Liber Abaci* or *Libro d'abacho*,[5] and even the famous book of Leonardo Fibonacci, "which was most influential in introducing the new algoristic method based on the Hindu-Arabic numerals to the scholars of Europe,"[6] bore the name *Liber Abaci*.

Taking into consideration all these facts the writer is convinced that the word *takht* ("board") is nothing else than the Persian-Arabic term for the abacus. The *Kitâb al-takht* is the Arabic *Liber Abaci*,[7] and the authors of

[1] The form *takht* should be used instead of *taḥt*, although the *Fihrist* and Hâji Khalfa (see hereinafter) adopted the latter. The *ḥ* is the Arabic letter *h* without a dot, whereas *kh* represents the same letter with a dot and is more guttural than the other. *Takht* is the more correct form and is therefore properly used by Woepcke, Ahlwardt (mentioned later), and Ibn al-Qifṭi. Suter first erroneously read *taḥt* and translated "method," but soon recognized that *takht* = "board" is the better reading; see his *Fihrist*, p. 70, note 232. *Takht* is merely an Arabic form of the Persian word *takhta* meaning "board" or "wood"; see Vuller, *Lexicon Persico-Latinum*, vol. 1, p. 425; Lane, *Arabic Lexicon*, p. 298.

[2] Smith, *History*, vol. 2, 166.

[3] Smith-Karpinski, p. 120; Cantor, vol. 1, chapter 40 on the abacists and algorists.

[4] Smith-Karpinski, *loc. cit.*

[5] Smith, *History*, vol. 2, p. 7; Cantor, vol. 1, chapter 40.

[6] Smith-Karpinski, pp. 129 seq.

[7] Leonardo Fibonacci, "the most noteworthy mathematical genius of the Middle Ages, ... the one most influential in introducing the new numerals to the scholars of Europe" (Smith-Karpinski, p. 129), was a disciple of the Arabs, and his *Liber Abaci* is only "eine freie Darstellung" (a free presentation) of the entire knowledge of the Arabs about arithmetic and algebra; see Hankel, loc. cit., pp. 293, 342. As we now see, the very title *Liber Abaci*, which was so common in the period of the abacists, is only a translation of the Arabic *Kitâb al-takht*. In the same way the title of another work of Fibonacci, *Practica Geometriae*, is also only a literal translation of the Arabic book titles *Al-Ṣinâ'a al-Handasiyya* and *Ṣinâ'at al-handasa* (cited by Woepcke, *La propagation*, p. 178, and by Ahlwardt, *Catalogue of Arabic manuscripts*, Berlin, vol. 5, p. 321, No. 5945 in fine). The more correct translation is *Ars Geometriae*,

those treatises are the Arabic abacists, the fore-runners of the European abacists. The book of Al-Mujtabâ was not the first of its kind, for he called it the great *Kitâb al-takht*. This indicates that there were already numerous such treatises, and that the name *Kitâb al-takht* was common and popular for works of this kind. Al-Mujtabâ undertook to write a compendious arithmetic which he called "The great *Liber Abaci*."[1]

Further evidence for the identity of the *takht* and the abacus may be found in the following book titles:[2] *Kâfiyya fî ḥisâb al-takht w'al-mîl*, "A compendium of arithmetic with board and stylus," written by Amîn al-Dîn, al-Abharî, who died in 1332; and the *Jawâmi' al-ḥisâb bi'l takht w'al-turâb*, "An encyclopedia of arithmetic with board and dust," written by Abû 'Abdallâh al-Zanâtî, whose date is unknown. Here we have all the requisites of the dust abacus, the tablet with dust and stylus.[3] Further corroboration is to be found in a note of Haji Khalfa,[4] which reads as follows: "There are many divisions of arithmetic. One of them is the science of arithmetic of the board and the stylus, *al-ṭakht w'al-mîl*. This is the science by which we learn how to perform the arithmetic operations by numerical symbols indicating the units, and also numbers exceeding the units, by the stages.[5] These symbols are attributed to the Hindus. . . . . This science is also called 'The board and the dust,' *al-takht w'al-turâb*." Here it is clearly stated that there is a science of "the board and the dust," or "the board and the stylus," which deals with the Hindu numerals and the place value, and this admits no other conception than that of the science of abacus arithmetic.

But the Arabs not only knew the abacus, they also knew the double etymology of the Greek word *abakion*, namely as "table, tray," and as "dust,"

---

sinâ'a meaning "art, handicraft." The *Liber Abaci* was also called by the Arabic name *Algebra et Almuchabala*; see Ball, *History of Mathematics*, 6th ed., p. 167. Further instances of the Arabic influence are evident in *regula elchatayn* and in the word *pensa* (weight, the Arabic *mizân*) for "proof"; see Cantor, vol. 2, 1st ed., p. 25; Tropfke, vol. 1, 2nd ed., p. 60. That Leonardo was under the influence of the Hebrew mathematicians Savasorda and Ibn Ezra has been satisfactorily proved, his *Practica Geometriae* being an imitation of the *Liber Embadorum* of Savasorda. The arrangement of the chapters is the same, and a great number of sentences and examples are verbally taken from the same work. On this point see Steinschneider, *Gesammelte Schriften*, pp. 405 seq., and in Orientalische Literatur Zeitung (1901), p. 93; Curtze, *Liber Embadorum* (in Abhandlungen zur Geschichte der Mathematik, 1902), p. 5. In his *Liber Abaci* Leonardo followed the same order and arrangement as in the arithmetics of Savasorda and Ibn Ezra; see Steinschneider, *Gesammelte Schriften*, *p.* 471. Woepcke, *Extrait du Fakhrî*, p. 3, proves that Fibonacci was also indebted to al-Karkhî.

[1] At the same time he expounded also the finger arithmetic which he called "The arithmetic with the hand without the abacus." Compare Heath, *Greek Mathematics*, vol. 1, p. 48: "In Aristophanes (*Wasps*, 656–64) Bdelycleon tells his father to do an easy sum *not with pebbles but with fingers*."

[2] See Ahlwardt, *Catalogue of Arabic manuscripts in Berlin*, vol. 5, p. 333, No. 5975; p. 352, No. 6013, 8.

[3] Or also the pointer; see Smith, *History*, vol. 2, pp. 157 seq., 177 seq.

[4] *Lexicon Bibliographicum et Encyclopaedicum*, the Arabic text with a Latin translation, by G. Flügel, in 7 volumes (Leipzig, 1835–1858). The note is in volume 3, pp. 61 seq.

[5] Or "by degrees," referring to place value.

from the Hebrew *abaq*.[1] While the eastern Arabs rendered it with the Persian term *takht*, "board, tray," the western Arabs called it *ghobâr*, "dust," and sometimes also *turâb*, which likewise means "dust." But this well-known term *ghobâr* does not mean "dust" or "dust-board," it is simply an attempt to arabicise the Greek term *abakion* (abacus). There is no more reason to restrict the *ghobâr* to the dust abacus alone than we have reason to restrict the Greek-Latin abacus to the dust abacus alone, because it is possibly derived from the Hebrew *abaq*. *Takht* and *ghobâr* are simply two Arabic terms for the translation of the word "abacus."[2] Now the word abacus may be derived from the ancient dust table, and may also, very probably, have another explanation. Certain it is that the word "abacus" was also applied to the counter- and line-abacus. Therefore the words *takht* and *ghobâr* must also have been used to signify these forms of the abacus.[3]

In making the assumption that the well-known *ghobâr* is nothing else than the technical term of the Arabs for the abacus, all the data and terms connected with the word *ghobâr* appear at once in a new light, fit more smoothly into the general history of mathematics, and throw more light on the data and terms used by Europeans.

We understand now why the *ḥisâb al-ghobâr* (abacus arithmetic) in Spain is like the *ḥisâb al-hindi* and the *ḥisab 'alâ'l takht* in the Orient.[4] We understand why the *ḥurûf al-ghobâr* (the abacus numerals) are identical with the symbols and characters of the medieval abacus.[5] We understand why there is a *ḥisâb al-ghobârî* (abacus arithmetic) in opposition to the *ḥisâb al-hawâî* (the mental

---

[1] See the Greek dictionaries; Smith, *History*, vol. 2, p. 156; Cantor, vol. 1, 4th ed., p. 131.

[2] The abacus was also known to the Hebrews, and there are two Hebrew terms for it: *pinqas* (the Greek *pinax*), "board, tablet," and *abaq*, *aphar*, "dust," corresponding to the Arabic *takht* and *ghobâr*. The *ghobâr*- or dust-numerals are called in Hebrew *mispar ha-abaq* and *mispar ha-aphar*. Abu Sahl ibn Tamîm (*c.* 955), better known as Isaac Israeli, wrote a book *Ḥisâb al-ghobâr beḥeshbôn anshê hôdô*, "The abacus arithmetic on Hindu arithmetic." This strange title, half Arabic and half Hebrew, sounds like the above mentioned *Kitâb al-takht fî ḥisâb al-Hindi*. Naturally, these Hebrew terms for the abacus have not been recognized as such until now, but have been misunderstood like the terms *ghobâr* and *takht*. On these Hebrew terms and upon Isaac Israeli and his book see Steinschneider, *Jewish Literature* (London, 1857), p. 363; *Hebräische Uebersetzungen*, p. 397, note 196, and pp. 400, 558; M. Reinaud, *Mémoire . . . . sur l'Inde*, p. 399. Reinaud, from whose book the European mathematicians drew their knowledge of these passages, owed his information to M. S. Munk; see Munk, *Notice sur Abou'l Walid*, p. 51.

[3] The terms *takht w'al-mîl* and *takht w'al-turâb*, mentioned above, will be best explained as attempts to combine the two etymologies of the table and the dust-table. Hence these terms cannot be used as proof for the real existence of the dust-abacus.

[4] See Woepcke, *La propagation*, pp. 59, 61, 182 seq.

[5] See Weissenborn, *Gerbert*, pp. 227, 237–239; the same author, *Zur Geschichte*, pp. 10 seq.; Hankel, p. 328, quoted above; Smith-Karpinski, pp. 117 seq.; Smith, *History*, II, pp. 74 seq.

or head-arithmetic).[1] We understand the terms *'ilm al-ghobâr* and *'ilm al-turâb*[2] which are like the *'ilm al-takht*[3] (the science of the abacus).

As already mentioned above, the term *turâb* ("dust"), is also used, like *ghobâr*, to signify the abacus. This will help us to better understand the famous passage in Alberuni's (Al-Bîrûnî's) *India* dealing with the Hindu numerals. The passage reads:[4] "As in different parts of India the letters have different shapes, the numeral signs, too, ... differ. The numeral signs which we [the Arabs] use are derived from the finest form of the Hindu signs. . . . . But the people of Kashmîr mark the single leaves of their books with figures which look like drawings or like the Chinese characters. . . . . However, they do not use them when reckoning in the sand." From the last sentence the historians usually inferred that the Hindus were in the habit of reckoning on the sand. But the sentence was misunderstood by Sachau, the English translator, as well as by Woepcke.[5] The Arabic text in Sachau's edition[6] reads: *walâ tasta 'malu fî'l-ḥisâb alâ'l-turâb*, meaning: "And they are not used in computing on the abacus." It refers to the Hindu-Arabic numerals mentioned above, and relates the well-known fact that the usual Hindu-Arabic numerals differ from the *ghobâr-* or abacus numerals.[7]

In view of all these facts it is hoped that historians will agree that the opinions of Hankel, Treutlein and Cantor to the effect that there is not the slightest trace of the abacus in the whole Arabic literature[8] will no longer be accepted, but that we shall recognize a new chapter in Arabic mathematics on the Arabic abacus and the Arabic treatises with the name *Liber Abaci*.[9]

---

[1] Woepcke, "Recherches," Extrait no. 13 from the Journal Asiatique (1854), pp. 11 seq.

[2] Woepcke, *ibid.*; Ahlwardt, *loc. cit.*, vol. 5, pp. 336, 352, nn. 5979, 6013, 7.

[3] Haji Khalfa, *loc. cit.*

[4] Alberuni's *India*, English text, vol. 1, p. 174.

[5] *La propagation*, pp. 95 seq.

[6] P. 83, lines 2–3.

[7] See Smith-Karpinski, pp. 68 seq., 98, 112 seq., 140.

[8] Cantor himself admits (Vol. 1, 4th ed., p. 807, quoted above) that Ibn al-Bannâ knew the column-abacus, but he says that it is proved that there was European influence. The writer does not know how or where this is done. But he well knows that this treatise of Ibn al-Bannâ is only an extract of a treatise of Ibn al-Ḥaṣṣâr, who lived in the twelfth century, and that Ibn al-Ḥaṣṣâr expressly states that he gives nothing of his own, having collected his material from the old Arabic mathematicians; see Suter, "Das Rechenbuch des Abû Zakarîja el-Ḥaṣṣâr," Bibliotheca Mathematica, (2), vol. 2 (1901), pp. 12 seq., 40. So we have, on the contrary, evidence of an old and genuine Arabic influence. See also Rodet in Journal Asiatique (7), vol. 16 (1880) p. 469.

[9] The writer wishes to express his appreciation of the assistance given by Professor David Eugene Smith in the preparation of his manuscript.

THE TWENTIETH ANNIVERSARY OF "SCIENTIA"

By DAVID EUGENE SMITH, Teachers College, Columbia University

It seems appropriate that The American Mathematical Monthly should call attention at this time to the fact that twenty years have elapsed since the first appearance of the international publication *Scientia*, and to extend to its sponsors congratulations upon the success which it has achieved. Founded in 1907 by Signori Bruni, Dionisi, Enriques, Giardina, and Rignano, it was published under their direction until 1925, since which time it has been under the editorship of Dr. Rignano alone.

During these two decades it has appeared uninterruptedly, even throughout the period of war, when the publication of so many journals was temporarily or permanently discontinued. Indeed, it was in the years of the great conflict that some of its most scholarly and forceful articles were written, many of them dealing with the causes of the war and with its political and economical consequences. To the impartiality of the editor and his advisers was due a notable series of such articles, prepared during the period of Italy's neutrality, written by scholars in England, France, Germany, Russia, Austria, Hungary, the United States, and the Scandinavian countries, and setting forth with great ability the views of the leaders in thought in these various parts of the world. Such material, important as it was then, will have even greater historical value in the years to come.

It would be presumptuous for this writer to attempt to compile from the impressive roll of contributors a brief list that should best represent the work of *Scientia* during these twenty years. Indeed, it would be difficult to find a man whose grasp of science, of history, of the arts, and of letters qualifies him to do so. There come to the mind at once, however, a few names that may serve to show the general standing of those who have made the journal what it is,—a compendium of the scientific thought of the time. These may be classified roughly as follows:

History of the Sciences: Bortolotti, Cajori, Carra de Vaux, Dreyer, Favaro, Heiberg, A. Loria, G. Loria, Karpinski, Kaye, Mieli, Milhaud, Vacca, and Zeuthen.

Philosophy of Science: Enriques, Poincaré, and B. Russell.

Astronomy: Abbott, Chamberlain, Crommelin, R. S. Dugan, Dyson, Eddington, Hagen, Jeans, Kapteyn, Lowell, MacMillan, Palatini, Puiseaux, Shapley, See, and Very.

Biology and Physiology: Bohn, C. M. Child, De Vries, Lillie, G. Lusk, Pearl, E. S. Russell, and J. A. Thomson.

Chemistry: Bruni, Ciamician, Crowther, Findlay, and Soddy.

Economics and Law: Barclay, Burdick, Cahns, T. N. Carver, J. B. Clark, I. Fisher, Gini, Knibbs, Mondiani, Parmalee, Price, Ryan, W. R. Scott, Seligman, and Virgilii.

Geology: G. H. Darwin, R. A. Daly, De Marchi, Gregory, C. R. Keyes, W. B. Wright.

Mathematics: Borel, Boutroux, Carslaw, Castelnuovo, Dickson, Carmichael, Jourdain, Marcolongo, Peano, Rey Pastor, and others included elsewhere in this list.

Physics: G. H. Bryan, De Broglie, Fabry, Larmor, La Rosa, E. P. Lewis, O. Lodge, Lorentz, Righi, Rutherford, Somigliana, and Zeeman.

Psychology: Claparede, C. Lloyd Morgan, Rignano, and Thorndike.

Sociology: Caetani, Cardinali, Guignebert, Kidd, Landry, Perozzi, and Pettazzoni.

To an American it is particularly interesting to see the number of names representing the western continents and to feel that the present century is giving to the New World a worthy place in the progress of science in its broadest aspect and in the history of its various branches.

Since *Scientia* is an international review of scientific synthetic thought, recognized throughout the world, our public and university libraries should have on their shelves the entire series, if complete sets are still available.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A Check Formula for the First Case of Oblique Triangles.

By L. J. PARADISO, Lehigh University

In connection with the solution of the oblique triangle when two angles and a side are given, it is desirable to check the results by a convenient formula. The tangent law is frequently used to check. If A, B, c, are the given parts and an error is made in getting the logarithm of c in applying the law of sines, then although a and b are wrong in consequence, the law of tangents will not detect this error. Mollweide's formulas have also been used as a check in this case but they also fail to detect an error made in getting either the logarithm of c or the antilogarithm of a, or of b.

The formula for the tan $(A/2)$ in terms of the sides of the triangle is sometimes used as a check in lieu of one of those already mentioned. Although

this formula is free from the objections given to those above yet it is obvious that it is a tedious one to use merely for a check.

The following formula, sometimes given as an exercise, has been found to give satisfactory results when used as a check. It is not open to the above objections:

$$(a+b-c)/(a+b+c) = \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B\,.$$

It can be proved directly from the formulas for $\tan\frac{1}{2}A$ and $\tan\frac{1}{2}B$ in terms of the sides or from the law of sines as follows:

$$\frac{a+b-c}{a+b+c} = \frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{4\sin\frac{1}{2}A \sin\frac{1}{2}B \cos\frac{1}{2}C}{4\cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C} = \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B.$$

For we have

$$\begin{aligned}\sin A + \sin B - \sin C &= \sin A + \sin B - \sin(A+B)\\ &= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+B)\\ &= 2\sin\tfrac{1}{2}(A+B)\left[\cos\tfrac{1}{2}(A-B) - \cos\tfrac{1}{2}(A+B)\right]\\ &= 2\cos\tfrac{1}{2}C(2\sin\tfrac{1}{2}A \sin\tfrac{1}{2}B)\,.\end{aligned}$$

Similarly for the denominator.

II. A Note on Partial Fractions

By Raymond Garver, University of Rochester

In any first or second course in the calculus, a topic of some importance is the reduction of a rational function of x, say $n(x)/d(x)$, to the sum of partial fractions whose denominators are factors of $d(x)$. If $d(x)$ can be factored completely into real, linear factors, the work can be carried through quite easily by any one of several familiar methods. But if there are quadratic factors of the form x^2+px+q $(p^2-4q<0)$, the work is much longer.

Thus to reduce the fraction

$$\frac{3x^3-4x^2-3x+5}{(x^2+x+1)(x^2-2x+3)} \quad \text{to the form} \quad \frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-2x+3},$$

we equate the two, since we have presumably proved that such a reduction is possible. Clearing of fractions, we have

$$(1)\quad 3x^3-4x^2-3x+5 = (Ax+B)(x^2-2x+3)+(Cx+D)(x^2+x+1),$$

which is an identity. There are then two usual ways of computing A, B, C, D. We may assign four convenient integral values to x, or we may equate corresponding coefficients. Either process leads to four equations in four unknowns, a fairly long problem in algebra.

It seems possible to simplify the work by giving x an imaginary value which makes one of the quadratic factors vanish. The work can be arranged so that we never have to find this root explicitly at all. Thus, if we put $x=x_1$, where x_1 is a root of $x^2+x+1=0$, we have $x_1^2=-x_1-1$, $x_1^3=-x_1^2-x_1=1$; and equation (1) reduces to

$$(2)\qquad \begin{aligned} x_1+12 &= (Ax_1+B)(-3x_1+2) = -3Ax_1^2+(2A-3B)x_1+2B, \\ &= (5A-3B)x_1+(3A+2B). \end{aligned}$$

Equating real and imaginary parts, $1=5A-3B$, and $12=3A+2B$, which gives $A=2$, $B=3$. In this example we can obtain $C=1$, $D=-4$, by inspection.

This method also has the advantage that it can be used to derive a formula for the reduction of any fraction similar to our example. Thus if we have

$$(3)\qquad \frac{ax^3+bx^2+cx+d}{(x^2+p_1x+q_1)(x^2+p_2x+q_2)} = \frac{Ax+B}{x^2+p_1x+q_1} + \frac{Cx+D}{x^2+p_2x+q_2}$$

where $p_i^2-4q_i<0$ and A, B, C, D are to be determined; and if we go through the above process we obtain

$$(4)\qquad \begin{aligned} A &= \frac{d(p_1-p_2)+(aq_1-c)(q_1-q_2)+(ap_1-b)(p_1q_2-p_2q_1)}{(q_1-q_2)^2+(p_1-p_2)(p_1q_2-p_2q_1)}, \\ B &= \frac{[q_1(aq_1-c)+dp_1](p_1-p_2)-[q_1(ap_1-b)+d](q_1-q_2)}{(q_1-q_2)^2+(p_1-p_2)(p_1q_2-p_2q_1)}, \end{aligned}$$

where the denominator does not vanish if the factors in the denominator of (3) are distinct, as we have tacitly assumed. The substitution in the formulas is easier than at first appears, since many of the terms are repetitions. If necessary, C and D can be found by interchanging subscripts, though in our example this was hardly necessary.

III. Is There A Student Standard of Truth?

By E. V. Huntington, Harvard University

Professor Osgood, in his letter in this Monthly for April, 1927, page 205, states a position in regard to the "truth" of a mathematical theorem which, as far as I know, is entirely new in the history of this Association. His position is that there is one standard of truth for a student and another standard of truth for a master of the science.

The occasion for this novel statement was the recent review by H. J. Ettlinger of F. S. Woods' *Advanced Calculus* (this Monthly, January, 1927 pp. 40–43). In this review, Professor Ettlinger had protested, a bit strenuously, against a certain theorem, which, according to hitherto accepted standards,

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Advanced Calculus. By W. F. OSGOOD. New York, The Macmillan Co., 1925. 16+525+4 pages. Price $6.00.

When an American teacher plans to give a second year of work in calculus, he is likely to do one of three things: first, he may pick over the forgotten morsels of the books used in the first course in calculus or a similar book, such topics for instance as evolutes, convergence of series, application of the calculus to the determination of asymptotes for curves given in polar coordinates, approximate integration or such topics always found in any self-respecting text but regularly slighted in the first course even by the author; or secondly, he may give an eclectic course without text or printed exercises, covering various topics in the theory of real variables, or special solutions of differential equations, or other topics, not adequately treated in any elementary calculus text; or thirdly, he may select the best text on "Advanced Calculus" that he can find, and except for the conscious omissions of certain chapters follow this text with fidelity. In this third case, unless he adopts the familiar "Goursat-Hedrick," he is almost sure to turn to Cambridge, Massachusetts, and select either a green or red or blue text labelled simply "Advanced Calculus."

Supposing that the teacher makes his selection on something other than his taste in colors, he may read the prefaces in turn of Wilson's, Osgood's and Woods' books. No one of these authors suggests the possibility of other texts of this title existing, in this country or abroad, so that differential claims are not made. Osgood's preface opens with the assertion that "Any course in advanced calculus must deal with partial differentiation and multiple integrals, with systematic integration, with improper integrals, and to some extent with complex quantities;" but this remark doubtless sounds more dogmatic than intended by the distinguished author; for certainly several other reputable mathematicians do not act on this basis.

Professor Osgood is to be commended for this scholarly work covering an imposing array of important topics all treated with a clarity of style and economy of language that, in the opinion of the reviewer, set this book in these respects above any others in the American field. The skill in writing equations in just the form most readily applied, to mention but one feature, while not of a sort to catch the eye, shows not merely familiarity with the subject but also an interest in and attention to the psychological processes of the student approaching the subject for the first time, only too rare in the more serious American texts. A concern for the logical niceties of the subject is what the

reader is prepared to find in all of Osgood's work, and he is not here disappointed. What is likely to surprise the teacher who examines the book is that so many topics, usually treated carelessly or reserved for the rigor of a graduate course, are neither avoided nor rendered formidable. The book is not a review from a higher view-point of elementary calculus, most of the material here given being entirely new to the student for whom this text is intended. The problems are numerous and not excessively difficult, the print is attractive and misprints are rare although, for example, Rodrigues is referred to in the text and index as Rodigues.

Among the topics deserving of special interest on account of their relative novelty in a work of this sort, may be mentioned Mercator's chart, Lagrange's multipliers, Landen's transformation of the elliptic integral, the flow of electricity in conductors, irrotational flow of an incompressible fluid, the equation of continuity (in connection with flux across a surface), Stokes's theorem in vector form, small oscillations of a system with n degrees of freedom, zonal harmonics, the differential equation of the vibrating membrane, Hamilton's principle, entropy, the gamma function, conformal mapping.

Reviewers have been in the past severely criticised by more than one irate author who insisted that the reviewer's own opinion as to desirable subject matter in a text is wholly beside the point, and that only the degree to which an author achieves his own purpose is to be considered. The present review, from this stage on, will consider the text under discussion from what is intended to be the college teacher's point of view. Admitting a fine piece of logic and exposition, the reviewer cannot forget this is a college text book, intended for use in classes, not a treatise nor a disquisition for the specialist who may have like interest with the author.

Under the probability of being misunderstood, the reviewer feels compelled to state that this excellent text gives the impression of being fitted for a rather uncommon group of students, uncommon as to preparation and outlook. The students are assumed to have previous knowledge and adequate understanding of the theory of linear dependence for which reference is made to Bôcher's *Algebra*. Even more definitely is assumed a fair mastery of all convenient topics in Osgood and Graustein's *Analytic Geometry* and in Osgood's *Introduction to the Calculus*. For instance, the student is supposed to know (of course references are given to these previous texts) the parametric equation for a system of confocal ellipses and hyperbolas, the law of the mean in integral calculus, the theory of simple harmonic motion, and Duhamel's theorem. On the other hand, it is regarded in this text as necessary to begin with the first definitions of polynomials, the degree of a polynomial, etc., and close this rather extensive work with a short chapter on complex numbers, assuming complete ignorance of the subject on the part of the student. Furthermore the

reader is within the covers of this book expected first to meet Taylor's theorem, L'Hospital's rule, double integrals, reduction formulas etc., so far as it appears from internal evidence. For a student brought up on Granville or an equivalent text, the coordination is not ideal. However this is a matter easily capable of adjustment.

A more serious disadvantage, from the point of view of the smaller college which cannot hope to compete with the many special attractions of Harvard, is the following. A second course in calculus is usually in practice designed either as a required course for engineers, physicists and chemists, or as an elective for students planning to continue toward university work. The small college seldom has a large group of students electing one more year of analytic equipment for general application to geometry or mathematical physics. For engineers of the usual sort, this text shows a brave and interesting effort, but one which the reviewer regards as at present futile. There is too much emphasis on logical structure and on general principles, and the problems are rather too difficult to prescribe as a final mathematical dose to engineering students carrying fifteen hours a week of laboratory. On the other hand for the future mathematician eager to prepare himself at once for special work, the scope of the subjects here covered is discouraging. Many little snatches of large theories are suggested but the work appears choppy, particularly so in contrast to Wilson's book. Many of the topics are illustrated by very few exercises, sometimes by none at all. The epsilon-delta notation that every good mathematician probably enjoys, and which is so beautifully treated in an "elementary" manner in Hardy's *Pure Mathematics*, is consciously avoided, just as one tries to avoid mathematical induction with a poor freshman class in college algebra. This book does not of course compete with Edwards' *Integral Calculus* in exhaustiveness in the traditional subjects of integral calculus, or with the better French "Cours d'Analyse" in developing mathematical theory. It does not attempt either.

If Arts students at Harvard can and do complete most of the substance of this book early in their undergraduate career, and can follow up their undergraduate work by extensive courses in projective geometry, linear transformations, differential equations, analytic mechanics, potential theory and function theory, they are to be envied; but if after all this they cannot define continuity in sharp cut epsilon-delta notation, they will still be pitied by many persons. If engineering students at Harvard appreciate the many logical features of this book, then the other engineering schools of the country are missing something very fine.

ALBERT A. BENNETT.

New Methods in Exterior Ballistics. By F. R. MOULTON. Chicago, University of Chicago Press, 1926. vi+257 pages. Price $4.00.

A projectile in its flight through the resisting medium, air, traces a path which cannot be algebraically expressed in terms of any known functions. Yet that path must be subjected to fairly exact mathematical treatment, if our big guns are to be accurately aimed and laid.

The fundamental differential equations of motion of a projectile in flight have been known ever since the days of Euler. Up to the World War, the progress of ballistics had consisted in devising approximate algebraic expressions whereby to wrench these equations into soluble form. These approximations have been improved and improved, but still could not keep pace with the development of modern artillery. So that finally Prof. F. R. Moulton of the University of Chicago, while serving as a Major in the U. S. Army during the World War, cut the Gordian knot by going back to the Eulerian equations and solving them in their original and exact form by means of numerical integration. Thus he laid the conerstone for an entirely new science of ballistics, and accordingly no one is better qualified than he to recount the pioneer steps of this new science. As such an account, his book is an invaluable addition to the history of mathematical physics.

Morse and Bell laid such a complete groundwork in their invention and development of the telegraph and telephone respectively, that these inventions have been able to progress beyond all resemblance to the original device. And this is equally true of Moulton and Ballistics, and is equally to his credit. His book recounts little beyond his own work of 1918, yet this work was so new and revolutionary at that time, that he may be pardoned the poetic license involved in entitling his book, published in September, 1926, *New Methods in Exterior Ballistics.* Col. Tschappat's treatise on ballastics in vol. 30 of the Encyclopedia Brittanica, published in 1922, and even the reviewer's own book, published in 1921, are more up-to-date than Moulton's *New Methods*, published in September, 1926.

The seven years since Major Moulton left the service have been crammed with progress. An orthogonal curvilinear coordinate system was officially substituted for Siacci's tangent plane (p. 10) in 1920, and consequently the curvature of the earth (p. 115) is no longer corrected for. The differential corrections, to which Professor Moulton devotes nearly the whole of Chapter IV, have been obsolete since 1919; being superseded by the Bliss system, in which Moulton devotes but one section (p. 134). Furthermore, in the winter of 1919–1920, all the Bliss symbols were changed, Franklin devised for the Bliss formulas a derivation much more interpretable physically than Bliss' adjoint system, and Gronwall discovered a new first integral which revolutionized the computations. Moulton says (p. 141) that the Bliss method "is not adapted

to determining all the terminal variations of a trajectory, or of determining even one variation for a series of points along a trajectory;" yet the Franklin physical derivation makes clear that the Bliss method *is* adapted to determining all the terminal variations, and its use for determining any variation for a series of points (anti-aircraft) was demonstrated to be quite feasible by Hedrick in 1922. Of all this basic development, Professor Moulton makes no mention.

Even such slight matters as his measuring azimuth from the south (p. 15) instead of north, his extending the z-axis to the right (p. 8) instead of to the left of the xy plane, and his use of F (p. 9) instead of E for the resistance function, put upon a reader trained under present official conventions an unnecessary burden of constant translation of signs and symbols.

The reviewer cannot speak with authority on the chapter on "Motion of a Rotating Projectile" (p. 172), but it does not appear to bear much resemblance to, or take into account, the very complete theory built up by Capt. R. H. Kent, as a result of painstaking research at Aberdeen from 1919 to date.

The reviewer has no intent to disparage Professor Moulton's colossal recasting of an entire science, nor the value of his book as an historical repository; but he feels that a more appropriate title would be "Wartime Contributions to Ballistics," and that, if the author had used more modern symbols throughout his book, this storehouse of information would be more available to those whose experience with the subject is post-armistice.

ROGER SHERMAN HOAR.

Mathematics of Accounting. By A. B. CURTIS and J. H. COOPER. New York, Prentice-Hall, Inc., 1926. xii+397 pages. Price $5.00.

This book is intended to develop an accountant from a beginning of only arithmetic up to a point where he can decide the purchase price of a bond to yield a given investment rate. All this is done by the use of set forms under which each arithmetic computation can be made. Teachers of the mathematics of finance in college will find a few suggestive problems.

C. F. CRAIG.

College Geometry. A second course in plane geometry for colleges and normal schools. By NATHAN ALTSHILLER-COURT. Richmond, Va., Johnson Publishing Company, 1925. xiii+254 pages.

This book is based on a series of lectures given by the author at the University of Oklahoma. It has for its object the presentation of material for an advanced course in plane geometry, comparable in extent with the traditional college course in advanced algebra. In view of the well known general lack of adequate preparation among our high school teachers of geometry, a course

which would give greater facility in the solution of so-called "originals and constructions" should be a welcome addition to our college curriculum.

While the present work is in no sense a book of methods, it is, after all, a sort of trail blazer, and the author has set about his presentation of such theorems and problems as he has chosen to explain, in a way to make of them patterns to be followed in the solution of similar or related exercises. Indeed in such solutions he is most detailed in his proofs, leaving little for the student to supply beyond the explicit wording of extremely elementary theorems of plane geometry. This very elaborateness of solution might be counted a serious fault in a text for a college student, save for the redeeming feature that over 500 exercises are given for the development of the student's own ingenuity.

The first chapter is concerned with the method of attack in the solution of geometric problems, special attention being given to geometric loci and problems involving indirect elements, for example, the difference of two sides of a triangle as a given quantity. The use of similar and similarly placed figures with their properties is also stressed.

The next three chapters are devoted to "Properties of a Triangle." To the immature student, whose high school geometry leads him to think he knows all there is to know about the triangle, these chapters will bring a revelation, and may cause him to exclaim with Crelle, "It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties. How many as yet unknown properties of other figures may there not be." Certainly such words as nine-point circle, Euler line, cyclic quadrilateral, sound strangely in the ears of the pupil from our ordinary high school classes, while Ptolemy's, Stewart's, and Feuerbach's theorems may show him that the science of geometry is the product of the minds of many men.

After a brief chapter on "Harmonic Section," there is introduced a discussion of the properties of circles. Inverse point, poles and polars, radical axes, coaxial circles and centers of similitude all receive attention. A chapter on "Inversion" and one on "Recent Geometry of the Triangle" complete the book.

The author's style is simple and direct, his notation is explicit and consistent. His choice of material is interesting and the exercises are varied enough to intrigue the student into further study. Typographically the book is excellent and the index seems adequate.

Let us revert now to the "Introduction," which, it might be said in passing, is not an introduction, but rather a second preface, in no sense the beginning of the subject matter. The author's expressed hope that "the vast army of high school teachers · · · might be worthily represented in this field of research," would seem to have greater chance of fulfillment if the student could learn a few

historical facts, even in footnotes. A knowledge of the chronological development of the subject would surely be worth while to any investigator.

But if the absence of dates seems only an omission and the student may be expected to search elsewhere for the genesis of his theorems, yet the misleading statement is not so to be excused. The author says: "In the last quarter of the nineteenth century, modern geometry became enriched by a splendid addition known as the 'geometry of the triangle' ," · · · "Take, for instance, · · · Ceva's theorem, the nine-point circle, · · · ." This is the only mention of time in the entire work, and as the extract from the following paragraph shows, the author is by no means confining himself to the period he mentions. Furthermore the term "modern geometry" taken in connection with these sentences is quite sure to catch the student's attention, and, finding in juxtaposition in the text, Simson's line, Menelaus' theorem, and Ceva's theorem, he may naturally consider them developments of the same generation and ask his teacher if, "the gentlemen mentioned were by any chance classmates at college."

HELEN B. OWENS.

Advanced Algebra. By H. E. BUCHANAN AND L. C. EMMONS. Boston, Houghton Mifflin Co., 1925. viii+185 pages.

This charming little book is a veritable story of adventure. The elusive Unknown is pursued through thickets of polynomials and labyrinths of determinants until, stayed by the calming influence of logarithms, through arithmetical and geometrical progression it attains an annuity at the end.

The figure is sober and nearly accurate. The enchanting quality of the book is largely due to its arrangement; to the many things which are not found in it; and to the introduction of new material in those early chapters which are supposed to furnish a review of parts of the more elementary algebra. Among other things this new material includes the use of the derivative, the definition of the conics with their equations, and maxima and minima.

The chapter on polynomials contains what the student needs for the solution of numerical quantics; Descartes' rule and Newton's and Horner's methods. In chapter VI, mathematical induction is defined and illustrated before its use in proving the binomial theorem. Permutations and probability are treated with Spartan meagreness. All necessary theorems of determinants aré given. The explanation of logarithms and of the method of using the four place table are as nearly fool-proof as can be imagined. The final chapter contains the progressions and applications to annuities, which students always find interesting, and either the first or the final proof that a table of logarithms may really be of value. There is an appendix of nine pages devoted to scales of notation.

It is conceivable that no teacher ever willingly admits that the textbook written by another hand is quite perfect. If that is true, while admitting all the merits of this book, he would probably lament that the treatment of permutations and of probability is almost sketchy and that nine pages are too few for these two subjects. He might also regret that more problems are not given in the chapter on determinants. He would surely find the introductory chapters so interesting that he would not willingly omit any part of them, and then he would be puzzled to cover what his conscience demands, in fifty hours, with his small army of conscripts who would gladly evade the carnage of this conflict.

However it seems impertinent and intrusive to attempt a review of this most excellent text, since the preface by the authors and the introduction of the editor say all that needs to be said, and with the clarity and precision that should distinguish the mental processes of a mathematician.

W. W. LANDIS.

Esercizi di Geometria Analitica raccolti a cura di AMEDEO AGOSTINI *ed* ENEA BORTOLOTTI. Parte prima, 1925, x+230 pages; Parte seconda, 1926, 469 pages. Bologna, Nicola Zanichelli editore.

These two volumes are companions to the *Lezioni di Geometria Analitica*, from the pen of Ettore Bortolotti, and reviewed in this *Monthly*, vol. 31 (1924) pp. 98, 99. As the title indicates, the *Esercizi* contain problems and exercises in the elementary and the higher field of analytic geometry. Many of them are new, others are classic problems of interest. To each problem is given the answer, accompanied when thought desirable by hints as to its solution. Here and there are given also theoretical matters not found in the *Lezioni*. The first volume deals mainly with points and right lines in the plane and in space; the second volume with conics, with surfaces and curves in space, and with the elements of differential geometry. The volumes will be found very serviceable to teachers and students in the fields of which they treat.

FLORIAN CAJORI.

The Theory of Measurements. By LUCIUS TUTTLE AND JOHN SATTERLY. London, Longmans, Green, and Co. 1925. xii+333 pages.

In the preface it is made quite clear that this book is intended, primarily, for students of physical science. Its chief use is as a supplement to the usual laboratory manual. The idea which serves to connect the subject matter into a consistent whole is *physical measurements*. The mathematics in it is brought in incidentally as it is needed. If it were judged only by the mathematics in it, it would appear as a collection of numerous disconnected subjects. This would be an unfair method of judging it.

There are twenty-two chapters followed by a set of miscellaneous exercises for student practice, a list of answers to these miscellaneous exercises and the numerous other exercises scattered throughout the book, nine short tables some of which are not ordinarily available for convenient reference, a short bibliography and finally an index.

The mental equipment of the student for whom this book is designed is not above that of a high school graduate. The mathematics presupposed is described as follows:

"A good course in algebra as far as the solution of equations of the first degree; also a sufficient knowledge of plane geometry to include the properties of perpendiculars, equal triangles, isosceles and similar triangles, the theorem of Pythagoras, and the properties of similar figures."

Among the chapter titles are the following: Weights and Measures (9 pages), Angles and Circular Functions (10 pages), Significant Figures (13 pages), The Slide Rule (10 pages), Graphic Representation (26 pages), Graphic Analysis (22 pages), Accuracy (11 pages), Measurements and Errors, Statistical Methods (20 pages), Indirect Measurements (11 pages), Least Squares (21 pages). Each is taken up in much detail. To one attempting to read the book through, independently of any simultaneous laboratory work, it gives much the same impression as one would get if he attempted to read through a dictionary; but that is not an adverse criticism any more than it is an adverse criticism of a dictionary.

The present reviewer has not had any recent experience with students in a physical laboratory, so that he does not feel competent to pass judgement from that point of view. His impression, however, is that it will be found to be very useful in laboratory courses, and it may be found advisable for the science departments of our colleges to require their beginning students to possess copies of it for convenient reference, and possibly a regular course of class instruction might be given to such students with this as a textbook. Evidently that is the practice of the author.

A somewhat hasty reading revealed no fundamental errors. The mathematics which the author has applied to the problems in hand is sound. A student of this book who should afterwards take up systematic courses in higher mathematics would not be forced to discard any concepts acquired from this book, but would find his acquisition of the advanced abstract mathematics helped by his experience with these concrete applications.

I believe the author and publishers have served a good purpose by giving this book to the educational world.

GEO. GAILEY CHAMBERS.

A Comment on the Review of Chambers "Statistical Analysis"

In the review of Dr. G. G. Chambers book "An Introduction to Statistical Analysis" in the January number of this Monthly, the reviewer has overlooked the fact that it was written to meet a very particular situation which exists at the University of Pennsylvania, as is stated in the preface. All members of the freshman class in the School of Education are required to take a half-year course in elementary statistics. The entrance requirements to this school are such that the student need only present two years of High School mathematics. Such a condition makes it necessary to have a text in which only very elementary mathematical concepts are used. The aim of the course is to give the students a sufficient background of the methods and terms in statistics so that they may be able to read intelligently articles and books dealing with educational research, and also to be able to make the necessary computations in a later course in "Tests and Measurements." In addition to this course there are a number of advanced courses in statistics given by the Mathematics department.

This text was written to meet the need of such a group of students and was not intended to be a text in mathematical statistics such as would be used with students who had had some college courses in mathematics and had thus acquired a mathematical background. The amount of space devoted in the first chapter to approximations might seem to be rather large, but to one who has taught the course it is not too great, as most of the students come with the idea that a measurement is an exact quantity.

It is the aim of the book throughout to impress the students with the fact that to go far in statistics a great amount of mathematics is required and that they are only getting an introduction so as to make them able to comprehend the meaning of the terms and methods of computation which will be used in later work.

H. M. LUFKIN.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) the titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 49, no. 1, January 1927: "Stability and the equations of dynamics" by G. D. Birkhoff, 1–38; "Quaternary quadratic forms representing all integers" by L. E. Dickson, 39–56; "Reduction formulas for the number of representations of integers in certain quadratic forms" by E. T. Bell, 57–66; "Generalization of certain theorems of Bohl" by F. H. Murray, 67–86; "Applications of the determinant and permanent tensors to determinants of general class and allied tensor functions" by C. M. Cramlet, 87–96; "Transformations leaving invariant certain partial differential equations of physics" by R. D. Carmichael, 97–116; "Transformations leaving invariant the heat equation of physics" by J. B. Goff, 117–122; "The application of fractional operators to func-

tional equations" by H. T. Davis, 123–142; "Classification of quadrics in hyperbolic space" by J. Pierpont, 143–151.

Annali di Matematica, ser. 4, volume 4, November 1926–1927: "Transformations of relations between numerical functions" by E. T. Bell, 1–6.

Bulletin of the American Mathematical Society, volume 33, no. 1, January–February 1927: "On the metrization problem and related problems in the theory of abstract sets" by E. W. Chittenden, 13–34; "A theorem on factorization" by D. H. Lehmer, 35–38; "On a problem in closure" by Virgil Snyder, 39–43; "A theorem concerning direct products" by L. Wiesner, 44; "A cubic curve connected with two triangles" by H. Bateman, 45–50; "On the integration in finite terms of linear differential equations of the second order" by J. F. Ritt, 51–57; "On small deformations of curves" by C. E. Weatherburn, 58–62; "Integers represented by positive ternary quadratic forms" by L. E. Dickson, 63–70; "A diophantine automorphism" by E. T. Bell, 71–80; "The Cauchy-Heaviside expansion formula and the Boltzmann-Hopkinson principle of superposition" by F. D. Murnaghan, 81–89; "Singularities of the Hessian" by T. R. Hollcroft, 90–96; "Three theorems on closure of biorthogonal systems of functions" by R. E. Langer, 97–105; "A connected and connected in kleinem point set which contains no perfect subset" by B. Knaster and C. Kuratowski, 106–109.

Bulletin of the American Mathematical Society, volume 33, no. 2, March-April 1927: "A mathematical critique of some physical theories" by G. D. Birkhoff, 165–181; "An assemblage theoretic proof of the existence of transcendentally transcendental functions" by J. F. Ritt and E. Gourin, 182–184; "The most general closed point set over which continuous functions may be defined by certain properties" by G. T. Whyburn, 185–188; "Invariant relations" by F. H. Murray, 189–192; "Determination of the number of subgroups of an abelian group" by G. A. Miller, 193–194 "The asymptotic osculating quadrics of a curve on a surface," by E. P. Lane, 195–200; "A new characterization of plane continuous curves" by W. L. Ayres, 201–208; "A method for accelerating the convergence in the process of iteration" by C. C. Camp, 209–220; "A quadratic algebra and its application to a problem in diophantine analysis" by G. E. Wahlin, 221–231; "Functions expansible in series" by L. E. Ward, 232–234; "Invariants of a poristic system of triangles" by J. H. Weaver, 235–240.

The Monist, volume 37, no. 1, January 1927: "Mathematical Reality" by J. B. Shaw, 113–119; "Mathematics and natural science" by A. Dresden, 120–130; "Infinity and the infinitesimal" by W. Parkhurst and W. J. Kingsland Jr., 131–150.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 2, February 1927: "On the closure of certain assemblages of trigonometrical functions" by N. Wiener, 27–28; "On a type of Lorentz transformations" by G. Y. Rainich, 29–30; "Cyclicly connected continuous curves" by G. T. Whyburn, 31–37; "Displacements in a geometry of paths which carry paths into paths" by L. P. Eisenhart and M. L. Knebelman.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 3, March 1927: "Dynamical economics" by C. F. Roos, 145–150; "On the proof of Shepard's corrections" by E. B. Wilson, 151–155; "Note on a generalization of Taylor's series" by D. V. Widder, 156–159; "A theory of matter and electricity" by G. D. Birkhoff, 160–164; "The hydrogen atom and the Balmer formula" by G. D. Birkhoff, 165–169; "Groups generated by two operators of order three whose product is of order six" by G. A. Miller, 170–174.

Rendiconti del Circolo Mathematico di Palermo, volume 50, September–December 1926: "The general class number relations contained in Jacobi's theta formula" by E. T. Bell, 368–374.

Science, volume 65, no. 1676, February, 1927: "A mathematical critique of some physical theories" by G. D. Birkhoff, 147–149.

Transactions of the American Mathematical Society, volume 29, no. 1, January 1927: "Sets of independent postulates for the arithmetic mean, the geometric mean, the harmonic mean, and the root-mean-square" by E. V. Huntington, 1–22; "Irregular difference systems of order two and the re-

lated expansion problems" by M. H. Stone, 23–53; "Some problems in the theory of interpolation by Sturm-Liouville functions" by C. M. Jensen, 54–79; "Singular ruled surfaces in space of five dimensions" by E. B. Stouffer, 80–95; "On a type of completeness characterizing the general laws of separation of point-pairs" by C. H. Langford, 96–110; "Integers and basis of a number field" by N. R. Wilson, 111–126; "Implicit functions and their differentials in general analysis" by T. H. Hildebrandt and L. M. Graves, 127–153; "Applications of the theory of relative cyclic fields to both cases of Fermat's last theorem" by H. S. Vandiver, 154–162; "Riemann integration and Taylor's theorem in general analysis" by L. M. Graves, 163–177; "Alternatives to Zermelo's assumption" by A. Church, 178–208; "On a general theorem concerning the distribution of residues and non-residues of powers" by J. M. Vinogradov, 218–226; "On convergence factors in multiple series" by C. N. Moore, 227–238.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Mathematical and Physical Society of the University of Toronto, Toronto, Canada.

The Mathematical and Physical Society was founded forty-five years ago and is almost half as old as the University of Toronto which this year is celebrating its hundredth anniversary. The fact that the Society has been decidedly successful in its functions of furthering interest in mathematics and physics and providing its members with certain social advantages is vouched for by its steadily increasing membership and popularity with graduates and under-graduates. The average attendance this year has been about one hundred and fifty.

The regular meetings are held fortnightly on Thursday afternoons in the Physics Building. Following refreshments it is the custom to have some outstanding man address the members on a subject of general interest. This year the executive has been very fortunate in securing for this purpose several leading members of the Faculty as well as a number of graduates and undergraduates who gave interesting talks. The very keen interest taken in the social activities of the society by the members has had the inevitable effect of making every event from the "Hike" to the "Open Meeting" an unqualified success.

Executive for 1926-1927

Hon. Pres. Dean A. T. DeLury, President C. A. Peachey 4th Yr., Vice-Pres. Miss M. Annetts 3rd Yr., Corr. Sec. Miss A. Keast 4th yr., Treasurer C. L. Bates 2nd Yr., Record. Sec. Miss W. Woollcombe, 3rd Yr., Grad. Rep., Miss M. Westman, B. A., 4th Yr. Rep., W. L. McCutcheon, 3rd Yr. Rep., R A.. Blythe, 2nd Yr. Rep., W. D. Douglas, 1st Yr. Rep., Miss I. Winter, N. M. Burns.

October 9, 1926. Annual hike to welcome freshmen and freshettes.

October 15. Opening meeting. "Outstanding scientific societies" by Professor C. McLennan, Director of the physics laboratory at Toronto.

October 29. "Newtonian philosophy" by Alfred Baker, Emeritus professor of mathematics.

November 13. Annual At-Home of the M. and P. Society.

November 26. A meeting devoted to impromptu talks by members of the society on subjects of scientific interest.

December 9. Voting on changes in the constitution. "Alloys and their properties" by Mr. Taylor. "Astronomical legends" by Miss Stone.

January 21, 1927. "Non-euclidean geometry" by Professor A. T. DeLury, Honorary President and Dean of the Faculty of Arts.

February 5. Annual skating party.

February 25. Open meeting. The relativity drama which appeared in this Monthly for June 1926 was presented with a few local insertions. There were also several musical numbers and readings. Refreshments and dancing.

March 17. Bicentenary of the death of Sir Isaac Newton. Addresses:—"Newton's optical view" by Professor J. C. McLennan of the department of physics. "Mathematical work of Newton" by Professor J. Chapelon of the department of mathematics. "Contributions of Newton to astronomy" by Professor Chant of the department of astronomy. This last meeting was open to the public and several hundred people attended.

(Report by Annie M. Keast)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OKLAHOMA, Norman, Oklahoma.

During the year 1925–1926, the officers of the Mathematics Club of the University of Oklahoma were Hazel Jones, president; C. E. Springer, vice president; Sarabeth Barbour, secretary-treasurer. The meetings were held in the afternoon and tea was served before the formal program began.

The following topics were discussed:

October 29, 1925: "Newton's methods of graphing curves" by Dr. Elsie J. McFarland.

November 12. "Complex numbers" by Kathleen Butt.

December 10. "Tetracyclical coordinates" by John C. Buxey.

February 4, 1926. "Bertrand Russell" by Mr. Howard Eaton, Department of Philosophy.

March 25. "History of the calculus" by Travis Groves.

May 6. "The geometrical properties of the hyperbola" by Wilma Gorton "Indeterminate equations" by Hazel James.

(Report by L. D. Montgomery, secretary)

MU THETA EPSILON, UNIVERSITY OF CALIFORNIA, Berkeley, California.

The following is the program of Mu Theta Epsilon, women's mathematical honor society of the University of California, Berkeley, California, for the year 1925–1926.

September 10, 1925. Mathematical recreations. Discussion.

September 23. "The logical foundations of algebra" by Mamie Giacomini and Lois Matzen. Welcome to new members. "The early history of Mu Theta Epsilon" by Dr. Pauline Sperry.

October 14. "Fundamental concepts of calculus" from J. W. A. Young's monographs. Discussion led by Bertha Vranna.

November 17. "Fundamental concepts of algebra" from J. W A. Young's monographs. Discussion led by Jessie Ramelli.

January 13, 1926. "Space" by Harriet Bowker. "The application of mathematics to statistics" by Elizabeth Lange.

February 10. "The determination of an orbit" by Dr. Sophia Levy.

March 10. "Functions of a complex variable" by Louise Kemp. "Roman methods of computation" by Beryl Britton.

March 24. Initiation of new members and banquet.

April 14. Summary of reconstructed history of Mu Theta Epsilon by Beatrice Ude. "The method of least squares" by Elizabeth Burroughs.

(Report by Harriet H. Bowker)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are expecially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3264. Proposed by L. S. Shively, Mount Morris College.

A square is subdivided into n^2 equal squares by drawing lines parallel to its sides, and all the diagonals of the squares thus formed are drawn. How many triangles are there in the configuration?

3265. Proposed by Norman Anning, University of Michigan.

Prove, or improve, the scout's rule for reading bicycle spoor: an inflexional tangent to the path of the hind wheel is an ordinary tangent to the path of the front wheel.

3266. Proposed by W. A. Jenkins, Chicago, Ill.

Cards are exposed one at a time, without being replaced, from an ordinary deck of playing cards. The draw ceases as soon as four cards of any one number (or Jack, Queen, or King) are exposed. Find the probability that the ith card exposed concludes the draw.

3267. Proposed by H. A. DoBell, Colgate University.

In a given triangle the line joining the ninepoint center and the orthocenter of the pedal triangle is parallel to the Euler line of the tangential triangle.

3268. Proposed by Tobias Dantzig, University of Maryland.

A and B are two fixed points on a conic and M and N, two variable points on the same conic, such that the cross-ratio $(A, B; M, N)$ with regard to the conic remains constant. Find the envelope of the line MN.

3269. Proposed by J. Rosenbaum, Milford, Conn.

On the sides A_1B_1, B_1C_1, C_1A_1, of a triangle, $A_1B_1C_1$, the points A_2, B_2, C_2 are taken so that

$$A_1A_2:A_2B_1 = B_1B_2:B_2C_1 = C_1C_2:\ C_2A_1 = \lambda,$$

thus forming a second triangle $A_2B_2C_2$. In this manner, by repeating the operation on the successive triangles a sequence of triangles $T_1, T_2, \cdots$, is obtained. Find under what conditions will (a) T_n be similar to T_1; (b) the shape of T_n approach a limit as n becomes infinite. (c) Find the point to which the three vertices, A_n, B_n, C_n converge when λ is a positive, proper fraction.

3270. Proposed by Nathan Altshiller-Court, University of Oklahoma.

If a triangle is self-polar with respect to a circle, the center of the circle is the orthocenter of the triangle.

State this proposition in a projective form and give a direct proof of the proposition so stated.

3271. Proposed by H. L. Olson, Michigan State College.

Given a positive integer r; find all the positive integers of which r is a primitive root.

SOLUTIONS

3198[3186; 1926, 279]. Proposed by A. A. Bennett, Lehigh University.

Let the twelve letters, $a, b, c, \cdots k, l$, denote in some unknown order, the twelve distinct residue classes taken modulo 12. Assume that the complete multiplication of 144 products, giving each product

as a letter of the set, is known, but nothing further is given concerning the sum or difference of two letters. Show that each residue class is completely identifiable. Determine whether or not a like theorem holds for the case when the modulus is 16, 18, or 24.

SOLUTION BY H. D. GROSSMAN, Brooklyn, N. Y.

We are given a table of congruences which we shall call

TABLE I

$a^2 \equiv \cdot \quad \cdot \quad \cdot$	$ba \equiv \cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$	$la \equiv \cdot \quad \cdot \quad \cdot$
$ab \equiv \cdot \quad \cdot \quad \cdot$	$b^2 \equiv \cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$	$lb \equiv \cdot \quad \cdot \quad \cdot$
$ac \equiv \cdot \quad \cdot \quad \cdot$	$bc \equiv \cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$	$lc \equiv \cdot \quad \cdot \quad \cdot$
$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$
$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$
$al \equiv \cdot \quad \cdot \quad \cdot$	$bl \equiv \cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$	$l^2 \equiv \cdot \quad \cdot \quad \cdot$

where the second term of each congruence is one of the letters, $a, b, \cdots, l$, taken positively, and where (as hereafter in this solution) "modulo 12" is understood after each congruence.

Now consider another table,

TABLE 2

$0\cdot 0 \equiv 0$	$1\cdot 0 \equiv 0 \quad \cdot$	$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$	$11\cdot 0 \equiv 0$
$0\cdot 1 \equiv 0$	$1\cdot 1 \equiv 1 \quad \cdot$	$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$	$11\cdot 1 \equiv 11$
$0\cdot 2 \equiv 0$	$1\cdot 2 \equiv 2 \quad \cdot$	$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$	$11\cdot 2 \equiv 10$
$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$
$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$	$\cdot \quad \cdot \quad \cdot$
$0\cdot 11 \equiv 0$	$1\cdot 11 \equiv 11$	$\cdot \quad \cdot \quad \cdot$	$11\cdot 11 \equiv 1$

If we compare these tables, it is obvious that in Table 1, there is one and only one column in which the second terms are all alike and all equal to the common multiplicand, which we may assume to be a. Then $a=0$. Substitute this value for a wherever found in the table.

It is equally obvious that in Table 1, there is one and only one column in which each second term is equal to the multiplier of the first term. Let the common multiplicand in this column be b. Then $b=1$. Substitute this value for b wherever found in the table.

Consider Table 2 again or rather those parts of it appended:

$6\cdot 0 \equiv 0$	$4\cdot 0 \equiv 0$	$8\cdot 0 \equiv 0$	$3\cdot 0 \equiv 0$	$9\cdot 0 \equiv 0$	$2\cdot 0 \equiv 0$	$10\cdot 0 \equiv 0$
$6\cdot 2 \equiv 0$	$4\cdot 3 \equiv 0$	$8\cdot 3 \equiv 0$	$3\cdot 4 \equiv 0$	$9\cdot 4 \equiv 0$	$2\cdot 6 \equiv 0$	$10\cdot 6 \equiv 0$
$6\cdot 4 \equiv 0$	$4\cdot 6 \equiv 0$	$8\cdot 6 \equiv 0$	$3\cdot 8 \equiv 0$	$9\cdot 8 \equiv 0$	———	———
$6\cdot 6 \equiv 0$	$4\cdot 9 \equiv 0$	$8\cdot 9 \equiv 0$	———	———	$2\cdot 4 \equiv 0$	$10\cdot 4 \equiv 4$
$6\cdot 8 \equiv 0$	———	———	$3^2 \equiv 9$	$9^2 \equiv 9$		
$6\cdot 10 \equiv 0$	$4^2 \equiv 4$	$8^2 \equiv 4$				

We already have all our zeros in Table 1. There will be one and only one column in which there are 6 zeros among the second terms. Let the common multiplicand in this column be c. Then $c=6$.

There are two and only two columns each having 4 zeros among its second terms. Let their multiplicands be d and e. The squares of these letters are both congruent to one of them. Let that one be d. Then $d=4$, $e=8$. Substitute these values for d and e wherever found in the table.

Proceeding similarly we will identify two other letters, say f and g, as 3 and 9 respectively.

Proceeding similarly we will find a pair of letters, say h and i, which will be equal in unknown order to 2 and 10. Their squares will be indistinguishable but the product of one of them by 4 is congruent to 8, that of the other is congruent to 4. The first is 2, the second 10.

The three letters remaining may be easily identified by checking their functions against these congruences:

$$5\cdot 2 \equiv 10;\ 7\cdot 2 \equiv 2;\ 11\cdot 2 \equiv 10;\ 5\cdot 3 \equiv 3;\ 11\cdot 3 \equiv 9.$$

To prove the second part of the theorem, namely, that it is not true for any multiple of 8 or 9, it is necessary only to prove that even if all the other letters are identified, there will still remain a pair, say x and y, the symmetry of whose like functions will render them indistinguishable. That is, if

(1) when $f(x) \equiv x$, $f(y) \equiv y$, and
(2) when $f(x) \equiv y$, $f(y) \equiv x$, and
(3) when $f(x) \equiv z$, $f(y) \equiv z$, where $z \not\equiv x$ and $z \not\equiv y$,

it is clear that x and y are indistinguishable, therefore individually unidentifiable.

I shall endeavor to show that for any number of the form $8m$, $2m$ and $6m$ constitute such a pair of numbers, and that for any number of the form $9m$, $3m$ and $6m$ constitute such a pair.

Let $x=2m$, $y=6m$, and the modulus be $8m$.

$f(x)$ and $f(y)$ must be of the forms nx, ny (where $n\not\equiv x$ and $n\not\equiv y$), x^2, y^2, or xy. Now $n\equiv 0$, 1, 2 or 3 (mod 4).

If $n \equiv 0 \pmod 4$, then $2mn \equiv 0 \pmod{8m}$ and $6mn \equiv 0 \pmod{8m}$, satisfying (3).
If $n \equiv 1 \pmod 4$, then $2mn \equiv 2m$ (mod 8m), $6mn \equiv 6m \pmod{8m}$, satisfying (1).
If $n \equiv 2 \pmod 4$, then $2mn \equiv 4m \pmod{8m}$, $6mn \equiv 4m \pmod{8m}$, satisfying (3).
If $n \equiv 3 \pmod 4$, then $2mn \equiv 6$m $\pmod{8m}$, $6mn \equiv 2m \pmod{8m}$, satisfying (2).

Further, $(2m)(6m)\equiv(2m)^2 \pmod{8m}$; and since $2m\equiv 1$ or $3 \pmod 4$ and $(2m)^2\not\equiv 2m$ or $6m \pmod{8m}$, it follows that $(2m)(6m)\not\equiv 2m$ or $6m \pmod{8m}$.

Also $(2m)^2\equiv(6m)^2 \pmod{8m}$, satisfying (3).

A similar proof may be worked out for moduli of the form $9m$.

Also solved by MICHAEL GOLDBERG.

3199[3187; 1926, 338]. Proposed by Frank Morley, Johns Hopkins University.

Taking six points, 1, 2, 3, 4, 5, 6, on a conic, let the areas of the triangles 612, 123, $\cdots$ be denoted by $a_1, a_2, \cdots, a_6$. Prove that the area, A, of the hexagon, formed by the points as ordered, is given by

$$(A-a_1-a_3-a_5)a_2a_4a_6=(A-a_2-a_4-a_6)a_1a_3a_5.$$

SOLUTION BY OTTO DUNKEL, Washington University.

The area of the triangle formed by the points P_i, P_j, P_k is

$$\pm\,(i,j,k) = \pm\,\tfrac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}, \tag{1}$$

where the + or − sign is used according as P_i, P_j, P_k occur in positive or negative rotation. If (x_z, y_z) denotes a variable point, then $(z,j,k)=0$ is the equation of the straight line through P_j and P_k. The equation of a conic passing through P_1, P_3, P_4, P_6 may be written

$$(1\ z\ 6)(z\ 3\ 4) - k(1\ z\ 3)(z\ 4\ 6) = 0, \tag{2}$$

and, if we set $z=2$ or 5, k is uniquely determined since all six points lie on the same conic (considered to be non-degenerate). Eliminating k from the two resulting equations, we have

$$(126)(234)(456)(135) - (231)(345)(561)(246) = 0. \tag{3}$$

It is geometrically evident that

$$A = (135) + (231) + (345) + (561) = (246) + (126) + (234) + (456). \tag{4}$$

If we eliminate (135) and (246) from (3) and (4), we have the equality in the problem.

Also solved by T. W. EDMONSON and HARRY LANGMAN.

3201[3189; 1926, 338]. Proposed by C. G. Latimer, Tulane University.

Find integers, h, k_i, c_i, n, where $|k_i|<h$, such that

$$\log(x+h)=\sum_{i=1}^{n}c_i\cdot\log(x+k_i)+\sum_{i=1}^{\infty}a_iz^i$$

where the a_i are constants and $z=b/f(x)$ where b is a constant and $f(x)$ is a polynomial of degree$\geqq 6$.

Borda's and Haro's series are similar to this except that they involve polynomials of the third and fourth degrees respectively.[1]

REMARKS BY E. B. ESCOTT, Oak Park, Illinois.

This question is answered by me in a paper on *Logarithmic Series* in the *Quarterly Journal of Mathematics* (1910), pages 141–156.

In this paper (page 152) is the solution

$$\begin{aligned}\log(x+51) &= \log(x+50)+\log(x+38)-\log(x+33)-\log(x+24)+\log(x+13)\\ &\quad+\log(x+7)-\log(x-7)-\log(x-13)+\log(x-24)+\log(x-33)\\ &\quad-\log(x-38)-\log(x-50)+\log(x-51)+2[6983776800/(x^7-4214x^5+4716649x^3\\ &\quad-1218798036x)+\cdots].\end{aligned}$$

[NOTE: In the paper there is a misprint in the known term of equation (52). There should be two zeros annexed to the known term as given.]

Other equations of the 7th degree similar to (52) are those having the following roots: 6, 19, -30, -40, 49, 51, -55; 8, 47, -66, -75, 87, 133, -134; 13, 23, -38, -105, 110, 120, -123; and others may easily be obtained.

In the above paper are given several series where $f(x)$ is of degree 6. Many of these series may be transformed into series involving antitangents instead of logarithms. On this subject see L. E. Dickson's *History of the Theory of Numbers*, vol. 2, page 716.

These series may also be used for interpolation. The connection will be seen by taking the formula $\Delta^3u_0=u_3-3u_2+3u_1-u_0$ and applying it to the function $u_n=\log(x+n+1)$.

3202[3190; 1926, 338]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Two given lines x, y are met in the points P, Q by a circle passing through their common point and through another fixed point in the plane. Find the locus of the point of intersection of the lines projecting P, Q upon y, x, respectively.

SOLUTION BY OTTO DUNKEL, Washington University.

Let A be the fixed point; then, since $\angle PAQ$ is equal to the angle between x and y (assumed to be different from 90°), the range of points P on x is projective with the range of points Q on y, and to the point at infinity on x corresponds the point at infinity on y. Hence the two sets of projecting lines, one set through the points P and the other through Q, are projective, and they have a self-corresponding ray in the line at infinity. Hence the locus of the points of intersection of corresponding rays is a straight line.

Also solved by THEODORE BENNETT, MICHAEL GOLDBERG, and the PROPOSER.

3205[3193; 1926, 338]. Proposed by W. H. Rasche, Virginia Polytechnic Institute.

Prove that two coplanar, copolar triangles are homological; and, conversely, two coplanar, homological triangles are copolar.

NOTE.

(1) In the proof of the foregoing theorem, use only Euclidean geometry and the positive and negative signs to distinguish the internal division of a linear segment from its external division.

NOTE BY THE EDITORS

The theorem above is commonly called Desargues' theorem, and proofs of the theorem may be found in most texts on projective geometry. The usual proof is given by constructing a triangle outside of the given plane which is in perspective with the two coplanar triangles by two centers on a line concurrent with the three lines through corresponding pairs of vertices (see Cremona's Proj. Geom. p. 7; Mathews' Proj. Geom. p. 26).

---

[1] See Goursat-Hedrick *Mathematical Analyses*, vol. 1, p. 133.

A proof of the kind desired in the problem is as follows: Let AA', BB', CC' meet in O, and let BC and $B'C'$ meet in X and cut OAA' in D and D', respectively. Then $(D, B, C, X) \barwedge (D', B', C', X)$, and hence $A(D, B, C, X) \barwedge A'(D', B', C', X)$. But these two pencils have a self-corresponding ray DD', and, therefore, the points $(AB, A'B')$, $(AC, A'C')$, $(AX, A'X)$ lie in a straight line. But the third point is precisely $(BC, B'C')$. The converse may be proved in a similar manner. It is very likely that this proof may be found in texts.

Also solved by the PROPOSER.

3208[1926; 385]. Proposed by V. M. Spunar, Chicago, Illinois.

Disprove the following: If a, n are any integers and $a^x \equiv 1 (\text{mod } n)$ for $x = n-1$ but not when x is an aliquot part of $n-1$, the integer n is a prime. (Lucas, *Theory of Numbers*, I, p. 441.)

SOLUTION BY D. H. LEHMER, University of California.

My object in what follows is not to "solve the problem" but to defend Lucas' theorem by giving a somewhat more evident proof of it.

Suppose that n were composite. Then, if we represent the totient of n by $\phi(n)$ we would have $\phi(n) < n-1$. Now $a^x \equiv 1$ for $x = n-1$ by hypothesis and for $x = \phi(n)$ by Euler's generalization of Fermat's theorem, (since a is evidently prime to n). Let δ be the greatest common divisor of $n-1$ and $\phi(n)$; that is, let δ be such that $n-1 = k\delta$ and $\phi(n) = l\delta$, with k and l integers prime to each other (or with $l=1$). It is always possible then, to find integers κ and λ such that

$$\kappa k - \lambda l = 1 \text{ and } \delta\kappa k - \delta\lambda l = \delta \text{ or } \kappa(n-1) - \lambda\phi(n) = \delta.$$

Then since $a^{n-1} \equiv a^{\phi(n)} \equiv 1, (\text{mod } n)$ we have $a^{\kappa(n-1)-\lambda\phi(n)} \equiv a^{\delta} \equiv 1 \ (\text{mod } n)$.

But, since δ is clearly an aliquot part of $n-1$, this last relation is contrary to the hypothesis. Hence n is not composite.

Also solved by J. L. REILLY.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor E. T. BELL of the California Institute of Technology has been elected a member of the National Academy of Sciences.

The University of Lemberg has conferred an honorary doctorate on Professor R. A. MULLIKAN of the California Institute of Technology.

Under the title *Mole Philosophy and Other Essays*, E. P. Dutton and Co. have recently published a volume of short essays from the pen of Professor CASSIUS J. KEYSER of Columbia University, mathematician and philosopher. Those acquainted with the author and his earlier publications need hardly be told that the essays exhibit a rare combination of mature wisdom and mellow humor. It is the sort of book one picks up when he has an odd ten minutes of spare time, and puts down reluctantly when the ten minutes have become an hour.

The list of doctorates conferred by American Universities during 1926, as published in the May issue of this Monthly, should have included the

following: GAYLORD M. MERRIMAN, Cincinnati, June, *Sufficient conditions for the summability of double series, with applications to the double Fourier series.*

Professor A. A. Bennett of Lehigh University has been appointed professor of mathematics at Brown University.

Professor A. B. COBLE of the University of Illinois has been appointed professor of mathematics at Johns Hopkins University.

Assistant Professor M. H. INGRAHAM of Brown University has been appointed professor of mathematics at the University of Wisconsin.

Professor C. J. KEYSER of Columbia University has retired, as Adrain professor emeritus.

Dr. LEE H. MCFARLAN, National Research fellow in mathematics stationed at the University of Chicago, has been appointed to an assistant professorship at the University of Washington, Seattle.

Professor F. R. MOULTON of the department of mathematical astronomy of the University of Chicago has recently resigned to go into business.

Dr. C. A. NELSON of Johns Hopkins University has been appointed to an associate professorship of mathematics at Rutgers University, New Brunswick, N. J. At the same institution, Mr. HOWARD PIXLEY, graduate student at the University of Chicago, has been appointed instructor in mathematics.

Assistant Professor J. F. RITT has been promoted to an associate professorship of mathematics at Columbia University.

Assistant Professor J. D. TARMARKIN of Dartmouth College has been appointed assistant professor of Mathematics at Brown University.

Professor R. F. BORDEN of George Washington University died on March 15, 1927.

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.
BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.
Twelfth Annual Meeting, Nashville, Tenn., December, 1927.
The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.
INDIANA, De Pauw University, April 29-30.
IOWA, University of Iowa, May 6-7.
KANSAS, Topeka, Kan., February 5.
KENTUCKY, Lexington, May 7.
LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.
MICHIGAN, April.
MINNESOTA, St. Peter, Minn., May 21.
MISSOURI, St. Louis, Mo., November 25-26.
NEBRASKA, Lincoln, May 14.
OHIO, Columbus, Ohio, April 8.
PHILADELPHIA, Philadelphia, Pa., November 26.
ROCKY MOUNTAIN, Colorado College, April 22-23.
SOUTHEASTERN, Columbia, S. C., April 15-16.
SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.
TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formula. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully γ η κ λ ν τ υ χ ω.
9. Among Greek capitals, use only Γ Δ Θ Λ Ξ Π Σ Φ Ψ Ω.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: 1̸. Similarly, 0 means "zero"; to indicate "cap O", backspace and overprint period; thus: 0̣ .
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma \cdots$ (all) A B $\Gamma \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma \cdots}$ (all except ι and o).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma \cdots}$ (all except ι and o);

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \varpi\ \varrho\ \vartheta\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q\ A\ B\ C\ D\ E\ F\ G\ H\ I\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$

* Boldface German—$\boldsymbol{\mathfrak{d\ A\ B\ D}}$

Script (special font) $\mathcal{A\ B\ C} \cdots$(all). No lower case manufactured.

* Hebrew—א אּ כּ ב troublesome to handle.

★ Dashed Italics—$\bar{A}\ \bar{a}\ \bar{B}\ \bar{b}\ \bar{C}\ \bar{c}\ \bar{E}\ \bar{e}\ \bar{F}\ \bar{f}\ \bar{G}\ \bar{g}\ \bar{H}\ \bar{I}\ \bar{\imath}\ \bar{K}\ \bar{k}\ \bar{M}\ \bar{m}\ \bar{N}\ \bar{n}\ \bar{P}\ \bar{p}\ \bar{q}\ \bar{R}\ \bar{r}\ \bar{s}\ \bar{t}\ \bar{u}\ \bar{v}\ \bar{X}\ \bar{x}\ \bar{Y}\ \bar{y}\ \bar{Z}\ \bar{z}$

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \bar{\Gamma}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

---

* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY
WILLIAM HENRY BUSSEY, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT
AUBREY JOHN KEMPNER

WITH THE CO-OPERATION OF

H. E. BUCHANAN
ELIZABETH CARLSON
W. B. CARVER
OTTO DUNKEL
H. J. ETTLINGER
H. S. EVERETT
B. F. FINKEL
H. W. KUHN
C. N. MILLS
F. D. MURNAGHAN
H. L. OLSON
D. E. SMITH

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 7, AUGUST-SEPTEMBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following thirty-six persons and six institutions have been elected to membership in the Association, their applications having been duly certified.

To Individual Membership

J. A. ANKEBRANT, A.M. (Michigan). Head of Dept., Western State Coll. of Colorado, Gunnison, Colo.

ALICE M. AUSTIN, A.M. (North Dakota). Head of Dept., Independence Jr. Coll., Independence, Kans.

D. L. BARRICK, A.B. (Okla. Baptist Univ.). Prof., Montezuma Coll., Montezuma, N. M.

WALTER BARTKY, Ph.D. (Chicago). Instr., Univ. of Chicago, Chicago, Ill.

A. W. BEAR, B.S. (Marquette). Instr., Marquette Univ., Milwaukee, Wis.

LEORA BLAIR. Asst. Prof., La. State Normal Coll., Natchitoches, La.

LUCILLE M. CHALMERS, A.M. (Michigan). Instr., Coll. of the City of Detroit, Detroit, Mich.

N. B. CONKWRIGHT, Ph.D. (Illinois). Instr., Univ. of Iowa, Iowa City, Iowa.

S. E. COOK, A.M. (Bates). Head of Dept., Springfield Tech. H. S. and Instr., Springfield Jr. Coll., Springfield, Mass.

TOBIAS DANTZIG, Ph.D. (Indiana). Asst. Prof., Math. and Mechanics, Univ. of Maryland, Baltimore, Md.

D. R. DAVIS, Ph.D. (Chicago). Asst. Prof., Univ. of Oregon, Eugene, Ore.

L. H. DENMAN, B.S. (Denison). Head of Dept., Shurtleff Coll., Alton, Ill.

H. G. FUNKHOUSER, A.M. (Columbia). Asst. Prof., Washington and Lee Univ., Lexington, Va.

H. L. GARABEDIAN, A.M. (Harvard). Instr., Univ. of Rochester, Rochester, N. Y.

AMELIA GAULDEN, Slidell, La.

J. T. HARWELL, A.B. Head of Dept., C. E. Byrd H. S., Shreveport, La.

R. W. JONES, B.S. (Delaware). Instr., Univ. of Delaware, Newark, Del.

H. K. JUSTICE, C.E. (Cincinnati). Asst. Prof., Univ. of Cincinnati, Cincinnati, O.

L. M. KELLS, Ph.D. (Columbia). Asst. Prof., U. S. Naval Academy, Annapolis, Md.

SYDNEY MARGERY KILPATRICK, A.B. (La. State Univ.). Teacher, Bolton H. S., Alexandria, La.

D. H. KYES, Ph.D. (Boston Univ.). Head of Dept., Math. and Physics, Dakota Wesleyan Univ., Mitchell, So. Dak.

G. A. LINHART, Ph.D. (Yale). Teacher, Riverside Jr. Coll., Riverside, Calif.

BIRDIE E. LONGMIRE, A.B. (Univ. of Nashville). Teacher, Bolton H. S., Alexandria, La.

H. R. MATHIAS, A.M. (Indiana). Prof., Indiana Central Coll., Indianapolis, Ind.

W. P. MORSE, A.B. (Maine). Instr., Univ. of Maine, Orono, Me.

K. E. MOYLE, A.B. (Bucknell). Student, Laceyville, Pa.

G. A. PARKINSON, A.M. (Ohio State). Instr., Univ. of Wis., Madison, Wis.

W. R. PAUL. Teacher, Rising City, Nebr.

J. W. PETERS, A.B. (Johns Hopkins). Grad. Student, Johns Hopkins Univ., Baltimore, Md.

C. M. SAVARY, C.E. Chief of Geodesy and Cartography, Dept. of Lands and Forests, Quebec, P. Q.

IDA K. SMITH. Teacher, Meridian H. S., Meridian, Miss.

P. K. SMITH, M.S. (Chicago). Head of Dept., Miss. Woman's Coll., Hattiesburg, Miss.

J. E. THOMPSON, A.M. (Columbia). Instr., Pratt Institute, Brooklyn, N. Y.

MORGAN WARD, A.B. (California). Teaching Fellow, Calif. Inst. of Tech., Pasadena, Calif.

C. R. WILSON, A.B. (W. Tex. State Teachers Coll.). Grad. Stud., Univ. of Iowa, Iowa City, Ia.

F. G. YOUNG, A.M. (Alberta). Instr., Provincial Inst. of Tech. and Art, Calgary, Alta.

To Institutional Membership

UNIVERSITY OF MINNESOTA, Minneapolis, Minn.
CARLETON COLLEGE, Northfield, Minn.
WELLS COLLEGE, Aurora, N. Y.
HUNTER COLLEGE OF THE CITY OF NEW YORK, New York City, N. Y.
UNIVERSITY OF SOUTH CAROLINA, Columbia, S. C.
THE RICE INSTITUTE, Houston, Tex.

C. H. YEATON, *Assistant Secretary*

THE FOURTH MEETING OF THE INDIANA SECTION

The fourth annual meeting of the Indiana Section of the Mathematical Association of America was held April 29–30, 1927 at DePauw University, Greencastle, Ind.

There were thirty present, including the following seventeen members of the Association:

W. C. Arnold, Gladys L. Banes, G. E. Carscallen, H. T. Davis, S. C. Davisson, J. E. Dotterer, W. E. Edington, P. D. Edwards, H. E. H. Greenleaf, F. H. Hodge, E. N. Johnson, Juna M. Lutz, T. E. Mason, C. K. Robbins, K. P. Williams, H. E. Wolfe, W. A. Zehring.

On Friday evening there was a banquet at which were present members of the Association and representatives from many of the science departments of the university. Professor W. M. Blanchard of the department of chemistry acted as toastmaster. At eight o'clock a public lecture, under the auspices of the Naperian Club and the Physics Seminar of DePauw University, was given by Professor Jakob Kunz, Professor of physics, University of Illinois, on the subject, "A popular review of the trend of modern physical science." The introduction was made by Professor O. H. Smith of the physics department of DePauw University. Professor Kunz first presented a picture of physics as it existed near the beginning of the present century as a result of the fundamental work of Faraday, Maxwell, Helmholtz, Kelvin and others. A set of principles had been formulated which seemed to establish an almost perfect foundation for an exact science. There were certain unsatisfactory features in this formulation, however, as, for example, the concept of action at a distance, the idea of an ether which was metaphysical rather than physical, and the second law of thermodynamics which makes no provision in nature for the re-creation of energy. During the last thirty years, all of our ideas have been undergoing a change. The classical foundations of physics have been attacked by fundamental experiments. The theory of relativity has replaced Newton's absolute space and time by a relative space and time. The Bohr theory of the atom has replaced the concept of a continuous flow of energy by a discontinuous flow of

7. In a series of m sets of n items or measurements, it is well known that $\sigma_A = \sigma/\sqrt{n}$ and $\sigma_s = \sqrt{(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2)}$, where σ_A is the standard deviation of the average, σ the standard deviation of the mn items, σ_s the standard deviation of the sum, and σ_i $(i=1, 2, \cdots, m)$, the standard deviation of the m sets. These formulas depend for their validity on the assumption that n is sufficiently large so that the sums of the product terms approach zero. In this paper Professor Edington determined the relation $\sigma_s^2 = m(1-n^{-1})\sigma^2 = m(n-1)\sigma_A^2$, which affords a check on the validity of the use of the formulas for σ_s and σ_A when n is not large.

At the conclusion of the meeting, a vote of appreciation was extended to the department of mathematics, the Naperian Club, and the Physics Seminar of DePauw University. The time and place of the next meeting was left in the hands of the executive committee.

H. T. DAVIS, *Secretary*

THE MAY MEETING OF MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-first meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association was held on Saturday, May 7, 1927, at the University of Maryland, College Park, Maryland, the morning session opening at 11 A.M. and the afternoon session at 2:30 P.M. Those attending the meeting were guests of the University of Maryland at luncheon between the sessions, and also were guests at various athletic events before and after the sessions. Chairman J. A. Bullard presided at both sessions.

There were 53 present including the following 38 members of the Association: O. S. Adams, R. N. Ashmun, H. G. Avers, L. M. Blumenthal, C. C. Bramble, J. A. Bullard, P. Capron, A. Cohen, G. R. Clements, J. A. Duerksen, P. J. Federico, G. L. Fentress, M. Goldberg, H. Gwinner, W. M. Hamilton, G. W. Hansen, P. E. Hemke, H. P. Kaufman, G. H. Keulegan, W. D. Lambert, A. E. Landry, F. Morley, F. D. Murnaghan, J. R. Musselman, J. W. Peters, E. C. Phillips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, R. E. Root, J. H. Schad, J. T. Spann, T. H. Taliaferro, A. A. Tomeldon, J. Tyler, P. Wernicke, E. W. Woolard.

The following program was presented:

1. "The problem of three vortices" by Prof. F. MORLEY, Johns Hopkins University.

2. "Euclidean invariants of the plane cubic" by P. J. FEDERICO and P. R. NEFF, U. S. Patent Office.

3. "Lagrange resolvents in Euclidean geometry" by L. M. BLUMENTHAL, Johns Hopkins University.

4. "The Lagrange resolvents of a triangle" by J. W. PETERS, Johns Hopkins University.

5. "The drag of wings with end plates" by Dr. P. E. HEMKE, Postgraduate School, U. S. Naval Academy.

6. "The contribution of Newton to pure mathematics" by E. W. WOOLARD, U. S. Weather Bureau.

7. "The contribution of Laplace to pure mathematics" by Prof. A. W. RICHESON, University of Maryland.

8. "On generalized free perspective" by Dr. P. WERNICKE, U. S. Patent Office.

9. "A note on the orthopole" by Prof. F. D. MURNAGHAN, Johns Hopkins University.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. A straight vortex filament imposes on a particle of fluid p a velocity inversely as the distance from p to the line and at right angles to both the distance and the line. Assuming that for parallel straight vortices the action of each on the other is of the same kind, the problem of the motion of three parallel and positively equal vortices is shown to be a simple problem of elliptic functions.

2. The method of applying the Lie theory of continuous groups to find the Euclidean invariants of second degree curves used by McDuffee in the May, 1926 issue of the MONTHLY was used to find the invariants of plane cubics. A complete system of seven invariants and two covariants of the cubic was found and the geometric interpretation of some of the invariants was explained.

3. A function of the vertices of two plane n-gons $x_1, x_3, x_5, \cdots, x_{2n-1}$ and $x_2, x_4, x_6, \cdots, x_{2n}$ which is invariant under translations $y=x+\alpha$ and rotations $y=tx$ is found to be $\sum_{k=1}^{n} \bar{x}_{2k}(x_{2k-1}-x_{2k+1})$. This invariant expresses the area A and norm N of the $2n$-gon $x_1, x_2, x_3, \cdots x_{2n}$. This function is then expressed in terms of the Lagrange resolvents

$$v_i = \sum_{k=0}^{n-1} \epsilon^{ik} x_{2k+1} \quad \text{and} \quad u_i = \sum_{k=0}^{n-1} \epsilon^{ik} x_{2(k+1)}\,,$$

(where ϵ is a primitive nth root of unity) of the vertices of the two n-gons; the result obtained is

$$\sum_{k=1}^{n-1} (1-\epsilon^{n-1}) v_k \bar{u}_k = 2n\,(N+iA)\,.$$

4. The equation $\alpha_0 x^3+3\alpha_1 x^2+3\alpha_2 x+\alpha_3=0$, when all the quantities are complex, presents a triangle when represented by points in the plane. Under homologies the equation has one invariant in terms of which the ratio of the two Lagrange resolvents is easily expressed. The geometric representation of

the Lagrange resolvents is closely connected with the triangle. Hence to express properties of the triangle invariant under homologies in terms of the invariant, first connect these properties with the Lagrange resolvents.

5. The induced drag of an airfoil is that part of the total resistance or drag of an airfoil depending upon the ratio of the span to the chord. Experiments have shown that the induced drag of airfoils may be considerably reduced by fastening end plates or shields to the airfoil so that the planes of these shields are perpendicular to the span. In this paper the effect of adding such end plates is calculated by using a conformal transformation to transform the flow around the airfoil with end plates to the flow around a straight line. The latter is known and the induced drag is then calculated by finding the kinetic energy of the flow set up in a fluid, otherwise at rest, by the motion of a straight line normal to itself.

The calculations are compared with experiments made at Göttingen University and the Langley Memorial Aeronautical Laboratory using single airfoils or monoplane cellules having end plates. The agreement warrants extending the calculations to biplane cellules with end plates and determining the reduction of induced drag for such arrangements.

6. The contribution of Newton to pure mathematics is chiefly the infinitesimal calculus, of which he was one of the founders. To him is due the analytical method with its advantages far exceeding any previous method—however the methods of the ancient Greeks were considerably superior in respect to soundness of theory. Neither Newton nor his contemporaries ever succeeded in placing the calculus on a satisfactory foundation; his expositions were vague and their validity often questioned. Newton originally based the calculus on infinitesimals, using these to obtain his so-called fluxions; he later altered the foundations several times—but never used the Method of Limits.

7. Laplace's contributions to pure mathematics are found in his works on determinants, differential equations, the theory of generating functions, finite difference equations and the evaluation of definite integrals by a method of approximation. In the theory of determinants Laplace gives an expansion formula which bears his name. He developed the idea of the generating functions in his Théorie Analytique des Probabilités, and it is here that he sets forth the greater part of his work in finite difference equations. Laplace's ability to handle analysis was phenomenal; however, he considered mathematics only as a tool with which to arrive at certain physical results.

8. The figure of two triangles (3-points) in perspective by linear projectors (s_1) meeting in a point (s_0), in space (s_3) is generalized. The "axis" of perspective is here called *base.* In *free* perspective of two n-points by means of p-dimensional projectors (s_p), each going through two "corresponding" vertices and all having a common "centric" s_c, the figure must be placed in an $S_{n(p-c)+c}$. It is

found that when $c=p-1$ there is a *basic* s_{n-p-1} containing the intersections (s_{n-p-2}) of pairs of corresponding sides (s_{n-2}). The converse theorem is also true.

9. The theorem of elementary plane geometry that the three altitudes of a triangle concur in a point may be generalized. Let D, E, F be the feet of the perpendiculars from the three vertices A, B, C of a triangle upon any line p in its plane. Then the three lines which pass through D, E, F and are perpendicular to BC, CA, and AB, respectively, concur in a point P which is known as the orthopole of the line p as to the triangle ABC. Thus the orthocenter of a triangle is the orthopole of each of the three sides of the triangle as to the triangle. Each altitude of a triangle is the Simpson line of that vertex of the triangle through which it passes. The following theorem is readily proved: If we have six points in a circle such that the Simpson lines of three of them with respect to the triangle formed by the remaining triad concur in a point, then the Simpson lines of the second triad with respect to the triangle formed by the first triad concur in the same point, and this common point of the six Simpson lines bisects the join of the two orthocenters of the triangles formed by the two triads.

Officers elected for the coming year were: *Chairman*, J. R. MUSSELMAN; *Secretary-Treasurer*, E. C. PHILLIPS; *Members of the Executive Committee*, T. H. TALIAFERRO and C. C. BRAMBLE. The next meeting will be held Dec. 3, 1927 at Georgetown University, Washington, D. C.

J. R. MUSSELMAN, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The regular spring meeting of the Minnesota Section of the Mathematical Association was held at Gustavus Adolphus College, St. Peter, Minnesota, on Saturday, May 21, 1927. Professor Inez Rundstrom of Gustavus Adolphus College, the Chairman of the Section, presided.

The attendance was 35 and included the following 18 members of the Association: W. O. Beal, A. Bogard, R. W. Brink, W. H. Bussey, Elizabeth Carlson, Sister M. Claudette, J. M. Earl, Sister Alice Irene Frieberg, Gladys Gibbens, C. H. Gingrich, W. L. Hart, D. Jackson, W. H. Kirchner, L. W. Moench, M. A. Nordgaard, Ella Thorp, Marion B. White, G. Winkelmann.

At the luncheon, after greetings had been extended by the Chairman and by President Johnson of Gustavus Adolphus College, Professor W. H. Bussey, University of Minnesota, Editor-in-Chief of the MONTHLY, spoke informally on the early history of THE AMERICAN MATHEMATICAL MONTHLY.

At the afternoon session the following officers were elected for the coming year: Chairman Sister M. CLAUDETTE, College of St. Benedict, St. Joseph, Minnesota; Secretary, A. L. UNDERHILL, University of Minnesota; an Execu-

tive Committee consisting of the Chairman, the Secretary and INEZ RUNDSTROM, Gustavus Adolphus College, F. J. TAYLOR, Hamline University, and A. BOGARD, College of St. Theresa.

A motion was passed expressing the appreciation of the Section for the hospitality of Gustavus Adolphus College.

The 1928 meeting will be held at the College of St. Benedict, St. Joseph, Minnesota.

The following six papers were read:

1. "Objective tests in descriptive astronomy," by Professor W. O. BEAL, University of Minnesota.

2. "Generalized evolutes in space," by Professor A. BOGARD, College of St. Teresa.

3. "Figures, and figures," by Professor W. H. KIRCHNER, University of Minnesota.

4. "The Pons-Winnecke comet," by Professor C. H. GINGRICH, Carleton College.

5. "On a problem arising out of the theory of a certain game," by Professor J. USPENSKY, Carleton College.

6. "On the equation tang $x=x$," by Professor USPENSKY.

Abstracts of these papers follow, the numbers corresponding to those in the list of titles.

1. Professor Beal reported on the comparative results obtained from the subjective and objective types of tests given to the same students as they progressed through a one quarter course in descriptive astronomy. This study covers a period of two years. A rather high correlation between the two methods was obtained. Samples of both types of questions and some of the statistics were thrown on a screen for the observation of the members of the section. He urged the teachers present to carry out similar experiments with objective tests in their classes in mathematics.

2. In Professor Bogard's paper two sets of rectangular coordinates are employed, one fixed in space and the other free to move along a curve C. The second set is the configuration consisting of the tangent, the principal normal, and the binormal, to C at any point. With respect to both sets of axes the parametric equations of a second curve Γ are derived. When the condition that the tangent to the curve Γ shall intersect the tangent to the curve C at right angles and at a fixed distance a from the point of tangency is introduced the equations of the generalized evolutes are obtained. Some properties of these curves are pointed out. The special case $a=0$ gives the equations of the evolutes of a space curve. Further specialization leads to the evolutes of plane curves.

3. In Professor Kirchner's paper, attention was called to the importance of figures in courses and text-books in mathematics. It was suggested that more

attention than is usually given to the subject in mathematics courses be paid to the methods of drawing figures. Some of the principles of isometric drawing were explained.

4. Professor Gingrich stated some facts pertaining to the approach of the Pons-Winnecke comet. He said they were drawn from a paper by Professor George Van Biesbroeck published in the May issue of "Popular Astronomy."

The comet belongs to the Jupiter family of comets and has a period of approximately six years. It will be nearest the earth on June 27 at which time its distance from the earth will be approximately three and a half million miles. It will be moving rapidly southward from June 27 to July 1. During this period it will be near opposition and consequently will be above the horizon nearly all night. At its brightest it may be visible to the naked eye, but will probably not be particularly conspicuous.

5. The theory of a certain game suggests the following problem: Is it possible to find n positive numbers $a, b, c, \cdots l$ in such a way that the n series[1]

$$\begin{array}{l} [a], [2a], [3a], \cdots \\ [b], [2b], [3b], \cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ [l], [2l], [3l], \cdots \end{array}$$

reproduce, without repetition, all the natural numbers $1, 2, 3, \cdots$? The author shows that, excluding the trivial case $n=1$, $a=1$, the problem is possible only when $n=2$. The necessary and sufficient condition that the two series (ma), (mb), for $m=1, 2, 3 \cdots$, reproduce without repetition all the numbers $1, 2, 3 \cdots$, is that a and b are both irrational numbers satisfying the condition $1/a+1/b=1$.

6. The positive roots of the equation tang $x=x$ are, as is well known, $x=(p+\frac{1}{2})\pi-\theta$, while the correction θ can be expressed by this power series in $\xi=1/(p+\frac{1}{2})\pi$ (p being $0, 1, 2, 3, \cdots$):

$$\theta=c_0\xi+c_1\xi^3+c_2\xi^5+c_3\xi^7+\cdots \text{ where } c_0=1,\ c_1=2/3,\ c_2=13/15\cdots,$$

are positive rational numbers. These series have been given by Euler but no thorough discussion of their convergence can be found in the existing literature. The author gives the following asymptotic expression for c_n

$$c_n=\Gamma(1/3)\cdot\pi^{-2/3}\cdot 12^{-1/6}\cdot(\pi/2)^{2n+1}\cdot(2n+1)^{-4/3}\cdot(1+\omega_n),\quad \omega_n\to 0,$$

which shows that the series is convergent even for the least root corresponding to $p=0$.

R. W. BRINK, *Acting Secretary*

---

[1] The symbol [K] denotes the largest integer which is less than or equal to K.

ON THE UPPER LIMIT TO THE REAL ROOTS OF AN ALGEBRAIC EQUATION[1]

By GLENN JAMES, University of California at Los Angeles

1. We consider the real equation, $f(x)=x^n+a_1x^{n-1}+\cdots+a_n=0$, in which not all the coefficients are positive.

A well known[2] upper limit of the roots of this equation is $1+G^{1/K}$ where K is the number of positive or zero coefficients preceding the first negative term and G is the greatest of the numerical values of the negative coefficients.

The essential steps in the derivation of this formula are: first, change $f(x)$ into a more wieldy polynomial, *equal to or less than $f(x)$ for x positive*, by dropping all the positive terms between x^n and the first negative term and changing the coefficients of the latter and all subsequent terms into $-G$; second, simplify this new polynomial under the assumption that x is greater than unity, and drop a troublesome positive term, G (see Equation 2 below). Then find an x_1, (equal to $1+G^{1/K}$), such that this simplified polynomial is positive for $x \geqq x_1$.

This proof is in no way invalidated if in the first part we retain any positive term we choose instead of x^n and discard whatever negative terms we can pair with preceding positive terms in such a way that each positive coefficient is equal to or greater than the numerical value of the corresponding negative coefficient. These modifications will in certain equations result in no reduction of the upper limit of the roots. However, equations may be written which will make the reduction arbitrarily large.[3] *An upper limit that is always lower can be secured by retaining the term, G.*

Upon the basis of these considerations we formulate the following theorem.

THEOREM: *If in the real equation $f(x)\equiv x^n+a_1x^{n-1}+\cdots+a_n=0$, where not all the coefficients are positive, we strike out an arbitrary number of such pairs of terms as*

$$a_px^p \text{ and } a_{p-q}x^{p-q}, \qquad a_p \geqq |a_{p-q}|, \quad a_{p-q}<0,$$

and if, in the resulting equation $a_{n-m}x^m$, where $a_{n-m}\neq 0$, is any term preceding all negative terms, $m-h$ is the power of the first negative term and g is the greatest of the numerical values of the negative coefficients, then each real root of $f(x)=0$ is less than any term of the sequence

$$1+(g/a_{n-m})^{1/h}\ ;\ \phi[1+(g/a_{n-m})^{1/h}]\ ;\ \phi\phi[1+(g/a_{n-m})^{1/h}],\ \cdots$$

where

$$\phi(x)\equiv 1+[(1-x^{-m+h-1})g/a_{n-m}]^{1/h}, \quad x>1.$$

[1] Presented to the Southern California Section of the Mathematical Association of America, Nov. 6, 1926.

[2] See L. E. Dickson, *First Course in the Theory of Equations.* p. 21.

[3] Compare Pascal's *Repertorium* (1910), vol. 1, p. 352.

PROOF: Assuming that x is greater than unity we can write

$$(1) \qquad f(x) \geqq a_{n-m}x^m - g(x^{m-h} + x^{m-h+1} + \cdots + 1) = \\ a_{n-m}x^m - [g(x^{m-h+1} - 1)/(x - 1)].$$

We now seek an x_1 (>1) such that $x \geqq x_1$ implies that

$$(2) \qquad a_{n-m}x^m(x - 1) - gx^{m-h+1} + g > 0.$$

Dividing this by x^{m-h+1} and transposing the g terms gives

$$a_{n-m}x^{h-1}(x - 1) > g - gx^{-m+h-1}.$$

Since

$$(3) \qquad (x - 1)^{h-1} \leqq x^{h-1}$$

we can replace the above inequality by

$$(4) \qquad a_{n-m}(x - 1)^h > g - gx^{-m+h-1}$$

which can be written

$$(5) \qquad x > 1 + [(1 - x^{-m+h-1})g/a_{n-m}]^{1/h}.$$

Obviously

$$(6) \qquad 1 + (g/a_{n-m})^{1/h} > \phi[1 + (g/a_{n-m})^{1/h}],$$

where $\phi(x)$ denotes the right member of (5). That is, (5) is satisfied by $x = 1 + (g/a_{n-m})^{1/h}$. It follows from (5) and (6) that

$$\phi[1 + (g/a_{n-m})^{1/h}] > \phi\phi[1 + (g/a_{n-m})^{1/h}].$$

That is, (5) is satisfied by $x = \phi[1 + (g/a_{n-m})^{1/h}]$.
Similarly it is satisfied by $x = \phi\phi[1 + (g/a_{n-m})^{1/h}]$ etc.

Returning to (1) we see that if any value of x, greater than unity, makes the second member of the equality positive, all greater values of x do the same. Consequently x, equal to or greater than any term of the sequence

$$1 + (g/a_{n-m})^{1/h}\,;\ \phi[1 + (g/a_{n-m})^{1/h}]\,;\ \phi\phi 1[+ (g/a_{n-m})^{1/h}]\,;\ \cdots\,;$$

makes $f(x) > 0$, which proves our theorem.

COROLLARY: *The limit of the ϕ sequence of the above theorem is either equal to or greater than the largest positive root of $f(x) = 0$.*

PROOF: It follows from the above theorem and the definition of a limit that the largest root could not be greater than the limit of the ϕ sequence. Hence we need only cite an instance in which the equality holds.

Let α be the limit of the sequence. Since the right member of the inequality (5) takes on the same sequence of values as x it also approaches α. Hence

$$\alpha = 1 + \left[(1 - \alpha^{-m+h-1})g/a_{n-m}\right]^{1/h}.$$

Thus α is a root of the equation obtained by using an equality sign instead of the inequality sign in (4).

Now in case $h=1$, the equality sign in (3) holds. Hence α is a root of the equation obtained by equating the left member of (2) to zero.

Finally, if
$$f(x) \equiv a_{n-m}x^m - g(x^{m-1} + x^{m-2} + \cdots + 1)$$
and if $f(x)=0$ has a root equal to or greater than unity, α is that root. Otherwise α is the root, unity, which was introduced in (2).

2. Examples. The following examples illustrate the essential features of the above theorem. The first one shows gain over the results given by the classic formula, $1+G^{1/k}$ and incidentally illustrates the corollary. The second example is a case in which the index of the radical in $\phi(x)$ is *increased* by dropping pairs of terms. The third is a case in which the radicand in $\phi(x)$ is *decreased* by the dropping of pairs of terms. Some gain *always* results from the use of the sequence, and the other features are not mutually exclusive:

Example 1. $x^3-x^2-x-1=0$. Applying our theorem we find $n=m=3$, $h=1$, $g=1$ and $a_{n-m}=1$. Hence the sequence of upper limits is

$$1 + 1 \;;\; 1 + (1 - 2^{-3}) = 15/8 \;;\; 2 - (8/15)^3 = 1.8483 \;;$$
$$2 - (1.8483)^{-3} = 1.8416 \;;\; 1.8399 \;;\; 1.83944 \;;\; 1.83932 \;;\; 1.83931 \;;\; \cdots$$

The term 1.83931 is a root with four decimal places correct. Using the formula $1+G^{1/k}$, we find 2 for an upper limit of the roots.

Example 2. $x^5+3x^4-2x^3+4x-32=0$. Here we drop the binomial $3x^4-2x^3$. Then $n=m=5$, $h=5$, $g=32$ and $a_{n-m}=1$. Hence the sequence of upper limits is

$$1 + 32^{1/5} = 3 \;;\; 1 + 2(1 - 3^{-1})^{1/5} = 2.84 \;;$$
$$1 + 2\left[1 - (2.84)^{-1}\right]^{1/5} = 2.82 \;;\cdots$$

If we apply the formula $1+G^{1/k}$ in this example we get $1+32^{1/2}=6.657$.

Example 3. $x^5-2x^4+22x^3+60x^2-73x+5=0$. In this case we separate $73x$ into $22x$ and $51x$, then strike out the binomials $22x^3-22x$ and $60x^2-51x$. The first term of our sequence of upper limits is then 3, while the formula $1+G^{1/k}$ applied to the original equation gives 74. In the above examples n was equal to m, but it is easy to select equations in which it is better to make $m<n$. The equation $x^6+15x^5-16x^4+4x^3+x-8=0$ is such a one. As it stands, $1+(g/a_{n-m})^{1/h}=5$. After striking out the first three terms, $1+(g/a_{n-m})^{1/h}$ $=1+2^{1/3}$.

3. Comments. If one is seeking merely a simple method of determining a low upper limit in numerical cases, the best procedure would seem to be that of the following rule: *Assume the upper limit of the roots of a given equation to be at least unity, modify the equation so that it can be easily solved* (*by annexing and dropping terms*), *in anyway whatever provided the highest degree term remaining be positive and the value of $f(x)$ for $x \geqq 1$ be not increased. Then take for an upper limit of the roots of the original equation, the largest positive root of the new equation if it exceeds unity, otherwise take unity.* The validity of the rule appears in its statement. It is pedagogically desirable because it offers no opportunity to "substitute in the formula" and requires versatility and common sense. Applying the rule to the above examples we change the three equations into $x^3-x^2-x-2=0$, $x^5-32=0$ and $x^5-2x^4=0$, respectively. These give us 2 for the upper limit[1] of the roots of each of the original equations. In such an equation as $x^5-x^2+4x-5=0$, we can drop $4x-4$ and replace the resulting -1 by $-x^2$. This gives $x^5-2x^2=0$; hence $2^{1/3}$ is our upper limit.

THE GREEK IDEA OF PROPORTION

By GEORGE W. EVANS

There were three definitions of proportion known to the Greeks of Euclid's time; one of these Euclid does not mention, and as to any logical connection between that and the other two that he does give, there is no explicit statement anywhere.

Euclid's two definitions are:

Book V, Definition 5:

Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in the corresponding order.

Book VII, Definition 20:

Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

The second of these two means that integers are proportional when the first is the same multiple or the same rational fraction of the second that the third is of the fourth.

---

[1] The upper limits 2, 5 and 3 are given for the three equations, respectively, by the familiar method of taking unity plus the largest quotient obtained by dividing the numerical value of each negative coefficient by the sum of all preceding positive coefficients. Obviously these are larger than the corresponding limits given by either method of this paper, except that the same result is obtained in the first by the above rule.

The other definition is found in Aristotle.[1] He is illustrating the importance of definition, and says of a certain geometrical figure "the areas and the lines have the same *antanairesis*, which is the definition of 'the same ratio.'" Some five centuries or so later, a theologian named Alexander of Aphrodisias, having a little time to spare from discussions of higher things, undertook to explain whatever might be obscure in Aristotle; when he came to this subject he said "magnitudes are proportional to each other which have the same *anthyphairesis*," and he gave *antanairesis* as Aristotle's synonym. The word *anthyphairesis* is used by Euclid in a different context, and is translated both by Heiberg and by Heath as "subtracting in turn."[2]

Heiberg says that Euclid's Definition 5 is a careful and exact paraphrase of *antanairesis*, and appears to be of Euclid's own devising. On the other hand Heath thinks that Definition 5 is due to Eudoxus, for "it is difficult to see how any mathematical facts could be deduced from such a definition" as *antanairesis*.

It is fortunate for us that the Greek writers of that golden age were so steeped in mathematics that they could not even talk politics without referring to it. There is a passage in Plato where Socrates[3] is explaining the desirability of teaching arithmetic to budding politicians, "for finding out things, not for commerce." These words of Plato come so pat to our need that we must be careful to have our interpretation of them sound. I give not only Jowett's translation, but also the original, and an attempt at a more literal translation for which I am solely to blame.

Jowett: "You know how steadily the masters of the art repel and ridicule anyone who attempts to divide absolute unity when he is calculating, and if you divide, they multiply, taking care that one shall continue one and not become lost in fractions."

More literally: "Yes, for you know the terrors at these things, how if anyone for his ratio attempts to cut the one itself, they jeer and refuse to accept it, but if you do split it, they multiply it, taking care lest ever the one appear not one, but many fragments."

The Greek:

Οἶσθα γάρ που τοὺς περὶ ταῦτα δεινούς, ὡς ἐάν τις αὐτὸ τὸ ἕν ἐπεχειρῇ τῷ λόγῳ τέμνειν, καταγελῶσί τε καὶ οὐκ ἀποδέχονται, ἀλλ' ἐὰν σὺ κερματίζῃς αὐτὸ ἐκεῖνοι πολλαπλασιοῦσιν, εὐλαβούμενοι μήποτε φανῇ τὸ ἓν μὴ ἓν ἀλλὰ πολλὰ μόρια.

If Socrates was thus finding the measure of a line without "cutting the unit for his ratio," as in finding the length of the diagonal of a unit square, he would necessarily proceed as follows: Supposing AB (Fig. 1) to be the diagonal, and

[1] See Heath's *Euclid*, vol. 2, p. 120.

[2] *Euclid*, X, 2 and X, 3.

[3] Republic, Bk. VII, 525 fin.

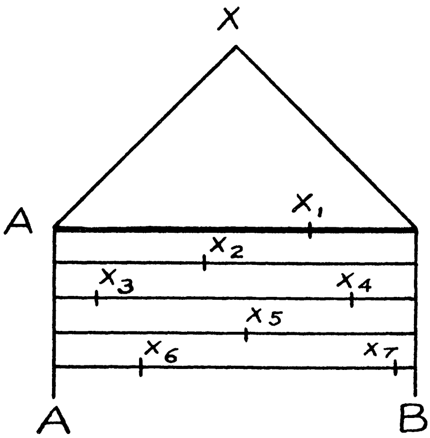


FIG. 1

AX the (unit) side, lay off AX on AB, X coming at X_1; X_1B then turns out to be smaller than AX, and the rest of AX is used for a second measuring of AB, so that X this time falls at X_2; and X_2B, the remainder of AB, is again too small to contain the whole of AX, X_3 being therefore located on the third trip along AB. Continuing in this way, on the seventh application of the unit he will have a rather small remainder X_7B; if he ignores that he can say that he has covered the diagonal 5 times in applying the unit 7 times. Thus he would have found for $\sqrt{2}$ the approximation 7/5, familiar of old.

Here we have the "subtracting in turn"; first the unit from the diagonal; then the remainder from the unit, then that remainder from the diagonal, then that remainder from the unit. We also have the process involved in Euclid's *Definition of Equal Ratios*, for we are comparing multiples of the unit with multiples of the diagonal. It is as if AB (Fig. 2) were marked off successively

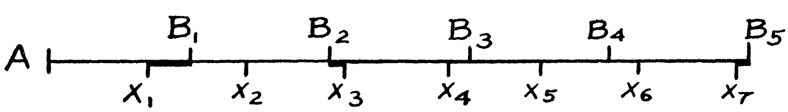


FIG. 2

on one line, and then AX; so that $B_1, B_2, B_3, \cdots$, would terminate multiples of AB, and $X_1, X_2, X_3, \cdots$, would terminate multiples of AX. Euclid's definition practically says, therefore, that the ratio of AB to AX is the same as the ratio of two other magnitudes, say P and Q, if all the approximate measures of AB in terms of AX are the same as the approximate measures of P in terms of Q.

This way of measuring one line with another suggests also the method given by Euclid[1] for finding the greatest common divisor; and therefore the approximations successively obtained by supposing each next smaller remainder to be zero are actually the convergents of the continued fraction in which the ratio may be developed. There are many places in the surviving texts where the Greek computers give approximations that forcibly suggest continued fractions, with which they were of course unacquainted.

Two well known examples of this kind are quoted by Heath[2] from Aristarchus

[1] Book X, Prop. 3.

[2] Heath's *History of Greek Mathematics*, vol. 2, pp. 11, 15.

of Samos; in his discussion of lunar eclipses he gives for the ratios 7921:4050 and 71755875:61735500 the approximate values 88/45 and 43/37, respectively. Heath says "it is difficult not to see in 43:37 the expression $1+1/(6+1/6)$ which suggests $\cdots$ a continued fraction." It turns out that we need not apologize for seeing it. If we apply the subtracting-in-turn idea,—there is no reason why it should not be applied to numbers,—and forget the fifth remainder in the first ratio and the third remainder in the second ratio, we get, as we ought, the approximations that looked so suspicious.

ON A SET OF PROBLEMS RELATED TO THE PROBLEM OF APOLLONIUS

By P. H. DAUS, University of California at Los Angeles

1. Introduction. The famous problem to construct a circle tangent to three given circles is discussed by Coolidge[1] from several distinct viewpoints. The problem is likewise discussed with more or less detail in several of our texts[2] on College Geometry, where the ten special cases, arising from allowing some of the circles to become points or lines, are exhibited.

It is the purpose of this paper to replace some or all of the conditions of tangency by the condition of orthogonality. Some of the resulting problems are conspicuously displayed in texts, others are implied and still others may be found scattered among sets of exercises, but the author does not know one book where they all appear. It is the aim of the author to collect them here, to illustrate a group of problems which utilize the same special methods developed to solve the original problem. We shall find however, that some of the solutions are as interesting as the solution of the problem of Apollonius itself.

2. Notation. The problem will be indicated by a symbol similar to $P_1O_cT_l$. P_i means that the required circle is to pass through i points; O_{jc}, that the required circle is to be orthogonal to j circles; and T_{kl}, that it be tangent to k lines, etc.[3] The problems, omitting the trivial cases and duplications of the ten problems referred to above, can conveniently be divided into three groups, according as the required circle is orthogonal to three circles, two circles, or one circle.

3. Orthogonal to three circles. The special cases are (1) O_{3c}; (2) P_1O_{2c}; (3) P_2O_c. The solution in each case is the radical circle. The methods of finding it are well known, but the uninitiated may consult the texts already cited.[4]

[1] Coolidge, J. L., *A Treatise on the Circle and the Sphere*, ch. 3.

[2] Godfrey and Siddons, *Modern Geometry*, pp. 83–84. For full details, see Altshiller-Court, *College Geometry*, pp. 173–181.

[3] Where j or k is equal to 1, it is omitted. Thus O_c is written instead of O_{1c} and T_l instead of T_{1l}.

[4] Godfrey and Siddons, p. 89; Altshiller-Court, pp. 171–172.

4. Orthogonal to two circles. The problems of this group lead to the special cases of the problem of Apollonius indicated by the symbols P_2T_l and P_2T_c. However, here the two points may be given by the intersection of a line with a circle or of two circles, and the solution is real even if the two points are not. Before we restate the problems, let us recall certain properties of coaxal circles.

If A and B are two points such that the tangents from them to two circles are equal, then the line AB contains all such points, and is called the radical axis of the circles. All circles which have the same radical axis, have their centers on a line perpendicular to the radical axis, and are said to be coaxal. Coaxal systems may be of two types. A system of the first type is such that all the circles have two real points in common; and that of the second is such that the circles have no real points, but two finite imaginary points, in common. In this case, there are two real point circles belonging to the system, known as the limiting points, L_1, L_2. This system is characterized by the fact that every circle of the system is orthogonal to the circle on L_1L_2 as diameter. Associated with a system of either type is a system of the other type, such that the common points (real or imaginary) of one are the limiting points of the other, and such that every circle of one system is orthogonal to every circle of the other system.

Because of these properties we may conveniently re-word the problems to read:—(a) Draw a circle of a coaxal system tangent to a given line. (b) Draw a circle of a coaxal system tangent to a given circle. If the two given points are real, we have a system of the intersecting type, otherwise of the non-intersecting type. The solutions below are so worded that they hold for both cases.

(a) Draw a tangent OP, from the intersection (O) of the given line with the radical axis of the system, to any circle of the system. On the given line locate T, such that $OT=OP$. A perpendicular to the line at T will cut the axis of the system in R, which is the center of the required circle, RT being the required radius.

(b) Draw any convenient circle of the system to cut the given circle, whose center is G, in P, Q. Let PQ cut the radical axis of the system in O, and OT be tangent to the given circle. GT will cut the axis of the system in R, which is the center of the required circle, RT being the required radius.

5. Special cases. The special cases are (1a) $O_{2c}T_l$; (1b) $O_{2c}T_c$; (2a) $O_lP_1T_l$; (2b) $O_lP_1T_c$; (3a) $O_cP_1T_l$; (3b) $O_cP_1T_c$; (4a) $O_cO_lT_l$; (4b) $O_cO_lT_c$. The solution in each case is a circle of the associated coaxal system, which may readily be constructed using the fundamental orthogonal property of such systems, and then (a) or (b) may be applied as the case indicates. In each case there are two solutions.

6. Orthogonal to one circle. If this one circle is a straight line, the subcases are (1) $O_lT_lT_c$; and (2) O_lT_{2c}; and by reflection on this line, the problems

reduce to special cases of the problem of Apollonius. In the first case we must solve $T_{2l}T_c$, which readers may recall, can be solved by the method of translation reducing it to $T_{2l}P_1$, and then using the method of similar figures, although the last part could be replaced by P_2T_l, that is (a) above. In the second case it is convenient to reduce one circle to a point before reflection and then we may apply (b) above. There are four solutions.

The remaining cases are (3) O_cT_{2l}; (4) O_cT_{2c}; (5) $O_cT_cT_l$. For the case (3) O_cT_{2l}, the given circle G and the bisector b of the angle between the lines, define a coaxal system. The required circle belongs to the associated system, since it is orthogonal to both b and G. Hence it can be constructed by use of (a). This problem well illustrates the necessity of considering the problem of P_2T_l when the points are not real, for even if the bisector does not cut the circle G, we get real solutions. The figure shows the construction when one bisector cuts the given circle and the other does not. There are four solutions.

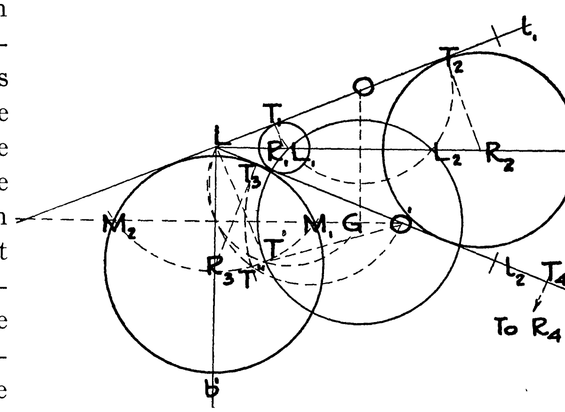


FIGURE for O_cT_{2l}

The method of inversion could have been applied to some of the problems above, but without gaining any advantage. In the case of (4) O_cT_{2c} however, inversion simplifies the analysis. If the required circle is orthogonal to C_1 and tangent to C_2 and C_3, we may invert with respect to any point on C_1, which reduces the problem to O_lT_{2c} already considered as (2) of this group. Or again, if we invert with respect to the circle C_1, the required circle will be tangent to C_2, C_3 and their inverses, and hence we have the problem of Apollonius itself. The first method indicates clearly the four solutions. We might suspect eight solutions due to the second method, but if we note that the contacts of the required circle with C_1 and its inverse are of the same type, we see that there are but four solutions.

The solution of the remaining case (5) $O_cT_cT_l$, is similar to case (4).

THE ANALYTIC DETERMINATION OF THE AREA OF A TRIANGLE IN TERMS OF ITS SIDES

By K. P. WILLIAMS, Indiana University

It is an interesting problem in the theory of functions to find an explicit expression for a function from its general properties. Thus we obtain the factor form for $\sin x$ from a knowledge of its zeros, and a series of fractions for $\cot x$ from a knowledge of its poles. In each case a delicacy of analysis is required in order to determine completely the desired expression, and it is necessary to know something about the function in the entire complex plane.

One can adopt a similar point of view towards a formula that gives the area or volume of a geometric figure, or one side in terms of other sides and angles. A difficulty now arises. The geometric configuration exists only for certain values of the variables, but the formula has an analytic sense for unrestricted ranges. There is consequently a question of uniqueness if a geometric formula is thought of in the broad unrestricted light of function theory. In this note we shall examine the area of a triangle as a function of its sides from the point of view of analysis.

Let the sides of the triangle be x, y, and z, and denote the area by A. We wish an analytic expression for A in terms of x, y, and z. The variables x, y, z are restricted to positive or zero values. But they are not independent, since the relation $|y-z|<x<|y+z|$ is to be satisfied. For a given y and z the excepted values of x are then of two types: (1) negative and complex values, (2) certain positive values. There is a distinctly different origin for these two exclusions. The first arises from the fact that we are concerned with lines; the second from the fact that the lines must form a triangle. It can be expected that the formula that gives A will not react in the same manner if numbers of the first and second excluded classes, respectively, be substituted. One does not care what the result of substituting a number of the first class may be. But one does expect that the value of A for an excluded positive value of x will itself indicate that no triangle exists for such a value. This leads to the postulate:

If positive values for x, y, z, that are inconsistent with x, y, z being the sides of a triangle, be substituted in the formula that gives the area A, the value found for A must be either negative or complex.

Other properties of A are obvious geometrically.

(1) A must be symmetric in x, y, and z.

(2) If y and z are regarded as fixed, A must have real positive values for $|y-z|<x<y+z$.

(3) A must vanish for $x=|y-z|$ and $x=y+z$.

(4) If x, y, z are all multiplied by k, A must be multiplied by k^2.

By considering a right triangle we have the boundary condition,

(5) If $x^2=y^2+z^2$ we must have $A=\frac{1}{2}yz$.

We construct a function $F(x, y, z)$ that satisfies the postulate and the five conditions. Write

$$F(x,y,z) = (y + z - x)(x - y + z)f(x,y,z).$$

Changing the variables cyclically, we obtain

$$F(y,z,x) = (z + x - y)(y - z + x)f(y,z,x).$$

Since we must have $F(x, y, z) = F(y, z, x)$, it is seen that $y-z+x$ must be a factor of $f(x, y, z)$ and $y+z-x$ a factor of $f(y, z, x)$. But $y+z-x$ is obtained from $y-z+x$ by the cyclic change.
Write then

$$f(x,y,z) = (y - z + x)\phi(x,y,z),$$

so that

$$F(x,y,z) = (y + z - x)(x - y + z)(y - z + x)\phi(x,y,z).$$

Since the first three factors on the right form a symmetric quantity, it follows that $\phi(x, y, z)$ is symmetric. If x, y, z be replaced by kx, ky, kz, the first three factors on the right are multiplied by k^3. This suggests that it may be possible so to determine $\phi(x, y, z)$ that $F(x, y, z)$ gives A^2 and not A. We can write

$$F(x,y,z) = (x^2 - y^2 - z^2 + 2yz)(y + z - x)\phi(x,y,z).$$

Substituting $x^2=y^2+z^2$ we see that $(y+z-x)\phi(x, y, z)$ must $=(yz)/8$ when $x^2=y^2+z^2$.

The simplest choice is obviously $\phi(x, y, z)=(y+z+x)/16$. It follows that the function $\frac{1}{4}[(y+z-x)(x-y+z)(y-z+x)(x+y+z)]^{1/2}$ satisfies conditions (1) . . . . (5). It also satisfies the postulate, since for $x<|y-z|$ or $x>y+z$ the function has an imaginary value. It is to be noted however that for $-|y+z|<x<-|y-z|$ the function has a real value, and vanishes for $x=-|y-z|$, and $x=-[y+z]$. Since, however these values of x are themselves totally inapplicable to any triangle, the fact does not, in accordance with the remark made before, arouse suspicion against the formula.

It remains to investigate the uniqueness of the function. For this purpose we consider

$$\tfrac{1}{4}[(y + z - x)(x - y + z)(y - z + x)(x + y + z)]^{1/2}\psi(x,y,z).$$

That some of the conditions would be satisfied by various choices of ψ is obvious. For instance if we put

$$\psi(x,y,z) = e^{\rho}, \text{ where } \rho = (x^2 - y^2 - z^2)(y^2 - z^2 - x^2)(z^2 - x^2 - y^2),$$

we shall have a function that satisfies (1), (2), (3), (5), and which furthermore has no positive zeros other than those given in (3).

In order to be consistent with (4) it is necessary that $\psi(kx, ky, kz) = \psi(x, y, z)$. Consider x, y, z as the rectangular coordinates of a point. It is evident that there is a bundle of rays lying in the positive octant forming a cone shaped solid, having the vertex at the origin, such that the coordinates of every point on one of the rays, and only the coordinates of such a point, can be the sides of a triangle. The last equation written shows that the function $\psi(x, y, z)$ is constant along any one ray. Assuming that $\psi(x, y, z)$ is continuous for $x=0$, $y=0$, $z=0$, we see that $\psi(x, y, z) = \text{constant}$. By considering a right triangle, it is found that the constant is 1. We therefore conclude that the area of the triangle is

$$A = \tfrac{1}{4}[(y + z - x)(x - y + z)(y - z + x)(x + y + z)]^{1/2}.$$

It could have been expected that in addition to the conditions (1), $\cdots$, (5) some other condition would need to be added. The restriction imposed is very mild and suffices to remove ambiguity.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A TYPE OF FUNCTION WITH k DISCONTINUITIES

By RAYMOND GARVER, University of Rochester

Professor Pierpont, in his *Theory of Functions of Real Variables*, gives an interesting expression[1] for a function which has the following properties: (I) It is defined for $x \geqq 0$. (II) It represents one arbitrary function in $0 \leqq x < 1$, and a second arbitrary function when $x > 1$. (III) It is, in general, discontinuous at $x = 1$. (It is the third property which I am particularly interested in here, though Pierpont stresses the second and does not mention the third.) The expression is

$$y = \lim_{n\to\infty} \frac{x^n f(x) + g(x)}{x^n + 1},$$

where $f(x)$ and $g(x)$ are supposed to be defined in the necessary intervals. We clearly have $y = g(x)$ when $x < 1$, $y = \frac{1}{2}[f(x) + g(x)]$ when $x = 1$, and $y = f(x)$ when $x > 1$. Property (III) follows, and of course there may be other discontinuities arising from the $f(x)$ and $g(x)$.

[1] Volume 1 (1905), p. 204.

We can extend the above to define a type of function possessing the first property as above, and two others which are extensions of (II) and (III). The function is, in general, discontinuous at $x=1, 2, \cdots, k$, (k any positive integer) and this seems to me to be its most interesting property. Further, it represents $(k+1)$ arbitrary functions in different parts of its interval of definition. The expression for the function is

$$F(x) \equiv \lim_{n\to\infty} \frac{x^n f_1(x) + \left(\frac{x^2}{2\,!}\right)^n f_2(x) + \left(\frac{x^3}{3\,!}\right)^n f_3(x) + \cdots + \left(\frac{x^k}{k\,!}\right)^n f_k(x) + f_0(x)}{x^n + \left(\frac{x^2}{2\,!}\right)^n + \left(\frac{x^3}{3\,!}\right)^n + \cdots + \left(\frac{x^k}{k\,!}\right)^n + 1},$$

where it would be more compact, but perhaps less clear, to employ summations. We now have at once

$$F(x) = f_0(x)\,,\ 0 \leqq x < 1\ ;\ \text{and } F(1) = \tfrac{1}{2}[f_0(1) + f_1(1)].$$

Dividing numerator and denominator by x^n, and letting n approach ∞, we obtain

$$F(x) = f_1(x)\,,\ 1 < x < 2\ ;\ \text{and } F(2) = \tfrac{1}{2}[f_1(2) + f_2(2)].$$

Multiplying numerator and denominator by $(2\,!x^{-2})^n$, and letting n approach ∞, we have

$$F(x) = f_2(x),\ \ 2 < x < 3\ ;\ \text{and } F(3) = \tfrac{1}{2}[f_2(3) + f_3(3)].$$

And in general

$$\begin{aligned} F(x) &= f_i(x)\,, \quad i < x < i+1, \quad i = 1, 2, 3, \cdots, k-1.\\ F(i+1) &= \tfrac{1}{2}[f_i(i+1) + f_{i+1}(i+1)].\\ F(x) &= f_k(x), \qquad k < x. \end{aligned}$$

II. A Note on Quaternary Forms

By Claiborne G. Latimer, Tulane University

In a recent paper[1] E. T. Bell showed that if $\phi(x') = bcx_1'^2 + acx_2'^2 + abx_3'^2$ then[2]

$$[x_0^2 + \phi(x')][y_0^2 + \phi(y')] = z_0^2 + \phi(z'),\ \text{where}$$

$$(1)\qquad \begin{aligned} z_0 &= x_0y_0 - bcx'y' - acx_2'y_2' - abx_3'y_3',\\ z_1' &= x_0y_1' + x_1'y_0 + a(x_2'y_3' - x_3'y_2'),\\ z_2' &= x_0y_2' + x_2'y_0 + b(x_3'y_1' - x_1'y_3'),\\ z_3' &= x_0y_3' + x_3'y_0 + c(x_1'y_2' - x_2'y_1'). \end{aligned}$$

[1] Annals of Mathematics (2), vol. 27, p. 99.

[2] Throughout this paper x, x', y, y', z, z', s represent respectively the sets of variables (x_1, x_2, x_3), $(x_1', x_2', x_3'), \cdots, (s_1, s_2, s_3)$.

This is a special case of the following result, due to Hermite.[1] Let $|a_{ij}|$ be the general symmetric third order determinant and let A_{ij} be the cofactor of a_{ij}. If $F(x)=\sum_{i,j=1}^{3}A_{ij}x_ix_j$, $f(x)=\sum_{i,j=1}^{3}a_{ij}x_ix_j$, $F^{(i)}(x)=\frac{1}{2}\partial F/\partial x_i$, $f^{(i)}(x)=\frac{1}{2}\partial f/\partial x_i$; then

$$
\begin{gathered}
[x_0^2+F(x)][y_0^2+F(y)] = z_0^2+F(z), \text{ where } \quad z_0 = x_0y_0 - \sum_{i=1}^{3} x_i\,F^{(i)}(y), \\
z_1 = x_0y_1 + x_1y_0 + f^{(1)}(s), \quad z_2 = x_0y_2 + x_2y_0 + f^{(2)}(s), \\
z_3 = x_0y_3 + x_3y_0 + f^{(3)}(s), \\
s_1 = x_2y_3 - x_3y_2, \; s_2 = x_3y_1 - x_1y_3, \; s_3 = x_1y_2 - x_2y_1.
\end{gathered} \tag{2}
$$

In his paper, Bell suggested that it might be possible to obtain Hermite's formulas, (2), by methods similar to those he used to derive (1). The object of this note is to show that Hermite's result may be deduced, by elementary methods, directly from Bell's formulas quoted above. Let $|a_{ij}|=D$. To $F(x)$ apply the transformation

$$
\begin{aligned}
x_1 &= x_1' - \frac{A_{12}}{A_{11}}x_2' + \frac{a_{13}}{a_{33}}x_3' \\
x_2 &= \qquad\quad\; x_2' + \frac{a_{23}}{a_{33}}x_3' \\
x_3 &= \qquad\qquad\qquad\quad x_3'
\end{aligned} \tag{3}
$$

Apply to $F(y)$ the same transformation, the new variables being y_i'. It will be found that $F(x)$ becomes $\phi(x')$ with $a=D/a_{11}$, $b=a_{11}D/A_{33}$, $c=A_{33}/D$. Then

$$
[x_0^2+F(x)][y_0^2+F(y)] = [x_0^2+\phi(x')][y_0^2+\phi(y')] = z_0^2+\phi(z'),
$$

where the z_i' and z_0 are given by (1) after replacing a, b and c by the expressions above. To $\phi(z')$ apply the inverse of the transformation (3), letting the new variables be z_i. The resulting form is of course $F(z)$. The z_i may be expressed in terms of x_i, y_i by means of (1), (3) and the two sets of equations obtained from (3) by replacing the x's by the corresponding y's and by the corresponding z's. It may be verified that the resulting expressions for the z_i may be written as in (2). Hence we have Hermite's result.

Bell's formulas (1) were derived from the well known result of Euler's that the sum of four squares repeats under multiplication. It follows that Euler's result, which is apparently a special case of Hermite's, is equivalent to the latter.

[1] Bachmann, *Die Arithmetik der Quadratischen Formen*, I. Abt., p. 13.

III. The Origin of the Name of the Devil's Curve

By Emile Borel, Professeur à la Faculté des Sciences de Paris

A la page 199 du "The American Mathematical Monthly" on demande pour quelle raison les géomètres français ont donné à une certaine courbe le nom de la "courbe du diable."

Je crois que cette raison est fort simple: On désigne, en effet en France, sous le nom de "diable" un jouet qui est formé par deux toupies juxtaposées.

Ce jouet très ancien était passé de mode et il a eu un renouveau il y a une vingtaine d'années, sous le nom italien de "diabolo."

La courbe en question dessine assez exactement la section de ce jouet qui a la forme d'une surface de révolution.

IV. Is There a Student Standard of Truth? A Reply

By William F. Osgood, Harvard University

To the Editors of the Monthly:

Professor Huntington[1] understands my words: "what, to the student, truth is" as tampering with the truth—setting up different standards of truth for mathematicians of different degrees of advancement in their science. I cannot grant that this interpretation is justified when one reads the whole letter. To enable the readers of the Monthly more easily to judge for themselves, I should like to point out precisely what the question at issue is.

First, as regards the mathematical material under discussion, it is necessary to review the case, since Professor Carver's note may create a wrong impression in the minds of some readers concerning the basis of Professor Ettlinger's criticism of Professor Woods's book.

Professor Woods is dealing with the sum of n positive infinitesimals, $\alpha_1+\alpha_2+\cdots+\alpha_n$, and a similar sum, $\beta_1+\beta_2+\cdots+\beta_n$. In the statement of the theorem, one of his hypotheses is that $\lim_{n=\infty}\beta_i/\alpha_i=1$. Let $\beta_i/\alpha_i=1+\epsilon_i$, or $\beta_i=\alpha_i+\epsilon_i\alpha_i$. Then the student naturally interprets the hypothesis as meaning that $\lim_{n=\infty}\epsilon_i=0$, no matter how i varies with n. This is the interpretation which the author intended, and on this interpretation is based an entirely correct proof of the theorem.

Now, if one demands less than the above; if one requires that ϵ_i approach 0 when i varies in a somewhat restricted manner with n, one can get a wrong theorem. Professor Ettlinger's whole criticism is based on the fact that Professor Woods did not formulate a difficulty which the student does not experience. What the student understands from Professor Woods's words is truth.

[1] E. V. Huntington, *Is there a student standard of truth?*, this Monthly, vol. 34 (1927), pp. 320–321.

The misunderstanding Professor Ettlinger fears is falsehood—which Professor Ettlinger is anxious to correct.

Turning now to Professor Huntington's charge, I would say that there was no question in my mind of "one standard of truth for a student, and another standard of truth for a master of the science." It is not an ethical question—honesty against dishonesty; the whole truth vs. a garbled statement. Professor Huntington is wrong at this point. He has picked out a half line from my letter and has put an interpretation on it which a careful reading of the letter as a whole would not bear out. My words: "what, to the student, truth is" were intended to express the extent of his understanding, the amount of light which breaks through on him, the depth of his insight, first things being put first. We all agree that a false statement is unjustifiable, no matter what the motive. But to make a true statement is not necessarily to create a true impression, or any impression at all. To reveal the truth to the student, it is necessary to focus his attention on the big, essential thing which he can and must understand at that stage of his work, omitting niceties which mean little or nothing to him.

Professor Huntington will say: "Ah, but that is pedagogy. I am talking about scientific truth." It was to meet this reply that I set forth the mathematical material under discussion at the beginning of my letter. Problems of abstract truth may interest the philosopher. The question at issue is whether the truth has suffered at the hands of Professor Woods.

WILLIAM F. OSGOOD

V. A NOTE ON THE COMPUTATION OF ARITHMETIC ROOTS

By OTTO DUNKEL, Washington University

The iterative formula for the approximation of numerical roots derived by Uspensky in the March number of this Monthly[1] was given by Peter Barlow in his *New Mathematical Tables*, London, 1814, p. 259. There is a misprint in the denominator of his formula for $x^{1/n}$. On pages xvii and xx are derivations of the formula for the cases $n=2, 3$. Other methods are given with numerical illustrations. There is no reference to an earlier mention of this formula, and, since Barlow gave references for other formulae, it is to be presumed that he knew of no earlier knowledge of it.

This formula may be described as an iterative one of the third degree. It is an easy matter to derive in a variety of ways a large number of such formulae of degrees extending from two upward and to obtain expressions for their respective errors. A large number of these formulae may be obtained from the long known method of approximations to the roots of algebraic equations

[1] J. V. Uspensky, *Note on the computation of roots*, vol. 34 (1927), pp. 130–134.

described in this journal[1] in the solution of problem 2803. For, if s is taken as an approximation to $N^{1/n}$ and x is the error in s, the binomial theorem gives this equation for x:

$$0 = (s^n - N) + ns^{n-1}x + \frac{n(n-1)}{2}s^{n-2}x^2 + \cdots + x^n.$$

Then, with the notation in that solution, $a_0 = s^n - N$, $a_1 = ns^{n-1}$, $\cdots$, and hence $s - a_0D(m)/D(m+1)$, where $D(m)$ is the determinant defined on page 229, gives an iterative approximation of the 2nd, 3rd, 4th, $\cdots$, degree according as $m = 0, 1, 3, \cdots$. The first of these is the well known rule of Newton or Raphson, the second is the one given by Barlow and Uspensky, while the others are more complicated but of higher and higher degrees as m increases.

As to approximations of the second degree one may easily obtain the following:

$$(1)\quad s + \frac{N - s^n}{ns^{n-1}}, \qquad (3)\quad s + s\frac{N - s^n}{(n+1)s^n - N},$$

$$(2)\quad s + s\frac{N - s^n}{nN}, \qquad (4)\quad s + s\frac{N - s^n}{(n-1)N + s^n}.$$

Of these the first two are of most interest. They may be stated as follows: Starting with a number s as an approximation to $N^{1/n}$, we find the percent of error in s^n; then the approximate percent of error in s is $1/n$th of this power percent. Add to s this last percent times s for a better approximation and repeat the process with this second approximation to find a third, and so on. If the percent of error in s^n is computed with respect to s^n, we obtain, after removing a factor s from the numerator and denominator, the first formula which gives a result too large. If the percent is reckoned with respect to N, we obtain the second which gives a result too small. A combination of the expansions which give two of these formulae may be made so as to remove the error term of the second degree, and we then fall upon the rule of the third degree mentioned above.

The first rule may also be written in the form of an average which is easy to remember and convenient for computation:

$$[(n-1)s + q]/n, \quad \text{where} \quad q = N/s^{n-1}.$$

In this form, a person with only a moderate knowledge of arithmetic and medium intelligence will perceive in the process a good and simple way for approximating a root. The rule may be stated thus for cube root. In order to find $N^{1/3}$ we try a number s; we raise it to the second power and we divide

[1] Vol. 33 (1926), p. 229.

N by this power, obtaining the quotient q. If $q=s$ then s is the exact root. If q is larger than s then s is too small and q is too large, and conversely. In this case the root lies between s and q and we instinctively take for the next trial an average of s and q, but since s is used twice and q only once in the product to produce N we take the weighted average $(2s+q)/3$. Each new quotient q gives an estimate of the correct figures in s used to produce it, while the average will in general give twice as many correct decimals as s. This fact may be verified by the next step of division and averaging.[1] This admirably simple and ancient rule is worthy of a place in elementary instruction, yet no text mentions it. Moreover, it furnishes the simplest way of generating two sequences of numbers which may be used to define $N^{1/n}$. For it may be shown that the two sequences converge to a common limit and that the sequences formed by the nth power of their terms converge to N. In an interesting paper by A. Hurwitz, *Ueber die Einführung der elementaren transzendenten Funktionen in der algebraischen Analysis*, Mathematische Annalen, vol. 70(1911), pp. 33–47, this rule is used in this manner as a basis for the development. The application of this rule to square root is attributed to Newton.

VI. A Simple Derivation of Hutton's Formula for the Computation of Roots

By Roy F. Newton, Purdue University

Mr. J. V. Uspensky[2] has described the use of Hutton's formula and given a derivation of it, stating that, while the method was probably not new, he was unable to find any reference to it. Davies and Peck[3] describe the method and attribute it to Hutton, but give no derivation of it. I have been unable to find the original of Hutton's work or any derivation other than Uspensky's. Perhaps the following derivation by the author may be of interest.

It is easily found by inspection and trial of the equation

$$s_1 = s[(ax + s^n)/(a + xs^n)] \tag{1}$$

that, if s is an approximate value for θ, the nth root of a, s_1 is a still better approximation, when $x>1$. If we write $s=\theta(1+e)$ and $s_1=\theta(1+e')$, then $|e'/e|<1$ for $x>1$. For x nearly equal to 1, e'/e is positive and nearly equal to 1,

---

[1] See Glenn James, *A rapid method of approximating arithmetic roots*, this Monthly, vol. 31 (1924), pp. 471–475; also Otto Dunkel, *A simple rule for extracting any root of a number*, School Science and Math. vol. 18 (1918), pp. 19, 20, and *A graphical representation of approximations for square roots*, ibid. vol. 18. (1918), pp. 621–625.

[2] J. V. Uspensky, *Note on the computation of roots*, this Monthly, vol. 34 (1927), pp. 130–134.

[3] Davies and Peck, *Mathematical Dictionary and Cyclopedia of Mathematical Science*, under "Extraction of Roots," p. 242. (1876, A. S. Barnes and Co.)

and for x very large, e'/e is negative and nearly equal to -1; we shall determine the value of x for which e'/e is small when e is small. Solving (1) for x:

$$x = \frac{s^{n+1} - s_1 a}{s_1 s^n - sa} = \frac{(1+e)^{n+1} - 1 - e'}{(1+e')(1+e)^n - 1 - e} =$$
$$(2) \qquad \frac{(n+1)e + (n+1)ne^2/2 + (n+1)n(n-1)e^3/6 + \cdots - e'}{(n-1)e + (n-1)ne^2/2 + (n-1)n(n-2)e^3/6 + \cdots + e' + nee' + \cdots}.$$

When both e and e'/e are small, we obtain $x = (n+1)/(n-1)$. Substituting in (1) and clearing of fractions, we obtain

$$(3) \qquad s_1 = s\frac{(n+1)a + (n-1)s^n}{(n-1)a + (n+1)s^n}$$

for which, when e is small, e'/e is also small.

To estimate the error in s_1, we shall substitute $s = \theta(1+e)$ and $s_1 = \theta(1+e')$ in (3), obtaining

$$1 + e' = \frac{n + 1 + ne + e + (n-1)(1+e)^{n+1}}{n - 1 + (n+1)(1+e)^n} =$$
$$(4) \qquad \frac{2n+(n+1)ne+(n+1)n(n-1)e^2/2+(n+1)n(n-1)(n-1)e^3/6+(n+1)\cdots(n-2)(n-1)e^4/24+\cdots}{2n+(n+1)ne+(n+1)n(n-1)e^2/2+(n+1)n(n-1)(n-2)e^3/6+(n+1)\cdots(n-2)(n-3)e^4/24+\cdots}$$
$$= 1 + (n+1)(n-1)e^3/12 - (n+1)(n-1)e^4/8 + \text{terms in } e^5 \text{ and higher},$$

or when e is small, $e' = (n^2-1)e^3/12$, from which it is evident that for comparatively small values of n, the new value of the root has about three times as many correct decimal places as the preceding one.

VII. On the Relative Accuracy of Simpson's Rules and Weddle's Rule A Question

By Raymond Garver, University of Rochester

In the March number of this Monthly[1] Professor Scarborough makes the unqualified statement that the accuracy in Weddle's rule is very much greater than that in Simpson's rules. Hence I have thought it would be interesting to call attention to the following exception.

Calculating $\int_0^1 (1+x^2)^{-1}dx$ by Weddle's rule and Simpson's one-third rule, we have

Correct value	.7853982	
Simpson's rule	.7853979	(using 7 ordinates)
Weddle's rule	.7853996	(using 7 ordinates)

Is this a case of exceptional accuracy for Simpson's rule, or just the reverse for Weddle's rule?

[1] Vol. 34 (1927), pp. 135–141.

VIII. On the Relative Accuracy of Simpson's Rules and Weddle's Rule A Reply

By J. B. Scarborough, U. S. Naval Academy

In order to explain the unexpected result found by Mr. Garver,[1] it is well to go back for a moment and consider the foundation of the process of interpolation on which the process of approximate quadrature is based. The theoretical validity of interpolation formulas, and hence of all approximate quadrature formulas derived from them, rests on the famous theorem of Weierstrass which states that any continuous function can be approximated over an interval, to any desired degree of accuracy, by a polynomial. The accuracy of the approximation in any given problem depends (a) upon the nature of the function under consideration and (b) upon the magnitude of the interval h between the equidistant values of the independent variable. The particular function $y=(1+x^2)^{-1}$, considered by Mr. Garver, happens to be one which is poorly approximated by a polynomial unless the interval h is taken very small. In fact, this function is the one used by writers on interpolation—Runge, Steffensen, and others—to illustrate the failure of interpolation formulas when applied over too wide an interval.

In the example cited by Mr. Garver the value of h is small enough to allow the function to be represented over the given interval by a Taylor series, but the series is slowly convergent. The following table of differences shows even more strikingly how poorly this function is represented by a polynomial for the value of h which he used.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.00000000						
0.97297297	−0.02702703					
0.90000000	−0.07297297	−0.04594594				
0.80000000	−0.10000000	−0.02702703	+0.01891891			
0.69230769	−0.10769231	−0.00769231	+0.01933472	+0.00041581		
0.59016393	−0.10214376	+0.00554855	+0.01324086	−0.00609386	−0.00650967	
0.50000000	−0.09016393	+0.01197983	+0.00643128	−0.00680958	−0.00071572	+0.00579395

It will be observed that the succeeding orders of differences decrease so slowly that the third differences cannot be considered constant beyond the second decimal place. Even the sixth difference is one-third as large as the third differences, and we should therefore not expect Weddle's rule to give much better results than Simpson's in this example. If we take $h=1/12$, compute the corresponding values of the function, and form a table of differences, we find that the third differences are not constant beyond the third decimal place, and the sixth differences cannot be considered constant beyond the fourth decimal place.

[1] See the discussion which immediately precedes this one.

We get additional light on this example by making use of the nth derivative of the given function. The nth derivative of $f(x)=(1+x^2)^{-1}$ is

$$f^{(n)}(x) = (-1)^n n!(1+x^2)^{-(n+1)/2}\sin(n\theta+\theta), \text{ where } \theta = \sin^{-1}(1+x^2)^{-1/2}.$$

Hence the remainder after n terms in the Taylor expansion of $f(a+3h)$ is

$$R_n = (3h)^n(-1)^n[1+(a+3\xi h)^2]^{-(n+1)/2}\sin\left[(n+1)\sin^{-1}\left\{1+(a+3\xi h)^2\right\}^{-1/2}\right].$$

For $h=1/6$ the remainder at the point $x=a=1/2$ (which is the mid-point of the given interval) is

$$R_n = (-1/2)^n[1+\tfrac{1}{4}(1+\xi)^2]^{-(n+1)/2}\sin\left[(n+1)\sin^{-1}\left\{1+\tfrac{1}{4}(1+\xi)^2\right\}^{-1/2}\right],$$

$0 \leqq \xi \leqq 1$. As $n\to\infty$, this remainder clearly approaches zero and the given function can therefore be represented in the given interval by a Taylor series.

The values of the 4th and 6th derivatives of the given function are

$$f^{\mathrm{iv}}(x) = \frac{24}{(1+x^2)^5}[5x^4-10x^2+1],\ f^{\mathrm{vi}}(x) = \frac{720}{(1+x^2)^7}[7x^6-35x^4+21x^2-1].$$

Therefore for the point $x=1/2$ we have

$$f^{\mathrm{iv}}(\tfrac{1}{2}) = -9.339, \quad f^{\mathrm{vi}}(\tfrac{1}{2}) = 327.9.$$

Substituting these values in the expressions for $E_{1/3}$ and E_w as given in my paper in the March number of this Monthly, we get

$$E_{1/3} = 0.0000400 - 0.0000474 = -0.0000074, \quad E_w = -0.00000836.$$

Since the actual errors as found by Mr. Garver and checked by me are

$$E_{1/3} = 0.000000218 \text{ and } E_w = -0.00000145,$$

it is evident that the principal part of the error is not given by the first term of the error series in either case. This means that the series representing the error is slowly convergent.

When we take $h=1/12$, we find the errors to be

$$E_{1/3} = 0.00000000332 \text{ and } E_w = -0.00000000596.$$

Here, again, Simpson's rule gives the more accurate result; but whereas the error in the result by Weddle's rule was nearly seven times as great as in Simpson's when $h=1/6$ it is less than twice as great when $h=1/12$. I venture the guess that if we should take $h=1/30$, we should find that Weddle's rule would give a far more accurate result than Simpson's, because in that case the series representing the errors would converge so rapidly that the principal part of the error in each rule would be given by the first term of the error series.

I think we can sum up the trouble in Mr. Garver's example somewhat as follows: The value of h was so much too large for this particular function that,

from the geometrical point of view, the arcs of three separate parabolas of the second or third degree (Simpson's rule) happened to coincide more closely with the graph of the given function than did the arc of a single curve of the fifth degree (Weddle's rule). I was aware of such singular cases at the time I wrote my paper for the Monthly, but I didn't think it worth while to mention them. If for any particular function h is taken so small that the error due to the quadrature formula is given largely by the first term of the series representing the error, then the accuracy of the two formulas (Simpson's and Weddle's) is as stated in my paper.[1]

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University to whom books and communications should be sent.

REVIEWS

Principles of Geometry. By H. F. BAKER. Volume IV, *Higher Geometry*. Cambridge, University Press, 1925. xvi+250 pages.

The first three books of Baker's series are entitled "Foundations," "Plane Geometry," and "Solid Geometry." These were undertaken largely to pave the way for the present text, "Higher Geometry." Before considering this last volume in detail, it may be well worth while to discuss briefly the series as a whole.

The prefaces of the first three volumes state in no uncertain terms that the exposition is intended as a first course in projective geometry, not only for future mathematicians but more especially for physicists and engineers. The author does not insist that students of applied mathematics continue through the present volume; yet we are very sure that he hopes they will do so. But even three volumes form an ambitious program for an engineer, perhaps even for a mathematician whose field is not geometry. And let there be no mistake about it, there is little in these volumes which can be termed light reading. The point of view is always general, always abstract, and the reader must possess or develop for himself the ability to follow page after page of close, intricate argument. Illustration of the abstract by the concrete is largely left to the reader in the form of exercises, a relaxation after the more robust text.

Now we cannot believe that there will be any immediate adoption of this type of course for the non-specialist in mathematics, at least in this country. Nor are we prepared to say that this is the best or even a desirable approach for the student of pure mathematics.

It is not so much the intrinsic difficulty of the treatment that the reviewer wishes to emphasize as the constant effort required for the reader to attain the

[1] This Monthly, vol. 34 (1927), pp. 135–141.

mathematical sophistication of the author. But these pedagogical considerations are far less attractive than the material of the book which the author entitles "Higher Geometry, being illustrations of the utility of the consideration of higher space, especially of four and five dimensions."

Section I of the first chapter deals with spaces of two and three dimensions. As an example, a correspondence between points of a plane and points on a quadric is given. A certain configuration of points on the quadric is shown to exist, the configuration is transferred to the plane and the result is MIQUEL'S THEOREM: *If D, E, F, be any points on the joins BC, CA, AB of three points of a plane, then the circles AEF, BFD, CDE meet in a point.*

Again in three-space, a relation between pairs of points on a quadric is established. This relation transferred to points of the plane becomes inversion, whereupon a surprisingly varied group of inversion theorems appears, dual to projective theorems for the quadric.

Next a correspondence is established between circles of a plane and points (or planes) in space. The author recalls two theorems:

WALLACE'S THEOREM: *If four lines be given in a plane, the four circles, each defined by the intersections of three of the lines, meet in a point;*

MÖBIUS' TETRAHEDRA: *There exist pairs of tetrads of points in space such that every point of either tetrad lies on the plane determined by three points of the other tetrad.*

It appears that the configurations of these theorems are identical under his correspondence, a result as unexpected as it is beautiful.

In Section II, comparisons of three and four dimensional geometry are given. A correspondence is shown between tetrahedral complexes (of lines in three-space) and planes meeting three lines in four-space. Another correspondence is between spheres of three-space and sections of a quadric three-fold in four-space. The most instructive example is the proof that two spheres cut everywhere at the same angle.

In Section III, the discussion deals principally with the well-known correspondence between lines of three-space and the points of a quadratic four-fold in five-space. A correspondence between spheres and the points of the quadratic four-fold is obtained, and thence Lie's line-sphere correspondence is inferred.

Chapter II is an end in itself. Here circles in a plane are in correspondence with plane sections of a quadric by projection. That this method of attack is successful may be judged from a brief selection of the theorems or constructions that are developed: The problem of Apollonius on circles touching three circles; the problem of Steiner on circles cutting three circles under given angles; the problem of Malfatti; Feuerbach's Theorem and Hart circles. Possibly the most striking thing in this chapter is the attack on the problem of Apollonius, an attack which leads inevitably to Gergonne's solution. It is of interest to

recall that Fiedler considered this problem in his *Cyklographie*, in which the correspondence was between circle in plane and point in space. Fiedler's method led also to Gergonne's solution.

In Chapter III, the correspondence is between the points of a plane quartic curve with two double points and the points of the curve of intersection of two quadric surfaces.

In Chapter IV, a correspondence is obtained between the twenty-seven vertices of a certain configuration in four-space and the lines on the general cubic surface. Like Chapter II, this is an end in itself, whereas Chapters I and III serve in the development of the later theory.

In Chapter V, we are introduced to a configuration in four-space, of fifteen points and fifteen lines, due to Segre. With this is associated a quadratic three-fold Σ and the tangent solids of Σ. The intersection of Σ and a tangent solid is the Kummer surface. The dual of the fifteen-configuration leads in similar fashion to the Weddle surface.

In Chapter VI, the correspondence is analogous to that in Chapter IV. The intersection of two quadratic three-folds in four-space is projected on a three-space, giving a quartic surface. This is a cyclide ; the general cyclide and the special Dupin cyclide are treated at some length.

Chapter VII brings the volume and the series to a brilliant conclusion. Segre's figure of Chapter V is shown to be in correspondence with a configuration in five-space and a correspondence is found for Σ. The points of a quadratic four-fold in five-space being in correspondence with the lines of three-space, and the analogon of Σ having been shown to be a part of this quadratic four-fold, Kummer's surface now appears as the locus of the singular points of a quadratic complex of lines. The mêlée now becomes general ; and the interplay between three-space, four-space, and five-space is constant, as the author comprehends under one point of view the cyclides, the surfaces of Weddle and of Kummer, the Wave Surface of Fresnel, and the line and sphere geometry.

B. H. Brown

Einführung in die Theorie der Analytischen Funktionen einer komplexen Veränderlichen. By Heinrich Burkhardt. Fünfte umgearbeitete Auflage besorgt von Dr. George Faber, Berlin and Leipzig, Walter de Gruyter and Co., 1921. x+286 pages, 41 figures.

Reviews of Burkhardt's *Theory of Functions of a Complex Variable* (Part II of Vol. I of his Einführung, etc.) were made by M. Bôcher for the first edition in the Bulletin of the American Mathematical Society, vol. 5 (1899), pp. 181–185 and by L. E. Dickson for the second edition, ibid, vol. 10(1904), pp. 317–321. The third edition appeared in 1907. The fourth edition appeared in 1912, was translated into English by S. E. Rasor, and was published by D. C. Heath and

Co. in 1913. A review of this translation was made by W. C. Brenke in this Monthly, vol. 21 (1914), p. 118. A fifth edition appeared in 1921 under the direction of G. Faber of the Tech. Hochschule zu München. A short notice concerning the fifth edition was made by H. J. Ettlinger in the Bulletin, vol. 28 (1922), p. 475.

It is the purpose of this note to call attention to some changes made by Faber in the fourth edition of Burkhardt's book. Burkhardt's demise occurred at about the time of the beginning of the World War. The last edition of his *Einführung* for the complex variable was the fourth. After the close of the war the socalled fifth edition was made by Faber. In reviewing a book one naturally turns to the preface first. On reading the preface to the fourth edition of this text and then the same preface as quoted in the fifth edition, one notices in the first few lines the absence of Burkhardt's own references to French, English, and American Treatises. The deleting of such references has been done throughout the book. One wonders why Burkhardt's own references to such names as Picard, Forsyth, Harkness and Morley, and Ch. Méray in the field of function theory could not at least be tolerated several years after the close of the war. Moreover, throughout the book practically all German words of English form or sound are changed to other words. For example, Resultat is replaced by Ergebnis, Deduktion by Schlüsse, Argument by Veränderlichen, etc. On page 2, section 2, the words, "double algebra sagen die Engländer" are omitted entirely. Again it seems to the reviewer that unnecessary omissions of sentences in whole or in part are made, often with detriment to the meaning. Thus is noted, at the beginning of section 3, the omission of the sentence "In elementary arithmetic, multiplication is considered a second way, etc." Again in Section 3, just preceding Theorem I, the words "double algebra" as used by Burkhardt are omitted entirely. Also the sentence following Theorem IV in Section 28 is omitted. Theorem V in Section 28 is quoted as in former editions except that "stetig" is used instead of "integrierbar",—an unnecessary change in view of the previous theorem. Such changes throughout the book remind one of the attempts to substitute a different bass in our traditional "Home Sweet Home." They may be musically correct but we no longer have "Home Sweet Home."

The numbering and naming of the various sections and paragraphs, except some rearrangements which seem ill advised and of doubtful value, remain practically the same. For example, analytic continuation is treated in Section 39a in the elementary development of power series instead of in the discussion in chapter VI where it seemed to fit into the general scheme of treatment. No figures are given as in previous editions, showing the domains concerned in the treatment of this topic.

In Chapter III on the real variable, many changes, additions, etc. have been made without destroying the general form and make up of the chapter. These,

on the whole, seem to have improved somewhat this short introduction to the theory of the real variable.

In general, the paging is poorly done. An example is page 4 where the equations are set too close. This contrasts sharply with the previous editions. The paper and ink are also of poor quality.

S. E. RASOR

Synthetische Zahlentheorie. By RUDOLPH FUETER. Second Edition, Revised. Berlin and Leipzig, Walter de Gruyter & Co., 1925. vii+277 pp.

The first edition of this work appeared in 1917. The present edition differs from the first in several respects. First, the exponential function is now introduced from the standpoint of the theory of functions of a complex variable; also, a complete proof is given of the existence of fundamental units in a cyclotomic field. Likewise, the law of cubic reciprocity is completely proved. The first edition included the demonstration of a special case only.

The book consists of two main divisions, the theory of rational fields and the theory of cyclotomic fields. The present review will be concerned with the first part only, covering the first sixty-seven pages of the text. In this the author introduces an innovation, as he develops the theory of rational fields entirely from the standpoint of the theory of algebraic fields. For example, in lieu of the divisibility of rational integers, the divisibility of ideals is considered and the elementary theory of rational integers and congruences is set forth entirely in terms of the language and concepts of algebraic number theory as applied to the rational field.

A modul is defined as a system of numbers with the property that the sum and difference of any two numbers therein is equal to a number of the system. An ideal is defined as a modul such that all numbers contained in it are rational integers. It is then shown by a process equivalent to the Euclidean algorithm that any ideal of the form $mx+ny$ where x and y range over all rational integers is equal to an ideal rz where z ranges in the same way, m, n and r being rational integers also. This theorem leads easily to the theorem that any ideal has an unique decomposition into prime ideal factors which result takes the place of the ordinary theorem concerning the unique expression of a rational integer as the product of prime factors.

Following the concept introduced into the theory of algebraic numbers by Weber and the author, a ray (Strahl) is defined as a system of rational numbers such that the product and quotient of two numbers therein also belong to the system. It is shown that every rational integer defines a ray of numbers of the form $1+xny^{-1}$ where x ranges over all rational integers and y ranges over all integers prime to n.

In analogy with the theory of ideal classes in an algebraic field, the author then defines the equivalence of two ideals with respect to the modul $\mathfrak{F}$. The symbol mod. $\mathfrak{F}$ stands for the modul consisting of all numbers of the form xfy^{-1} where x ranges over all rational integers and y over all integers prime to f, considering a ray defined by the integer f which consists of unity plus the modul $\mathfrak{F}$. As all ideals can be reduced to the form nx where n is given and x ranges over all rational integers, we write $nx=(n)$.

Two ideals $\mathfrak{N}_1=(n_1)$, $\mathfrak{N}_2=(n_2)$, $n_1>0, n_2>0$, are called equivalent with respect to the modul $\mathfrak{F}$ when the quotient n_2/n_1 is a member of the ray defined by f. From these definitions the laws of equivalence for ideals mod $\mathfrak{F}$ are worked out. It will be seen that the equivalence of two ideals $\mathfrak{N}_1$ and $\mathfrak{N}_2$ corresponds to the condition that n_1 and n_2 belong to the same residue class modulo f.

Exactly in line with the theory of algebraic numbers, the rational number a is said to be congruent to zero modulo $\mathfrak{F}$ if a belongs to the modul $\mathfrak{F}$ and the notation

$$a \equiv 0(\mathrm{mod.}\,\mathfrak{F})$$

is used. One advantage of this definition over that employed in ordinary elementary number theory is that fractions are used in congruences in addition to integers. Persons who have had considerable experience in dealing with congruences will realize the convenience of including fractions as one of the concepts used in their definition.

Some of the properties of congruences are then derived by the author.

On page 40 the author begins the consideration of the theory of the abstract finite groups, devoting eight pages to the development of some of the principles of that theory together with the special case of substitution groups. These results are then used to prove Fermat's theorem, and to obtain the properties of primitive roots and indices.

Fueter then associates with every ideal $\mathfrak{N}=(n)$, prime to l, a permutation as follows,

$$\pi(\mathfrak{N}) = \begin{pmatrix} 1 & 2 & 3 \cdots & l-1 \\ n_1 & n_2 & n_3 \cdots & n_{l-1} \end{pmatrix},$$

where the n's are the least positive residues of n, $2n$, $3n$, $\cdots$, $(l-1)n$, modulo the prime l. This notion was first used by Zolotareff in 1872 in connection with a certain proof of the law of quadratic reciprocity. The author employs it in various ways, in particular to a derivation of the quadratic character of 2. (p. 65). The law of quadratic reciprocity is not proved in this part of the text, but is obtained as an application of the theory of cyclotomic sub-fields. (p. 258).

It would have been entirely in keeping with the spirit of this text if some account had been given of the theory of finite fields, developed by Moore and Dickson, for the particular case of residue classes with respect to a prime modulus.

Many numerical examples are given illustrating the general theorems and the treatment is quite clear and systematic. After a person has digested the first sixty-eight pages of this book, he is in a position to go into the theory of algebraic numbers without any abrupt modification of the ideas he has obtained concerning rational integers. I think the book will be very valuable in spreading a knowledge of some of the modern arithmetical theories, in particular several which have been developed within the last thirty years, notably by the author himself.

H. S. VANDIVER

The Law of Diminishing Returns. By W. J. SPILLMAN and EMIL LANG. Yonkers, N.Y., World Book Company, 1924.

The book is divided into two parts, "The Law of The Diminishing Increment," by Dr. W. J. Spillman, and the "Law of The Soil", by Dr. E. Lang. The latter is a translation of the original article in the Landwirthschaftliche Jahrbücher, vol. 55 (1920).

For a great many years, scientists were unable to reconcile the results they obtained with successive applications of fertilizer by experimental trial with the well known law of diminishing returns. They insisted that with additional quantities of plant food, there was a proportional increase in the yield of crops. If the quantity of fertilizer was doubled, crop yields doubled. Gradually the "law of the soil" was evolved, which stated that increasing applications of fertilizer did not result in proportional increases in crop yields. Later the "law of the minimum" was evolved. According to this law, the yield was supposed to be determined wholly by the growth factor farthest removed from optimum conditions.

These questions resulted in an enormous amount of research work, especially in Germany. From 1839 to 1909, chemists tried to reconcile these laws. Finally, Dr. Mitscherlich, of the Königsberg Experiment Station, in a series of experiments showed that the law of minimum was wrong, and that the law of diminishing returns was the true relationship.

Dr. Spillman, using the results of published data on fertilizer experiments, fattening hogs, steer feeding, growth of children, growth of capons, developed mathematical expressions (decreasing geometric curves) of the law of diminishing returns.

The book is of interest to the natural scientist, the economist and the mathematician. It presents the gradual evolution of the problem, indicating the enormous amount of time, work and controversy that is necessary to finally evolve a principle. Progress is a slow and painful process.

F. A. PEARSON

Plane Curves of the Third Order. By HENRY S. WHITE. Published with the cooperation of the National Research Council. Cambridge, Harvard University Press, 1925. xii+168 pp. Price $2.75.

This treatise deals with cubics exclusively from the projective point of view. The purpose of the author is twofold, to develop the projective properties of the cubic and to introduce the subject of higher plane curves through the study of the cubic. In former treatises on cubic curves, general curves are first discussed and their properties later specialized for the cubic. In many cases the discussion applied directly to the cubic is much simpler than that for the general curve. In such cases by adopting this method, the author, in discussing properties of the cubic, gives a simple introduction to the study of similar properties of curves of higher order as well.

After a short introductory chapter on the real branches of the cubic, the intersections of two cubics and related properties are treated with great rigor. Then follow polar curves, the Hessian, the Cayleyan, poloconics, inflections, invariants, cross-ratio, conjugate points, the general net of conics as a polar net, all treated algebraically. The generation of a cubic, including the methods of Chasles, Grassmann, Schroeter and others, is fully discussed.

The next twenty seven pages contain the most valuable contribution of the book. Noether's fundamental theorem and the use of elliptic functions as parameters are here applied to the cubic, thus greatly simplifying an intricate portion of curve theory. The book closes with a discussion of covariant curves and apolar curves.

Even the projective properties of the cubic are too numerous to be fully covered in a brief treatise. Having decided upon brevity, the author did well in choosing to treat fully the more important properties rather than to cover a wider field less thoroughly. The topics included are well chosen, but it does not seem that a desire for brevity should be permitted to exclude from a book of this kind such topics as linear systems of cubics and point correspondences on a cubic that leave the curve invariant.

The table of contents is fully itemized and there is a serviceable index. The printing is excellent and remarkably free from typographical errors.

The book is a worthy contribution to the projective theory of cubic curves and an excellent introduction to certain important portions of the general theory of algebraic curves. It is of especial interest to those working in algebra of invariants, as the treatment is largely by that method.

TEMPLE RICE HOLLCROFT

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical

papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Bulletin of the American Mathematical Society, volume 33, no. 3, May-June 1927: "The Gibbs lecture for 1926: Mathematics and the Biological Sciences" by H. B. Williams, 273–294; "On a generalization of the secular equation" by J. Pierpont, 294–296; "A generalized two-dimensional potential problem" by J. N. Carson, 296–299; "A generalization of Waring's theorem on nine cubes" by L. E. Dickson, 299–301; "An elementary proof by mathematical induction of the equivalence of the Cesàro and Hölder sum formulas" by T. Fort, 301–305; "Some properties of continuous curves" by G. T. Whyburn, 305–309; "The dual of a logical expression" by B. A. Bernstein, 309–311; "The Heaviside operational calculus" by H. W. March, 311–319; "Extensions of Waring's theorem on fourth powers" by L. E. Dickson, 319–327; "Tests for primality by the converse of Fermat's theorem" by D. H. Lehmer, 327–341; "On analytic solutions of differential equations in the neighborhood of non-analytic singular points" by B. O. Koopman, 341–352.

Journal of Mathematics and Physics, Massachusetts Institute of Technology, volume 6, no. 3, April 1927: "The spectrum of an array and its applications to the study of the translation properties of a simple class of arithmetical functions" (Part One) by N. Weiner, 145–158; "On the translation properties of a simple class of arithmetical functions" (Part Two) by K. Mahler, 158–164; "The expression of a tensor or a polyadic as a sum of products" by F. Hitchcock, 164–189.

Mathesis, volume 41, no. 4, April 1927: "Les points hessiens d'un triangle" by F. D. Murnaghan 155–161.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 4, April 1927: "On the expansion of harmonic functions in terms of harmonic polynomials" by J. L. Walsh, 175–180.

Transactions of the American Mathematical Society, volume 29, no. 2, April 1927: "Singular case of pairs of bilinear, quadratic, or Hermitian forms" by L. E. Dickson, 239–254; "Triads of ruled surfaces" by A. F. Carpenter, 254–276; "Certain uniform functions of rational functions" by E. P. Starke, 276–287; "On rejection to infinity and exterior motion in the restricted problem of three bodies" by B. O. Koopman, 287–332; "A connected and regular point set which has no subcontinuum" by N. L. Wilder, 332–341; "Meromorphic functions with addition or multiplication theorems" by J F. Ritt, 341-361; "Real functions with algebraic addition theorems" by J. F. Ritt, 361–369; "Concerning continua in the plane" by G. T. Whyburn, 369–401; "Extremals and transversality of the general calculus of variations problem of the first order in space" by J. Douglas, 401–431; "A figuratrix for double integrals" by P. M. Rider, 421–429; "Manifolds with a boundary and their transformations" by S. Lefschetz, 429–463; "On sets of functions of a general variable" by L. L. Dines, 463–470.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3272. Proposed by L. M. Berkeley, New York City.

Solve the differential equation,

$$(\tan x + m \sin y)\, dy = (\sin y - m \tan x) \cos y\, dx,$$

where m is a constant.

3273. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The three points of intersection of the symmedians of a triangle with its circumcircle determine a triangle whose circles of Apollonius are identical with those of the given triangle.

3274. Proposed by C. N. Mills, Normal, Illinois.

The axes of three mutually perpendicular right circular cylinders intersect in a common point. If the radii are equal, find the common volume.

3275. Proposed by J. A. Shohat, University of Michigan.

Solve the differential equation,

$$xy\frac{dy}{dx} = y^2 + a^2 - ax \pm [(y^2 + a^2 - ax)^2 - y^2\,(y^2 + a^2 - 2ax)]^{1/2}.$$

3276. Proposed by L. L. Silverman and J. Tamarkin, Dartmouth College.

Prove that if ν is an integer greater than or equal to 1, then

$$\int_0^\infty \frac{(1+z)^{-\nu}dz}{\log^2 z + \pi^2} = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu}dt, \text{ where } \binom{t}{\nu} = \frac{t(t-1)\cdots(t-\nu+1)}{\nu!}.$$

3277. Proposed by R. H. Sciobereti, Berkeley, California.

Find the motion of a weighing particle of mass 1 moving in a vertical plane and attracted by a point of this plane with a force k/r^2, k being a positive constant. Consider applications of the result to the two special cases defined by the following initial conditions: (i) at $t=t_0$, the coordinates of M are $(0, b)$, $b>0$, and velocity, given by $v_0^2=2(gb^2+k)/b$, is parallel to the axis of x. (ii) at $t=t_0$ the coordinates of M are $(0, b)$, with $(k/g)^{1/2}<b<0$, and $v_0^2=2(gb^2-k)/b$, with v_0 parallel to the axis of x.

What can be said of the stability of the motion in these two particular cases?

3278. Proposed by J. V. Uspensky, Carleton College.

For every set of real values $\xi_1, \xi_2, \cdots, \xi_n$ let the function $\phi(\xi_1, \xi_2, \cdots, \xi_n)$ be defined as the greatest of $|\xi_1-h_1|$, $|\xi_2-h_2|$, $\cdots$, $|\xi_n-h_n|$, where $h_1, h_2, \cdots, h_n$ are given real constants. Find the integral values of $\xi_1, \xi_2, \cdots, \xi_n$, which minimize $\phi(\xi_1, \xi_2, \cdots, \xi_n)$, while their sum is equal to a given number N, so that the following condition holds:

$$\xi_1+\xi_2+\cdots+\xi_n = N.$$

3279. Proposed by J. Rosenbaum, Milford, Conn.

Given n points, $P_1, P_2, \cdots, P_n$, and a ratio, λ, to construct an n-gon $A_1, A_2, \cdots, A_n$, such that each P_i lies on the side A_iA_{i+1} and the ratio A_iP_i/P_iA_{i+1} shall be equal to λ ($i=1, 2, \cdots, n$, and $A_{n+1}=A_1$).[1]

SOLUTIONS

3194[3182; 1926, 278]. Proposed by D. H. Lehmer, University of California.

Prove the following two theorems and show how they may be used to advantage in finding the factors of R.

THEOREM 1. Let R be a non-square integer of the form $8n+k$ and represent by $2^\alpha(2m+1)$ any even denominator of a complete quotient occurring in the expansion of $\sqrt{R}$ in a continued fraction. Then, if $k=1$, $\alpha\geqq 3$; if $k=4$ or 0, $\alpha=2$; if $k=5$, $\alpha=2$; and if $k=2$, 3, 6, or 7, $\alpha=1$.

THEOREM 2. If R contains a square factor C^2, then every multiple of C appearing as a denominator of a complete quotient must also contain a factor C^2.

---

[1] This is a generalization of problem 3128 (1925, 204). The solution by J. P. Ballantine (1926, 164) for the case $\lambda=1$, is, with a slight modification, applicable to this more general case.

SOLUTION BY THE PROPOSER.

PROOF OF THEOREM 1. Suppose, first, that R is odd; that is, $k=1, 3, 5$, or 7. Let A/B be the convergent to $\sqrt{R}$ which precedes the partial quotient under consideration. Then we have the well known equality:

$$A^2-RB^2=\pm 2^\alpha(2m+1).$$

Since A and B are mutually prime, they cannot both be even. From the fact that R is odd and the right hand member of the above equation is divisible by 2, it follows that A and B are both odd. Then A^2 and B^2 are of the form $8n+1$. From this we have:

$$A^2 \equiv 1 \pmod{8},\ RB^2 \equiv k \pmod{8},\ \mp 2^\alpha(2m+1) \equiv k-1 \pmod{8}.$$

Hence we have:

$$\text{for } k=1,\ \ \alpha \geqq 3;\ \ \text{for } k=5,\ \ \alpha=2;\ \ \text{for } k=3 \text{ or } 7,\ \ \alpha=1.$$

A similar proof holds for an even R in which case A is even and B is odd.

PROOF OF THEOREM 2. Let $R=C^2n$ and let the denominator in question be Cm. Then we have:

$$A^2-C^2nB^2=\pm Cm.$$

Since the right hand member is divisible by C, the left side must also contain a factor C. This means that A^2 is divisible by C. But in that case A^2 contains the factor C^2. (See the note at the end of this solution). Hence the entire left hand member is divisible by C^2 and this must be likewise true of the right side. That is, the denominator of the complete quotient contains the factor C^2, which proves the theorem.

These theorems are useful when we wish to factor R by the method of quadratic residues. The denominators of the complete quotients taken with the proper sign are quadratic residues of R. These residues are normally too large to use but smaller ones can be found by making use of the fact that if $\alpha^2\beta$ is a residue of R then β is a residue. Denominators containing square factors are therefore much preferred. If R is included in one of the forms $8n+0$, 1, 4, 5, every even denominator will contain the square factor 4. If R does not belong to one of these forms it can easily be made to do so by multiplying by some small factor.

NOTE It should be pointed out that in the proof of THEOREM 2 we tacitly assumed that C itself does not contain a square factor. In case it does, the theorem is not true. In this case however we obtain a denominator to a complete quotient which contains a square factor, which we can reject.

3207 [1926; 385]. Proposed by C. N. Mills, Normal, Illinois.

Prove that $\frac{1}{4}m^2\sqrt{3}$ is the maximum area of a triangle which can be formed with the lines a, b, c, subject to the condition that $a^3+b^3+c^3=3m^3$.

SOLUTION BY OTTO DUNKEL, Washington University.

The proof of a generalization of the theorem in this problem may be based upon certain elementary theorems of algebra and geometry which are more important than the theorem of the problem. Proofs of these theorems will be given; they are simple enough to be employed in elementary instruction. All of the theorems relating to three quantities are easily extended by the same reasoning to any given number of quantities.

THEOREM I. *If n is a positive integer greater than unity, x is any positive quantity, and d is positive or negative but not zero and such that $x+d$ is positive, then*

$$(x+d)^n > x^n + ndx^{n-1}. \tag{1}$$

For $n=2$ this is obviously true since then the left side exceeds the right by d^2. If each side of the inequality for $n=2$ is multiplied by $x+d$, it will follow that it is true for $n=3$, and this process continued leads to the theorem above.

THEOREM II. *If a, b, c are any three positive quantities,*

$$[(a+b+c)/3]^3 \geqq abc \tag{2}$$

where the equality sign is used if and only if $a=b=c$.

If a, b, c are not all equal, we may suppose that they are written in ascending order of magnitude. If we set $a+b=2m$, then $c>m$; and we may set $c=m+d$, $d>0$. The left side of (2) becomes then $(m+\frac{1}{3}d)^3$, and by (1) this is greater than $m^3+m^2d=m^2c$. But $m^2c\geqq abc$, for $m^2=ab+(a-b)^2/4\geqq ab$.

THEOREM III. *If a, b, c are any positive quantities and n is a positive integer greater than unity*

$$(a^n + b^n + c^n)/3 \geqq [(a + b + c)/3]^n, \tag{3}$$

where the equality sign is used if and only if $a=b=c$.

Set $a+b+c=3m$, $a=m+a'$, $b=m+b'$, $c=m+c'$, where $a'+b'+c'=0$. By (1), $a^n\geqq m^n+na'm^{n-1}$, $b^n\geqq m^n+nb'm^{n-1}$, $c^n\geqq m^n+nc'm^{n-1}$, where the equality sign is used in all three cases if and only if $a=b=c$. Hence the left side of (3) is greater than $m^n+\frac{1}{3}nm^{n-1}(a'+b'+c')=m^n$ which is the right side of (3).

THEOREM IV. *The equilateral triangle has the greatest area of all triangles having the same length of perimeter.*

Let the given perimeter be $2s=a+b+c$, and let S be the area of the triangle with these sides. From geometry we have $S^2=s(s-a)(s-b)(s-c)$. Then by (2), if a, b, c are not all equal,

$$(s - a)(s - b)(s - c) < [(s - a) + (s - b) + (s - c)]^3/27 = s^3/27.$$

Hence $S^2<s^4/27$, or S is less than $s^2/3^{\,3/2}$, the area of the equilateral triangle of side $2s/3$.

THEOREM V. *If the sides of a triangle satisfy the equality*

$$a^n + b^n + c^n = 3m^n, \tag{4}$$

where n is a positive integer greater than unity, the equilateral triangle of side m has the greatest area.

For suppose that a, b, c are unequal and satisfy (4), and let S be the area of the corresponding triangle. Set $a+b+c=3p$, and let E_p be the area of the equilateral triangle of side p. Then by IV, $S<E_p$; and, by (3) and (4), $m^n>p^n$, or $m>p$. Hence, if E_m is the area of the equilateral triangle of side m, $E_m>E_p>S$.

This completes the proof of one generalization of the problem.

If we suppose that the sides of the triangle satisfy a relation similar to (4) where the exponent is $1/n$, $n=$ any positive integer, the above reasoning does not apply. But this case may be treated by replacing IV by another geometric theorem and by extending III. We shall first extend III.

THEOREM VI. *If a, b, c are positive and n is any positive integer greater than unity*

$$\left(\frac{a^n + b^n + c^n}{3}\right)^{1/n} \geqq \frac{a + b + c}{3} \geqq \left(\frac{a^{1/n} + b^{1/n} + c^{1/n}}{3}\right)^n \geqq (abc)^{1/3} \tag{5}$$

where the equality signs are used if and only if $a=b=c$.

The last inequality follows by replacing a, b, c by $a^{1/n}$, $b^{1/n}$, $c^{1/n}$ in (2). The middle inequality follows from (3) by the same substitution while the first one is (3) with a slight change.

THEOREM VII. *Of all triangles inscribed in a circle of given radius, the equilateral triangle has the greatest area.*

First consider all inscribed triangles having a given chord, not a diameter, for base. It is easily shown that the isosceles triangle having the given base and containing the center of the circle in its interior has the greatest area.

Let S be the area of an inscribed triangle which is not equilateral, and let c' be the shortest side. Then the isosceles triangle $A'B'C$, $A'B'=c'$, which contains the center, has an area $S'>S$, or $S'=S$ if the two are the same triangle. We shall now compare the area E of the inscribed equilateral triangle with S' by a figure. Let M' be the middle point of $A'B'$, and $COM'D$, the diameter of the circle dividing S' into equal parts, where O is the center of the circle. Draw through the middle point M of OD the chord AMB perpendicular to OD. Then CAB is an inscribed equilateral triangle. Let CA' cut AM in K, and take L on AC so that $AL=AK$. Then LAK is equilateral. Draw the chord AA'. Then $\angle ALK=\angle AA'K=60°$, while $\angle KAA'<60°$ and $\angle AKA'>60°$. The two triangles LAK and $A'AK$ may be inscribed in the same circle with the same base AK (if one of them is turned over about the side AK). Hence, by the first part of the proof, the area of the first is greater than that of the second. Hence the altitude KN of the first is greater than the altitude $A'P$ of the second. Since $CA=2AM$, the area CAK is $AM\cdot NK>AM\cdot PA'=$ area $MKA'M'+$ area $KAQA'$, where Q is the fourth vertex of the rectangle $AMM'Q$. Hence $E-S'>2KAQA'$, and therefore $E>S'\geqq S$, and this completes the proof.

THEOREM VIII. *Of all triangles whose sides satisfy the equality*

$$abc = m^3, \tag{6}$$

the equilateral triangle with the side m has the greatest area.

For, from elementary geometry, if R is the radius of the circumscribing circle of the triangle with unequal sides satisfying (6) and if S is its area, $abc=4RS=m^3$. Let E be the area of the equilateral triangle of side e inscribed in the same circle, then by VII $E>S$, and then $e^3=4RE>4RS=m^3$, or $e>m$. Let E_m be the area of the equilateral triangle of side m and let R_m be the radius of its circumscribing circle. Then from the last inequality, $R>R_m$. But $4RS=m^3=4R_mE_m$, and hence $S<E_m$.

THEOREM IX. *Of all triangles whose sides satisfy the equality*

$$a^r + b^r + c^r = 3m^r, \tag{7}$$

where r is a positive integer or the reciprocal of a positive integer the equilateral triangle of side m has the greatest area.

For, let S be the area of a triangle with unequal sides which satisfy (7), and let $abc=t^3$. Then, if E_t is the area of the equilateral triangle with side t, $E_t>S$ by VIII. Also, by (7) and (5), $m=[(a^r+b^r+c^r)/3]^{1/r}>(abc)^{1/3}=t$, or $m>t$. Hence, if E_m is the area of the equilateral triangle of side m, $E_m>E_t>S$. This includes a second proof of IV which does not use the area formula in the previous proof.

THEOREM I may be extended by elementary algebraic methods to any real exponent α greater than unity. It is then easy to show that $[(a^\alpha+b^\alpha+c^\alpha)/3]^{1/\alpha}$ decreases when α decreases, always remaining greater than $(abc)^{1/3}$, if a, b, c are not all equal.

Also solved by H. L. SLOBIN, MICHAEL GOLDBERG, and J. B. REYNOLDS.

3210[1926; 385]. Proposed by Thurman Andrew, University of North Dakota.

Find the general solution of the probability of throwing any number with any number of dice. Dice are here taken to be polyhedrons, with any number, k, of faces numbered consecutively from 1 to k. (Assume all faces equally likely to turn up.)

SOLUTION BY MICHAEL GOLDBERG, Washington, D.C.

Expand $(x+x^2+x^3+\cdots+x^k)^n$, $n<k$. Then the coefficient of x^s is the number of permutations of 1, 2, 3, 4, $\cdots$, k, taken n at a time, which have the sum s, every sum containing n terms. The above expression may be written as $x^n(1-x^k)^n(1-x)^{-n}$. The coefficient of x^{s-n} must then be found in the expansion of

$$\left\{1 - nx^k + \binom{n}{2}x^{2k} - \binom{n}{3}x^{3k} + \cdots\right\}\left\{1 + nx + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \cdots\right\}.$$

The coefficient of x^{s-n} is obtained as follows: Multiply the first term of the first factor by $\binom{s-1}{s-n}x^{s-n}$ of the second factor; the second term of the first factor by $\binom{s-k-1}{s-n-k}x^{s-n-k}$ of the second factor; the third term of the first factor by $\binom{s-2k-1}{s-n-2k}x^{s-n-2k}$ of the second factor, etc. The coefficient of x^{s-n} then equals

$$\binom{s-1}{s-n} - \binom{n}{1}\binom{s-k-1}{s-n-k} + \binom{n}{2}\binom{s-2k-1}{s-n-2k} - \cdots, \text{ which may be written}$$

$$\frac{(s-1)!}{(s-n)!(n-1)!} - \binom{n}{1}\frac{(s-k-1)!}{(s-n-k)!(n-1)!} + \binom{n}{2}\frac{(s-2k-1)!}{(s-n-2k)!(n-1)!} - \cdots.$$

Since the total number of cases is k^n, the probability of throwing the number s with n dice of k sides each is then

$$P_s = \frac{1}{k^n}\left\{\frac{(s-1)!}{(s-n)!(n-1)!} - \binom{n}{1}\frac{(s-k-1)!}{(s-n-k)!(n-1)!} + \binom{n}{2}\frac{(s-2k-1)!}{(s-n-2k)!(n-1)!} - \cdots\right\}.$$

Also solved by J. M. Barbour and the Proposer.

3211[1926; 385]. Proposed by J. A. Bullard, U. S. Naval Academy.

Find by integration the area of the ellipse $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Solution by H. L. Slobin, University of New Hampshire

Solving the given equation for y, we have for the difference of the two solutions,

$$y_2 - y_1 = 2[(h^2 - ab)x^2 + 2(hf - bg)x + (f^2 - bc)]^{1/2}/b. \tag{1}$$

Since the curve is a real ellipse, the quadratic in x within the brackets must have two distinct real roots, r_1 and r_2 where $r_2>r_1$, and for values of x between the roots the quadratic must be positive. From this it follows that the coefficient h^2-ab must be negative and that $b(af^2+bg^2+ch^2-2fgh-abc)$ must be positive.

Hence the required area is given by

$$A = \frac{2(ab - h^2)^{1/2}}{b}\int_{r_1}^{r_2}[(x - r_1)(r_2 - x)]^{1/2}dx. \tag{2}$$

If we set $x-r_1=(r_2-r_1)\sin^2\theta$, $r_2-x=(r_2-r_1)\cos^2\theta$, we obtain by elementary methods

$$\mathrm{A} = \frac{\pi(af^2 + bg^2 + ch^2 - 2fgh - abc)}{(ab - h^2)^{3/2}}. \tag{3}$$

Also solved by Theodore Bennett, J. M. Earl, J. S. Georges, Michael Goldberg, J. L. Riley, W. H. Richert, J. B. Reynolds, Victor d'Unger, and the Proposer.

3213[1926; 429]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Prove the proposition: If AB, CD are two harmonic segments, the harmonic conjugate of the middle point of AB with respect to the couple C, D is identical with the harmonic conjugate of the middle point of CD with respect to the couple A, B. Generalize.

I. Solution by F. D. Murnaghan, Johns Hopkins University.

Naming points by means of a complex variable, four points (a, b, c, d) are harmonic when

$$(c - a)/(c - b) = -(d - a)/(d - b) \quad\text{or}\quad ab - \tfrac{1}{2}(a + b)(c + d) + cd = 0.$$

The harmonic conjugate of any point, x, with respect to the pair (a, b) is, accordingly,

$$y = [x(a + b)/2 - ab]/[x - \tfrac{1}{2}(a + b)]$$

and, in particular, the harmonic conjugate of the middle point $(c+d)/2$ of the segment CD is

$$[(a + b)(c + d) - 4ab]/2(c + d - a - b)$$

and, by the use of the hypothesis that the pair (c, d) is harmonic to the pair (a, b), this becomes $(cd-ab)/(c+d-a-b)$. An interchange of the rôles played by the pairs (a, b) and (c, d) does not affect this point, which proves the theorem.

An obvious generalization is suggested by the fact that the middle points of the segments AB and CD are the harmonic conjugates of the point at infinity with respect to the corresponding pairs (a, b) and (c, d). If we take any point x and its two conjugates y and z in the harmonically conjugate pairs (a, b) and (c, d), respectively, the harmonic conjugate of y with respect to the pair (c, d) coincides with the harmonic conjugate of z with respect to the pair (a, b). A calculation similar to that given above identifies this point as

$$[x(ab - cd) - ab(c + d) + cd(a + b)]/[x(a + b - c - d) + cd - ab].$$

II. Solution by the Proposer.

The two circles (P), (Q) having the segments AB, CD for diameters are orthogonal; hence the polar of the center P of (P) with respect to (Q) is the common chord XY of the two circles, and $Z\equiv(XY, PQ)$ is the harmonic conjugate of P with respect to C, D. For analogous reasons Z is the harmonic conjugate of Q with respect to A, B, which proves the proposition.

Remarks: I. The circles (P), (Q) being orthogonal, we have

$$PQ^2 = PX^2 + QX^2 = (AB/2)^2 + (CD/2)^2,$$

i.e.: *The distance between the mid-points of two harmonic segments is equal to half the diagonal of the rectangle having the two segments for its sides.*

II. This property gives an immediate solution of the following problem: *To place two segments of given lengths on a given line so that they shall be harmonic.*

III. This last problem may also be stated as follows: *In an hyperbolic involution to find a pair of conjugate points which shall be a given distance apart.*

Also solved by R. E. Morris.

Note by Otto Dunkel, Washington University.

The generalization of the first solution may also be proved as follows: Each of the five sets of points on the same line, $(ACBD)$, $(AxBy)$, $(CxDz)$, $(CyDu')$, $(AzBu'')$ is harmonic. If x moves on the line while A, B, C, D remain fixed we have the projective ranges $u' \barwedge y \barwedge x \barwedge z \barwedge u''$. If $x=A, B, C, D$, then $y=A, B, D, C$, $z=B, A, C, D$ and $u'=B, A, D, C=u''$. Hence the two projective ranges of points u' and u'' are identical.

A second proof is obtained by considering the circles P, Q, Y, Z, Y' with diameters, respectively, AB, CD, xy, xz, yu'. Then Q is orthogonal to P, Z, Y' and P is orthogonal to Y. If this system is inverted with respect to Q, the circles P, Z, Y' invert each into itself, while Y inverts into a circle orthogonal to P and tangent to Y' and Z at u' and z. Hence $(AzBu')$ is harmonic.

3214[1926; 429]. Proposed by the late Laenas G. Weld.

A block sliding without friction and a ball rolling without friction start together down an inclined plane with the same initial velocity: Determine their subsequent relative motion.

Solution by J. B. Reynolds, Lehigh University.

I doubt if this problem is worded as Mr. Weld intended it to be. If the block and the ball both move without friction there will evidently be no relative motion. If, however, the block slides without friction and the ball rolls without slipping on a plane of inclination A the acceleration a_1 of the block is $g \sin A$ and the acceleration a_2 of the ball is $(5/7)g \sin A$. Then the relative acceleration a is given by $a=a_1-a_2=(2/7)g \sin A$. The relative velocity $v=(2/7)g \sin A\ t$ and the relative displacement, s, or the distance between the ball and the block at any time is $s=(1/7)g \sin A t^2$.

3215[1926; 429]. Proposed by R. M. Mathews, University of Illinois.

When a quadrangle is inscribed in a central conic so that two of its opposite sides pass through the foci, then the tangent pairs at points one on each of these sides meet on the bisectors of the angles formed by the sides.

When a quadrangle is inscribed in a central conic so that two of its opposite sides are symmetric with respect to the bisectors of the angle subtended at their intersection by the foci, then the tangents at point pairs, one on each of those sides, meet on the said bisectors.

Dualize. Also modify for parabola.

Solution by Nathan Altshiller-Court, University of Oklahoma.

A point L of the plane of a conic (S) is the center of an involution of pairs of rays conjugate with respect to the conic. If L lies at a finite distance and is not a focus of (S) this involution has one and only

one rectangular pair of elements r, r'. The lines r, r' are the bisectors of the angles formed by the lines LF, LF' joining L to the foci F, F' of (S), and therefore separate these lines harmonically.

The two pairs of points A, B; C, D in which the lines LF, LF' joining L to the foci F, F' of (S) meet the conic (S) determine a complete quadrangle inscribed in (S) whose diagonal points $L\equiv(AB, CD)$, $M\equiv(AC, BD)$, $N\equiv(AD, BC)$ are the vertices of a triangle LMN self-polar with respect to (S). Thus the lines LM, LN are conjugate with respect to (S) and separate harmonically the lines $LFAB$, $LF'CD$. But the lines r, r' fulfill the same two conditions, and there can be only one pair of lines through L having both of these properties, hence $LM\equiv r$, $LN\equiv r'$.

The point of intersection of the tangents to (S) at the points A, D is the pole of the line AD with respect to (S), and since AD passes through N, the pole of AD lies on the polar $LM\equiv r$ of N with respect to (S). This proves the first proposition.

Similarly we see that the tangents at B and C also meet on r. The tangents at A and C meet on the bisector r', also those at B and D.

Let LEI, LGH be a pair of lines through L symmetric with respect to r, r', and meeting (S) in the points E, I; G, H. These four points determine a complete quadrangle inscribed in (S) whose diagonal points are $L\equiv(EI, GH)$, $P\equiv(EG, IH)$, $Q\equiv(EH, IG)$. By exactly the same reasoning as above it may be shown that the lines LP, LQ coincide with the lines r, r', and furthermore that the tangents, say at E and H, to (S) intersect on LP; similarly the tangents at I and G meet on LP. This proves the second proposition.

In the case of the parabola one of the foci is at infinity and coincides with the center of the curve, i.e., is determined by the direction of the axis of the curve. But this special position of the focus does not affect either the propositions or the above considerations, so that it is not essential to modify the propositions of the problem, unless an implicit reference to a point at infinity seems particularly objectionable. In such a case it will be necessary to replace in the first proposition the words: "two opposite sides pass through the foci," by the words: "one of its sides passes through the focus and the opposite side is parallel to the axis" (of the parabola). Similarly for the second proposition.

The dual of the quadrangle $ABCD$ is the quadrilateral formed by four tangents to the conic (S) such that two opposite vertices X, X' lie on the polars f, f' of the foci, i.e. on the directrices of (S). The line $XX'\equiv l$ is then the dual of the point L, and the pair of points R, R' on l conjugate with respect to (S) and separated harmonically by the points X, X' are the duals of the lines r, r'. But the points R, R' do not play with regard to the other points of the line l any such special role as do the lines r, r' with regard to the lines through the point L, for the good reason that the orthogonal and bisecting properties of r, r', not being projective, are lost in the process of dualizing. The duals of the propositions of the problem do not therefore invite the same attention as the propositions themselves.

An interesting special case, namely, when the line l is tangent to (S), and which may, in a way, be regarded as a dual of the first proposition, was considered before in this Monthly (1923, p. 403).

3216[1926; 430]. Proposed by S. A. Corey, Des Moines, Iowa.

Let L, M, and N be any three unit vector space co-ordinates. Also let X, Y, and Z be three other vector space co-ordinates such that $X=a^2pL+acmM+c^2nN$, $Y=b^2pL+bdmM+d^2nN$, and $Z=2abpL+(ad+bc)mM+2cdnN$; a, b, c, d, p, m, and n being ordinary scalars. Then prove that 4(tensor of X) (tensor of Y) $(\cos \widehat{XY})-$(tensor of $Z)^2=(ad-bc)^2\,(4pn\cos \widehat{LN}-m^2)$.

Solution by T. C. Esty, Amherst College.

Since L, M, and N are unit vectors, $L^2=M^2=N^2=1$, and $N\cdot L=\cos \widehat{LN}$. Also

$$4(\text{tensor of } X)(\text{tensor of } Y)(\cos \widehat{XY}) = 4X\cdot Y.$$

Now

$$4X\cdot Y = 4a^2b^2p^2 + 4abcdm^2 + 4c^2d^2n^2 + 4(ad+bc)abpmL\cdot M + 4(ad+bc)cdmnM\cdot N + 4(b^2c^2+a^2d^2)npN\cdot L.$$

Also

$$Z^2 = 4a^2b^2p^2 + (ad+bc)^2m^2 + 4c^2d^2n^2 + 4(ad+bc)abpmL\cdot M + 4(ad+bc)cdmnM\cdot N + 8\,abcdnpN\cdot L.$$

Therefore,

$$4X\cdot Y - Z^2 = (ad-bc)^2(4pn\cos \widehat{LN} - m^2).$$

3218[1926; 480]. Proposed by F. M. Garnett, Savannah, Georgia.

A rectangular lighter $a=40$ ft. by $b=20$ ft. moves upstream due west, at the uniform speed $v_1=8$ miles per hour. What time will be required for a swimmer who begins at the southeast corner of the lighter to swim around it while it is in motion, if his rate upstream $v_2=10$ miles per hour, and downstream $v_3=16$ miles per hour?

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Since the man swims with a velocity $\frac{1}{2}(v_3+v_2)$ in still water, the stream flows with velocity $\frac{1}{2}(v_3-v_2)$, and the lighter moves with velocity v_1 upstream, we find the time required to go a distance b across-stream then overtake the lighter by swimming upstream to be $(b \tan \frac{1}{2}\theta)/(v_2-v_1)$, in which θ is the angle his velocity relative to the stream makes with the upstream direction. The time required to swim a distance b across in front of the lighter then back to it is $(b \cot \frac{1}{2}\theta)/(v_3+v_1)$. These results show that the least time will be consumed by the swimmer if he keeps always as near as possible to the lighter.

He therefore crosses in front and behind the lighter perpendicularily relative to the lighter, that is, he heads at arc sin $(v_3-v_2+2v_1)/(v_2+v_3)=\text{arc sin } 11/13$ with the back edge of the lighter and has a cross-stream velocity of $4\sqrt{3}$ mi. per hr. To swim across in front and behind the lighter therefore takes $2b/4\sqrt{3}$ hours. (In this solution, a and b are the dimensions of the lighter expressed in miles.) To gain on the lighter its length upstream takes $a/(v_2-v_1)=\frac{1}{2}a$ hours, and to swim downstream the distance a relative to the lighter takes $a/(v_3+v_1)=a/24$ hours. The total time required is $(13a/24)+(b/2\sqrt{3})$ hours $=18.7$ seconds.

Also solved by J. M. BARBOUR.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor E. W. BROWN of Yale University has accepted the invitation of the American Mathematical Society to give the fifth Josiah Willard Gibbs Lecture in connection with the meetings of the Society and the American Association for the Advancement of Science at Nashville, in December, 1927.

Dr. H. Y. BENEDICT, Dean of the College of Arts and Sciences, and Professor of Applied Mathematics and Astronomy, has been elected President of the University of Texas.

The following awards of Guggenheim fellowships for the coming academic year in the field of mathematical physics are announced: Dr. C. H. ECKERT, of the California Institute of Technology, for research in quantum theory; Professor G. E. GIBSON, of the University of California, for the study of the theory of band spectra; Dr. W. B. HOUSTON, of the California Institute of Technology, for the study of quantum mechanics as applied to the explanation of spectra; Dr. F. C. HOYT, of the University of Chicago, for research in quantum theory and its relation to radiation and atomic structure; Professor V. F. LENZEN, of the University of California, for the study of statistical mechanics; Professor M. S. VALLARTA, of the Massachusetts Institute of Technology, to study the

connection between Schrodinger's wave mechanics and the Einstein theory of radiation.

Princeton University has established a professorship in memory of CHARLES A. YOUNG, who was professor of astronomy at the University from 1887 to 1908. Professor H. N. RUSSEL, director of the Princeton Observatory, has been made the first incumbent.

Yale University plans to honor the memory of Josiah Willard Gibbs by the establishment of a Gibbs Fund of $250,000, the income of which will be devoted to work of the departments of chemistry, physics, and mathematics.

Dr. C. R. ADAMS of Brown University has been promoted to an associate professorship of mathematics.

Mr. W. B. CAMPBELL of Cornell University has been appointed assistant professor of mathematics at Colgate University.

Dr. C. C. CAMP of the University of Illinios has been appointed associate professor of mathematics at the University of Nebraska.

Professor TOBIAS DANTZIG of the University of Maryland lectured during the academic year 1926–1927 at the Bureau of Standards on the mathematical theory of elasticity.

Mr. VICTOR D'UNGER, of the actuarial department of the Pyramid Life Insurance Company, has been elected a member of the Calcutta Mathematical Society.

Professor W. C. EELS, of the department of applied mathematics at Whitman College, has been appointed associate professor of education at Stanford University.

Assistant Professor W. W. ELLIOTT of Duke University has been promoted to a full professorship of mathematics.

Associate Professor ARNOLD EMCH of the University of Illinois has been promoted to a full professorship of mathematics.

Professor TOMLINSON FORT of Hunter College, New York City, has been appointed professor of mathematics and head of the department of mathematics and astronomy at Lehigh University.

Dr. P. A. FRALEIGH of Cornell University has been appointed assistant professor of mathematics at the University of Vermont.

Assistant Professor C. IRVING GAVETT of the University of Washington has been promoted to an associate professorship of mathematics.

Dr. H. M. Gehman of Yale University has been promoted to an assistant professorship of mathematics.

Professor R. L. Green of Stanford University has retired.

Dr. C. M. Huber of Rutgers University has been promoted to an assistant professorship of mathematics.

Dr. L. Hulburt, collegiate professor of mathematics at Johns Hopkins University, has retired.

Assistant Professor Howard Justice of the University of Cincinnati has been promoted to an associate professorship of mathematics.

Associate Professors Joseph H. Kindle and Edward S. Smith of the University of Cincinnati have been promoted to full professorships of mathematics.

Assistant Professor R. E. Langer of Brown University has been appointed professor of mathematics at the University of Wisconsin.

Professor D. D. Leib, of the department of mathematics at the Connecticut College for Women, has been awarded the honorary degree of Doctor of Science by Dickinson College.

Dr. Harry Levy has been appointed assistant professor of mathematics at the University of Illinois.

Dr. Florence M. Mears of Cornell University has been appointed professor of mathematics at Alabama College.

Miss Muriel Metz has been appointed assistant professor of mathematics at the University of Cincinnati.

Dr. Oystein Ore of the University of Oslo has been appointed assistant professor of mathematics at Yale University.

Assistant Professor Bruce H. Redditt of Kenyon College has been granted leave of absence for a year's study in Johns Hopkins University.

Associate Professor W. D. Reeve of Teacher's College, Columbia University, has been promoted to a full professorship of mathematics.

Dr. Bernard P. Reinsch of the University of Illinois has been appointed associate professor of mathematics at the Southern Methodist University.

Dr. D. E. Richmond has been appointed assistant professor of mathematics at Williams College.

Mr. H. C. Shaub of Cornell University has been appointed assistant professor of mathematics at Washington and Jefferson College.

Dr. C. A. SHOOK has been promoted to an assistant professorship of mathematics at Columbia University.

Dr. MARION E. STARK of Wellesley College has been promoted to an assistant professorship of mathematics.

Mr. H. S. THURSTON of Brown University has been appointed professor of mathematics at Acadia University.

Assistant Professor L. A. H. WARREN of the University of Manitoba has been promoted to a full professorship of mathematics.

Mr. P. D. WILKINS of Case School of Applied Science has been appointed assistant professor of mathematics at Bates College.

Associate Professor W. A. WILSON of Yale University has been promoted to a full professorship of mathematics.

The following appointments for 1927–28 from the University of Chicago are announced:

1. Those having the master's degree.

Mr. H. C. BILLINGS, 1927, Instructor in Mathematics at the Oklahoma Agricultural and Mechanical College, Goodwell, Okla.

Mr. C. R. WORTH, 1927, Instructor in Mathematics at the University of Arkansas, Fayetteville, Ark.

Mr. M. G. BOYCE, 1926, Instructor in Mathematics at Adelbert College, Western Reserve University, Cleveland, Ohio.

Mr. H. H. PIXLEY, 1927, Instructor in Mathematics at Rutgers University, New Brunswick, New Jersey.

Mr. M. A. BASOCO, 1925, Instructor in Mathematics at the University of California at Los Angeles, California.

Miss GLADYS H. FREEMAN, 1920, Instructor in Mathematics at Illinois State Teachers College, DeKalb, Illinois.

2. Those holding fellowships, 1926–27.

Miss MARIE JOHNSON, Instructor in Mathematics at Pennsylvania State College, State College, Pennsylvaina.

Miss ROSA LEA JACKSON, Instructor in Mathematics at Leland Stanford University, Stanford University, California.

Mr. R. H. MARQUIS, Instructor in Mathematics at the University of Michigan, Ann Arbor, Michigan.

Mr. SAMUEL SILBERFARB, on special fellowship from the University of Manitoba, Instructor at the University of Akron, Akron, Ohio.

3. Those receiving the doctorate.

Miss Lois W. Griffiths, Ph.D., 1927, Instructor in Mathematics at Northwestern University, Evanston, Illinois.

Mr. R. G. Archibald, Ph.D., 1927, Instructor in Mathematics, Columbia University, New York City.

Mr. R. J. Garver, Ph.D., 1926, Instructor in Mathematics at Rochester University, Rochester, New York.

Miss Marguerite Darkow, Ph.D., 1924, recently instructor at the University of Indiana, Instructor at Pennsylvania State College, State College, Pennsylvania.

Mr. H. L. Smith, Ph.D., 1926, Assistant Professor of Mathematics at Louisiana State University, Baton Rouge, Louisiana.

Mr. H. R. Phalen, Ph.D., 1926, Professor of Mathematics, St. Stephens College, Annandale-on-Hudson, New York.

Mr. R. W. Barnard, Ph.D., 1926, at present National Research Fellow located at Princeton University, Assistant Professor of Mathematics, University of Chicago, Chicago, Illinois.

Mr. V. A. Tan, Ph.D., 1925, Professor of Mathematics, University of the Philippines, Manila, P. I.

Mr. F. R. Bamforth, Ph.D., 1927, Instructor at the University of Chicago, Chicago, Illinois.

Mr. Walter Bartky, Ph.D., 1926, Assistant Professor of Mathematical Astronomy, University of Chicago, Chicago, Illinois.

Mr. H. S. Everett, Ph.D., 1922, Extension Professor of Mathematics at the University of Chicago, in charge of home study and down town classes.

4. National Research Fellow located at Chicago.

Mr. L. H. McFarlan, Ph.D., 1925, University of Missouri, has been appointed to an assistant professorship at the University of Washington, Seattle, Washington.

The following appointments to instructorships in mathematics are announced:

Brown University, Mr. A. O. Hickson;

Cornell University, Mr. E. H. Hadlock, Mr. R. L. Jeffery, Mr. P. M. Swingle, Mr. F. G. Williams;

Hunter College, Miss Mary V. Kenny;

Kenyon College, Mr. Benedict Williams;

Yale University, Mr. H. T. Engstrom, Mr. T. H. Rawles.

TWO NOTABLE GIFTS TO THE ASSOCIATION

I. THE CARUS MATHEMATICAL MONOGRAPHS.

The entire expense for publishing and distributing these MONOGRAPHS is provided by MRS. MARY HEGELER CARUS as a gift to the ASSOCIATION. The sale of these books at cost to its members by the ASSOCIATION is thus made possible, and the receipts from such sales are used to build up an endowment fund of the ASSOCIATION to be known as the CARUS PUBLICATION FUND. Hence, when a member buys a CARUS MONOGRAPH he not only gets full value at minimum cost but he also contributes to a fund which will ultimately be of the utmost value to the ASSOCIATION. Can any member show good reason for not rendering this service to the ASSOCIATION? The first and second Monographs are still available to members at the cost price.

II. THE RHIND MATHEMATICAL PAPYRUS.

CHANCELLOR ARNOLD BUFFUM CHASE, of Brown University, who has repeatedly shown his vital interest in the Association by cash contributions to its depleted budget, has now made a notable gift which was fully explained in the September, 1926 issue of the MONTHLY. He has done the ASSOCIATION signal honor by publishing at great expense his RHIND MATHEMATICAL PAPYRUS under its auspices. The entire receipts from the sale of this work will be devoted to an endowment fund of the ASSOCIATION to be known as the ARNOLD BUFFUM CHANCE FUND. Individuals and institutions not now members of the ASSOCIATION may secure the special rate to members by making application for membership before the sale begins, early in October.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.
Twelfth Annual Meeting, Nashville, Tenn., December, 1927.
The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.
INDIANA, De Pauw University, April 29-30.
IOWA, University of Iowa, May 6-7.
KANSAS, Topeka, Kan., February 5.
KENTUCKY, Lexington, May 7.
LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7 and Georgetown University, December 3.
MICHIGAN, April.
MINNESOTA, St. Peter, Minn., May 21.
MISSOURI, St. Louis, Mo., November 25-26.
NEBRASKA, Lincoln, May 14.
OHIO, Columbus, Ohio, April 8.
PHILADELPHIA, Philadelphia, Pa., November 26.
ROCKY MOUNTAIN, Colorado College, April 22-23.
SOUTHEASTERN, Columbia, S. C., April 15-16.
SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.
TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

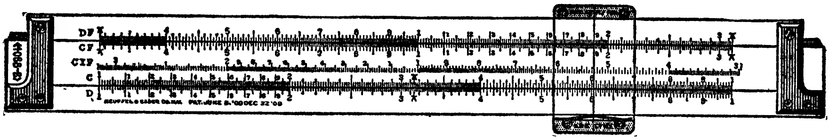
NUMBER 8, OCTOBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others



THE EIGHTH ANNUAL MEETING OF THE ILLINOIS SECTION

The eighth annual meeting of the Illinois Section of the Mathematical Association was held at Illinois Wesleyan University, Bloomington and Normal University, Normal, on May 13–14, 1927, under the chairmanship of Professor E. C. Kiefer. There was an attendance of 48 at the dinner and 57 at the meetings, including the following thirty members of the Association: Beulah Armstrong, Edith I. Atkin, T. Bennett, Lillian I. Brown, C. C. Camp, C. E. Comstock, A. R. Crathorne, D. R. Curtiss, A. E. Gault, J. S. Georges, Mary G. Haseman, Mildred Hunt, E. C. Kiefer, Mayme I. Logsdon, E. B. Lytle, W. D. MacMillan, Bessie I. Miller, C. N. Mills, E. J. Moulton, Mary W. Newson, H. P. Pettit, H. L. Rietz, H. A. Simmons, H. E. Slaught, Mildred E. Taylor, E. H. Taylor, C. A. Van Velzer, J. I. Vass, Alice Winbigler, F. E. Wood.

The following officers were elected for the ensuing year: C. N. MILLS, Illinois Normal University, Chairman; A. E. GAULT, Bradley Polytechnic Institute, Vice-Chairman; BESSIE I. MILLER, Rockford College, Secretary-Treasurer; E. H. TAYLOR, Eastern Illinois State Teachers' College, Member Executive Committee. The next meeting will be held May 4–5, 1928 at the Eastern Illinois State Teachers' College at Charleston. A motion was adopted assessing each member of the section fifty cents to meet expenses incurred in inviting special speakers. A motion was also adopted appointing a committee for investigating methods of securing in Illinois a policy more favorable to emphasizing the necessity that high school teachers should have some college training in any subject which they are required to teach. Professors Comstock, Lytle, and E. H. Taylor were appointed as the committee. A motion of appreciation of the hospitality of the two universities was passed.

The following papers were read, abstracts with corresponding numbers being given below:

1. "Curve tracing" by Dr. MARY G. HASEMAN, University of Illinois.
2. "The preparation of high school teachers of mathematics" by Professor E. H. TAYLOR, Eastern Illinois State Teachers' College.
3. "Associativity conditions for division algebras corresponding to Abelian groups" by Dr. J. S. GEORGES, University of Chicago.
4. "Logarithms of large numbers" by Dr. C. C. CAMP, University of Illinois.
5. "Personal recollections of J. J. Sylvester" by Professor C. A. VAN VELZER, Carthage College.
6. "The evolution and dissolution of matter" (an illustrated lecture on Friday evening) by Professor W. D. MACMILLAN, University of Chicago.

7. "Some remarks on the Third Carus Monograph" by Professor H. L. RIETZ, University of Iowa.

8. "Stereographic projection" by Dr. THEODORE BENNETT, University of Illinois.

9. "Algebraic geometry and the Italians" by Dr. MAYME I. LOGSDON, University of Chicago.

10. "Apparent maximum numbers in the evolution of certain Diophantine problems" by Dr. H. A. SIMMONS, Northwestern University.

11. "Some queries suggested by the proposed twelve unit entrance requirements" by Professor C. E. COMSTOCK, Bradley Polytechnic Institute.

1. The methods of algebra, trigonometry and calculus are used in a systematic scheme for tracing a complete algebraic curve in x and y of degree n.

2. Dr. Taylor's paper suggested that the preparation of the high school teachers of mathematics should serve these ends: (1) Give an adequate knowledge of the subject. One object of courses taken by prospective teachers should be to contribute ideas and methods, and to open fields of interest that may be of use in teaching elementary algebra and geometry. For example, a course in college geometry might well be required that would improve the mastery of the methods of synthetic geometry and stimulate interest in the subject. (2) Foster a permanent interest in mathematics. Too many teachers of high school mathematics do not advance their knowledge of the subject after leaving colleges and in the classroom seldom get outside the text. (3) Give a view of modern aims and methods of instruction of high school mathematics. (4) Give training in teaching. We may well attend to the care taken in the preparation of teachers in foreign countries where the instruction in secondary schools is superior to ours.

3. General associativity conditions for division algebras, corresponding to the Galois group G of an equation $f(x)=0$ of degree N and irreducible in a field F, are considered when the group G is an abelian group. These general conditions are reduced in detail to a simple set of four independent conditions, when the abelian group G has three independent generators. The paper is based on Dickson's theory of division algebras.

4. Briggs's tables are extremely scarce and the extensive manuscripts of the Tables du Cadastre and Lang are still unpublished. The scarcity of Vlacq's and Vega's tables has been compensated for by the new ten place tables of Peters but the demand for larger tables still exists. A. J. Thompson is compiling a standard twenty place table, one chiliad of which is published. When a table is available there remain two possible difficulties. One is the obvious inconvenience due to the size, and the other the excessive amount of numerical work necessary in computing a logarithm. It is with a view to overcoming these that the author of the present paper compares the methods used

Auxiliary tables such as those of Borgen, Andoyer, Steinhauser, Gray, and Thoman, involving numbers to 10 or 11, 13, 20, 24, and 27 places, respectively, are investigated. The simplest method for a ten figure logarithm is believed to be the use of Steinhauser in connection with Schron's seven decimal table. A more extensive logarithm can easily be calculated with a slight increase in the arithmetic.

5. Professor Van Velzer's recollections were arranged to show four characteristics of Sylvester: physical peculiarities, versatility, modesty and intensity of interest.

6. In a popular form this paper discussed the hypothesis that the radiant energy of space is condensed in space into the atoms which are the constituent elements in the vast amounts of nebulosity that are recorded on our astronomical photographs; that this nebulosity is gathered in by the stars in their travels through space and that the energy which they contain is the source of the energies of the stars. The property of mass is acquired when the energy is locked up in the formation of the atom and is lost again when the atom passes out of existence in the interior of some star. This conception of the evolution and dissolution of matter leads to entirely new ideas as to the evolution of stars and galaxies.

7. Professor Rietz began with a discussion of the scope and limitations of the material in the third Carus Monograph which is an introduction to mathematical statistics. He described two general types of problems with which mathematical statistics deals. The first type of problem is concerned primarily with the description and characterization of a random sample drawn from a class or "population" of items. The second type of problem is concerned mainly with the question of making and testing the validity of inferences about the properties of the class or population from a knowledge of a random sample. After a sort of parallel presentation of certain fundamental concepts such as relative frequency relating to the sample and probability relating to the population, the paper gives a summary of the material in the monograph which deals with the following three topics which have been of dominant interest in recent progress in mathematical statistics: Generalization of frequency curves, correlation theory, and random sampling theory.

8. Two fundamental theorems concerning stereographic projection state that circles are transformed into circles, and that the transformation is conformal. The author proves these facts by using some theorems from projective geometry, keeping in mind the special metric properties associated with the absolute, i.e., the sphere circle at infinity. The proofs are interesting in that the projective theorems employed are of an elementary nature; for the most part they are the fundamental theorems concerning quadric surfaces and their reguli.

9. Some notes concerning algebraic geometry as developed by the Italians, and opportunities for study in Italy.

10. This paper treats three problems. First it exhibits a solution of the Diophantine equation $\sum 1/(x_1x_2 \cdots x_r)=1$ whose left member consists of the sum of the reciprocals of the terms of the elementary symmetric function E_r of the n unknowns $x_1, x_2, \cdots, x_n$. This solution is then shown to contain the maximum x that can occur in a solution in positive integers of the equation. Furthermore this solution is the only one which contains the maximum x. The second problem treated is that of finding the maximum x (when $x_1 \leqq x_2 \leqq \ldots \leqq x_n$) that can occur in a solution in positive integers of the cyclo-symmetric equation

$$\frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\cdots+\frac{1}{x_{n-1}x_n}+\frac{1}{x_nx_1}=1\,.$$

A unique result is obtained by the method which was used in handling the first problem. The third part of the paper exhibits an apparent maximum for Problem 2 in the list of problems proposed to readers of the Monthly by the author in his paper on "Diophantine Problems in Weighing."[1]

11. What effect will the 12 unit entrance requirements proposed by the North Central Association of Colleges and Secondary Schools have upon the present requirement of one unit in geometry and one unit in algebra? Will this plan result in such a lowering of standards that the number of students applying for admission without algebra or geometry will be increased? How shall such students be treated? Which of several suggested ways should be followed? (a) admit without two units of mathematics and require no mathematics in college; (b) admit without two units but require them to be made up without college credit, an opportunity to make up such work being (1) provided by the college, or (2) not provided by the college; (c) admit without two units, but opportunity given in college to make up such work with college credit.

BESSIE I. MILLER, *Secretary*

THE SIXTEENTH MEETING OF THE IOWA SECTION

The sixteenth regular meeting of the Iowa Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Iowa Academy of Science at the University of Iowa, Iowa City, May 6 and 7, 1927.

The attendance was about forty including the following twenty-five members of the Association: R. P. Baker, A. H. Blue, E. W. Chittenden, Julia T.

---

[1] This Monthly, vol. 34 (1927), pp. 4–22.

Colpitts, N. B. Conkwright, Marian E. Daniells, Annie W. Fleming, D. Jackson, J. V. McKelvey, F. M. McGaw, C. A. Messick, E. E. Moots, E. A. Pattengill, J. F. Reilly, F. Reusser, H. L. Rietz, E. R. Smith, J. Theobald, J. S. Turner, L. E. Ward, C. W. Wester, A. E. White, R. Woods, C. C. Wylie.

The Chairman of the Section, J. V. McKelvey, presided at both the Friday afternoon and the Saturday morning sessions.

Dinner was enjoyed together Friday evening at the Iowa Memorial Union. At the business meeting which followed the program, the following were elected officers for 1927–1928: Chairman, ROSCOE WOODS, University of Iowa; Vice-chairman, E. E. MOOTS, Cornell College; Secretary-Treasurer, J. F. REILLY, University of Iowa.

The program consisted of nine papers and two addresses. The papers were as follows:

1. "Note on the hyperbola" by Professor ROSCOE WOODS, University of Iowa.

2. "Fitting of conic sections to intersecting straight lines," by Professor E. R. SMITH, Iowa State College.

3. "On folk algebra and problem solving," by Professor C. W. WESTER, Iowa State Teachers College.

4. "Thorndike's problem," by Professor WESTER.

5. "The expansion of an infinite product of polynomials into an infinite series," by Mr. C. A. MESSICK, University of Iowa.

6. "The meteor of January 2, 1927," by Professor C. C. WYLIE, University of Iowa.

7. "On the expected value of the product of certain statistical variables," by Professor H. L. RIETZ, University of Iowa.

8. "Asymptotic series," by Professor JULIA L. COLPITTS, Iowa State College.

9. "Note on the mean value theorem in the finite difference calculus," by Professor J. F. REILLY, University of Iowa.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. This note is the solution of a recent problem, No. 2409, proposed in Mathesis of December, 1926. It is proved that it is possible to draw four circles through the center of the hyperbola and a fixed point M on the curve and tangent to the hyperbola, the four points of contact being distinct from the point M. Further that the four points of contact are concyclic and that this circle passes through the center. The locus of this circle is found as M traces out the hyperbola.

2. This paper considers the properties of the family of conics which are tangent to two given intersecting lines at fixed points. The centers of the curves all lie on a straight line through the intersection of the given lines and

through the mid-point of the segment which joins the fixed points of tangency. The equation of the family, involving a parameter, may be readily obtained, and the values of the parameter for which a parabola, ellipses, hyperbolas, and a pair of straight lines are obtained may be determined. Geometric constructions are available for constructing the conic corresponding to any value of the parameter.

3. An algebra of positive numbers used more or less unconsciously by everybody solving arithmetic problems is here exhibited in the notation of ordinary algebra. Its chief principle is stated, and applications are given to problems to show its power and wide scope. It is offered as a substitute for the present confusion of methods for problem solving in arithmetic, being far simpler than ordinary algebra and incomparably more so than the current arithmetics.

4. The problem of rating computers where speed and accuracy are both taken into account has been a troublesome one. Various workers in the educational field have proposed tentative solutions, arrived at empirically. Thorndike proposed that it be solved by determining which of two or more computers would finish first a set of computations, if each computation must be checked and rechecked till two agreeing results are obtained. A mathematical solution of this problem is here given; it indicates that the only mark needed as a rating is the number right in a given time.

5. Let $\pi[1+V_n(x)]$ be absolutely convergent, and let $V_n(x)$ be a polynomial of fixed degree r. By use of the logarithmic and exponential series, expand the product into a power series $1+c_1x+c_2x^2+\cdots$. A formula is derived for each value of r which gives c_n in terms of the simple series $\sum a_n^\alpha b_n^\beta \cdots k_n^\kappa$ of the coefficients $a_n, \cdots k_n$, of V_n.

6. At about 6:02 P.M. January 2, 1927, a brilliant meteor passed over eastern Iowa. A total of considerably more than one hundred observations have been received by the University of Iowa. The observers at Iowa City considered it a little brighter than the full moon, while most of those at Dubuque, slightly farther away, considered it somewhat less bright. A computation of its path shows that it must have appeared about twenty-five miles north of Waterloo and at a height of about eighty miles, and crossed the Mississippi river just north of Burlington at a height of about twenty miles. The exact point of disappearance is not well fixed as we have insufficient data from Illinois. Our preliminary estimate for the velocity is about forty miles per second, which indicates that it came almost certainly from outside the solar system. It came from the direction of the constellation, Draco, near the point towards which the solar system is moving.

7. This paper by Professor Rietz deals with the expected value of the products of certain relative frequencies in which the expected value of the product of the variables is not equal to the product of their expected values.

8. In this expository paper Professor Colpitts discussed the history of asymptotic series, computation by means of them, and some of their properties.

9. In his paper Professor Reilly gave the analogues of Rolle's theorem and the mean value theorem for the finite difference calculus, and suggested that, if the mean value theorem be extended as in the infinitesimal calculus, Newton's formula of interpolation would result.

After the Friday evening dinner Professor J. V. McKelvey gave the retiring chairman's address on "Discontinuities and prerequisites," Professor E. R. Smith, Iowa State College, presiding. This was followed by a general and enthusiastic discussion.

At the Saturday morning session Professor Dunham Jackson, University of Minnesota, who was present by invitation, gave an address on "Trigonometric interpolation," Professor E. W. Chittenden, University of Iowa, presiding. This address is published in this number of Monthly.

Professor J. S. Turner, Iowa State College, exhibited a very old book on Trigonometry and Logarithms, "Mirifici logarithmorum canonis descriptio . . .," published at Edinburgh in 1614, and dedicated to the Prince of Wales (afterwards Charles I).

J. F. REILLY, *Secretary*

THE TWELFTH MEETING OF THE KANSAS SECTION

The twelfth regular meeting of the Kansas Section of the Mathematical Association of America was held at Topeka, February 5, 1927, in connection with the annual meeting of the Kansas Association of Mathematics Teachers. Two sessions were held. The first was a joint session with the Kansas Association. Professor W. H. Garrett, Chairman of the Kansas Section, presided part of the time during the forenoon (joint) session and all of the time during the afternoon session.

The attendance was fifty-six including the following twenty-seven members of the Association: C. H. Ashton, W. H. Andrews, Alice Austin, Wealthy Babcock, Florence Black, Lucy Dougherty, P. L. Evans, E. F. Farner, W. H. Garrett, R. W. Hart, Ina Holroyd, Emma Hyde, C. F. Lewis, W. H. Lyons, Anna Marm, Nina McLatchey, U. G. Mitchell, H. S. Myers, A. W. Philips, T. I. Porter, C. A. Reagan, Ethel Rumney, J. A. G. Shirk, G. W. Smith, Edith Steininger, W. T. Stratton, J. J. Wheeler.

The following officers were elected for the coming year: Chairman, U. G. MITCHELL, Vice-Chairman, H. S. MYERS, Secretary-Treasurer, LUCY DOUGHERTY.

The following papers by members of the Association were presented:

1. "The problem of the college freshman with no high school mathematics," by Professors J. A. G. SHIRK and H. S. MYERS.

2. "Relation of principals and teachers to the recent changes in high school mathematical requirements," by Principal E. F. FARNER.

3. "Certain tests for primality of integers," by Professor C. A. REAGAN.

4. "Some important points in the development of the theory of determinants," by Professor WEALTHY BABCOCK.

5. "The work of Euler, Laplace, and Lagrange relating to the stability of the solar system," by Professor PIUS PRETZ (representing an institutional member).

6. "An appreciation of the life and work of Sir Isaac Newton," by Professor G. W. SMITH.

Comments on the first two papers and abstracts of the others, numbered as in the above list of titles, follow:

1. and 2. These papers were presented at the joint session with the Kansas Association of Mathematics Teachers and were in the nature of discussions of the situation created in Kansas by a recent ruling of the State Board of Education permitting students to graduate from accredited high schools without any high school mathematics. Full summaries of these papers appeared in the April number of the Bulletin of the Kansas Association of Mathematics Teachers (Edited by Professor Ina E. Holroyd and published at Manhattan, Kansas).

3. Professor Reagan showed that all integers of the form $6n+1$ are primes except when n is of the form $6ar \pm (a+r)$. For a given integer of the form $6n+1$ the $6ar \pm (a+r)$ may be equated to the numerical value of n and a test for primality made by testing the resulting indeterminate equations for positive integral solutions. For certain rather large primes the labor involved in making such a test seemed not to be formidable. Similar forms for testing numbers of the form $6n-1$ were developed.

4. Professor Babcock's paper outlined the theory of determinants somewhat in the order of its historical development. Of the early work the contributions of only four men, Leibnitz, Cramer, Vandermonde and Cauchy, were considered. Cayley's expansion theorem, Jacobi's theorems, Sylvester's theorem, and theorems concerning the vanishing of a determinant and the independence of the minors of a determinant were cited as important theorems in the later development of the theory. A few special determinants, including symmetric determinants, Jacobians, and Wronskians, were defined and some theorems concerning them given. Brief mention was made of the theory of infinite determinants, developed since the publication of Hill's paper in 1877, and of the Fredholm's determinant and its application to integral equations.

5. Prior to the eighteenth century many theories had been advanced to account for the order, harmony, and stability of the solar system. Most notable among them were the *deus ex machina* of Newton, maintaining that a "powerful hand" intervened from time to time to set the celestial machinery right, and the "perpetual miracle" of Leibnitz. Professor Pretz attempted to show, in a non-technical way, that the work of Euler, Lagrange, and Laplace in mathematics, mechanics, and astronomy placed sufficient material in the hands of Laplace to prove that the order, harmony, and stability of the solar system were due to the operation of natural laws, the laws of gravitation.

6. Professor Smith pointed out that the two-hundredth anniversary of the death of Newton would occur on March 20, 1927, and that various mathematical societies were devoting meetings to a commemoration of his life and work. He then briefly recalled the chief events in Newton's life and his most important discoveries in mathematics and physics, and gave a short discussion of his law of universal gravitation in connection with recent discoveries and developments.

U. G. MITCHELL, *Secretary*

SOME NOTES ON TRIGONOMETRIC INTERPOLATION[1]

By DUNHAM JACKSON, University of Minnesota

The object of this paper is to call attention to certain relations involving the formulas of trigonometric interpolation with equidistant ordinates. Some of these relations, while they may not be new, are at any rate less familiar than the most fundamental facts in the theory of such interpolating formulas.

Let $f(x)$ be a given function of period 2π, for simplicity continuous. Let m be an arbitrary positive integer, and let $x_i = 2i\pi/m$, so that $x_0 = 0$, $x_m = 2\pi$, and the interval $(0, 2\pi)$ is divided into m equal parts by the points $x_1, \cdots, x_{m-1}$. If coefficients a_k, b_k are defined by means of the formulas

$$a_k = \frac{2}{m}\sum_{i=1}^{m} f(x_i)\cos kx_i, \qquad b_k = \frac{2}{m}\sum_{i=1}^{m} f(x_i)\sin kx_i,$$

it is well known that the expression

$$\begin{aligned}\phi_n(x) = \tfrac{1}{2}a_0 &+ a_1\cos x + a_2\cos 2x + \cdots + a_n\cos nx\\ &+ b_1\sin x + b_2\sin 2x + \cdots + b_n\sin nx\end{aligned}$$

if m is odd, $m = 2n+1$, or

$$\begin{aligned}\psi_n(x) = \tfrac{1}{2}a_0 &+ a_1\cos x + \cdots + a_{n-1}\cos(n-1)x + \tfrac{1}{2}a_n\cos nx\\ &+ b_1\sin x + \cdots + b_{n-1}\sin(n-1)x\end{aligned}$$

---

[1] Presented to the Iowa Section of the Association at Iowa City, May 7, 1927.

if m is even, $m=2n$, is a trigonometric sum of the nth order coinciding in value with $f(x)$ at each of the points x_i, and is the only sum of the nth order, or the only sum of the nth order lacking the term in $\sin nx$, which has this property.

For comparison, let the Fourier coefficients of $f(x)$ be denoted by

$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad \beta_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx.$$

If $\Delta x = 2\pi/m$, it is seen that

$$a_k = \frac{1}{\pi}\sum_{i=1}^{m} f(x_i)\cos kx_i\,\Delta x, \qquad \text{with a corresponding formula for } b_k.$$

Hence, if k is held fast, $\lim_{m=\infty} a_k=\alpha_k$, $\lim_{m=\infty} b_k=\beta_k$; the interpolating coefficients are approximations to the Fourier coefficients, to the extent that a sum of rectangles is an approximation to the area under a curve.

The formulas for the a's and b's, however, are deserving of closer examination. Apart from the factor $1/\pi$, each is of the form

$$\sum_{i=1}^{m} F(x_i)\Delta x, \tag{1}$$

where $F(x)$ is a function of period 2π. The trapezoid formula for approximating to the integral

$$\int_0^{2\pi} F(x)\,dx \tag{2}$$

would be

$$\Delta x\left[\tfrac{1}{2}F(x_0) + F(x_1) + \cdots + F(x_{m-1}) + \tfrac{1}{2}F(x_m)\right]. \tag{3}$$

But as $F(x_0)=F(x_m)$, by the hypothesis of periodicity, (3) is identical with (1). *Each interpolating coefficient is an approximation to the corresponding Fourier coefficient, not merely in the sense of the rectangle formula, but in the sense of the trapezoid formula of mechanical quadrature at the same time.*

The application of Simpson's rule to the integral (2), when m is even, yields

$$\tfrac{1}{3}\Delta x\left[F(x_0) + 4F(x_1) + 2F(x_2) + \cdots + 4F(x_{m-1}) + F(x_m)\right],$$

which is the same as

$$\tfrac{1}{3}\Delta x\left[4F(x_1) + 2F(x_2) + \cdots + 4F(x_{m-1}) + 2F(x_m)\right]. \tag{4}$$

This is not by any device of interpretation the same as (1). But let the corresponding quadrature formula be written down for the integral $\int_{x_1}^{x_1+2\pi} F(x)\,dx$, the value of which is the same as that of (2), since the integral of a periodic

function over a period interval is independent of the situation of the interval. The formula is

$$\tfrac{1}{3}\Delta x[4F(x_2) + 2F(x_3) + \cdots + 4F(x_m) + 2F(x_{m+1})],$$

which is the same as

$$\tfrac{1}{3}\Delta x[2F(x_1) + 4F(x_2) + 2F(x_3) + \cdots + 4F(x_m)], \tag{5}$$

since $F(x_{m+1}) = F(x_1)$. *The average of* (4) *and* (5) *reduces to* (1) *again; the interpolating coefficient is the average of the results obtained by two applications of Simpson's rule to the integral defining the Fourier coefficient*, and so represents an approximation at least as good, on general principles, as that to be expected from the use of Simpson's rule directly.

The same form of interpretation can be extended to higher formulas of mechanical quadrature, if m is divisible by the number of sub-intervals that are grouped together in setting up the quadrature formula, that is, by the degree of the polynomials from which the quadrature formula is obtained. And it is worthy of note that the values of m most likely to be used in practice, namely the number of hours in a day and the number of months in a year, are divisible by 2, 3, 4, 6, and 12.

Even if m were not divisible by 2, say, in connection with Simpson's rule, that rule could still be applied to the integral from 0 to 4π, which is equal to the double of the integral from 0 to 2π, because of the periodicity, and the interpretation of the interpolating coefficient as a result of quadratures by Simpson's rule would still be justified. For a_k, for example, may be regarded as defined by the formula

$$a_k = \frac{1}{2\pi}\int_0^{4\pi} f(x)\cos kx\,dx.$$

It must be admitted that attention has been directed to the excellence of the approximations, rather than to their shortcomings. It is apparent on renewed consideration that the approximations are unquestionably good only if m is large in comparison with k, and this is true only for the early terms in the interpolating sum of any specified order. Suppose for example that $f(x)$ is everywhere positive; then $f(x)\cos kx$ changes sign $2k$ times in a period, and if k is only a little less than $\frac{1}{2}m$ the quadrature formula uses only about one point for each change of sign, and promises, superficially at least, only a decidedly crude result.

This deficiency is particularly apparent on comparison of the formulas for the sum of the whole interpolating expression on the one hand and the partial sum of the Fourier series on the other. For simplicity let attention be restricted to the case that $m = 2n+1$. The formulas are respectively

$$\frac{1}{\pi}\sum_{i=1}^{2n+1} f(x_i)\frac{\sin (n+\frac{1}{2})(x_i-x)}{2\sin\frac{1}{2}(x_i-x)}\Delta x \quad \text{and} \quad \frac{1}{\pi}\int_0^{2\pi} f(t)\frac{\sin (n+\frac{1}{2})(t-x)}{2\sin\frac{1}{2}(t-x)}dt.$$

There is just one term in the sum for each arch of the sine in the numerator.

In the light of the last remarks, it is perhaps all the more noteworthy that the convergence properties of the interpolation formula and of the Fourier series correspond as closely as they do.[1] One is led to inquire, though with a rather definite expectation as to the result, whether the convergence properties would be preserved if the interpolating coefficients were replaced by quadrature formulas using a larger number of points, and so presumably approximating more closely to the actual values of the Fourier coefficients. A conclusion with regard to this problem may be formulated in the following

THEOREM. *For each positive integral value of p, let*

$$\phi_p(x) = \tfrac{1}{2}a_{0p} + a_{1p}\cos x + a_{2p}\cos 2x + \cdots + a_{pp}\cos px$$
$$+ b_{1p}\sin x + b_{2p}\sin 2x + \cdots + b_{pp}\sin px,$$

where

$$a_{kp} = \frac{2}{m_p}\sum_{i=1}^{m_p} f(x_i)\cos kx_i, \qquad b_{kp} = \frac{2}{m_p}\sum_{i=1}^{m_p} f(x_i)\sin kx_i,$$
$$x_i = 2i\pi/m_p,$$

the numbers m_p, for the successive values of p, being subject merely to the requirement that $m_p > 2p+1$. Then $\phi_p(x)$ will converge uniformly toward $f(x)$, as p becomes infinite, provided that $f(x)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta=0}\omega(\delta)\log\delta = 0$, *that is, if $f(x)$ satisfies one of the standard sufficient conditions for the convergence of the ordinary interpolation formula or the Fourier series.*

The condition with regard to the modulus of continuity means that $[f(x_2)-f(x_1)]\log|x_2-x_1|$ approaches zero uniformly when x_2-x_1 approaches zero. A detailed presentation of the proof of the theorem is unnecessary for the technical worker in the field, and would be of little interest to other readers. Briefly, the proof is as follows.

If $f(x)$ were itself a trigonometric sum of order p, $\phi_p(x)$ would be identical with $f(x)$. For in any case $\phi_p(x)$ is merely the partial sum of the interpolating expression for $f(x)$ corresponding to a subdivision of the interval $(0, 2\pi)$ into m_p equal parts, through terms of order p; and when $f(x)$ is a trigonometric sum of order $p < \frac{1}{2}(m_p - 1)$, the whole interpolating expression is identical with $f(x)$, and has no terms of order higher than p.

[1] Cf., e.g., Faber, *Über stetige Funktionen*, Mathematische Annalen, vol. 69 (1910), pp. 372–443; D. Jackson, *On approximation by trigonometric sums and polynomials*, Transactions of the American Mathematical Society, vol. 13 (1912), pp. 491–515; *On the accuracy of trigonometric interpolation*, the same Transactions, vol. 14 (1913), pp. 453–461.

Secondly, if $f(x)$ has M as an upper bound for its absolute value, but is otherwise unrestricted, $|\phi_p(x)|\leqq CM\log p$, where C is a constant independent of p, m_p, and the form of the function $f(x)$. For ϕ_p can be written by means of a well-known identity in the form

$$\phi_p(x) = \frac{1}{m_p}\sum_{i=1}^{m_p} f(x_i)\frac{\sin\,(p+\frac{1}{2})(x_i-x)}{\sin\frac{1}{2}(x_i-x)}\,.$$

The number of terms in this sum for which $\frac{1}{2}(x_i-x)$ differs from 0 or from π by not more than $1/p$ is within one unit of the number of times that an interval of length π/m_p is contained in an interval of length $2/p$, and does not exceed a constant times m_p/p; the absolute value of the quotient of sines can not be more than $2p+1$; and so the sum of this group of terms, with the factor $1/m_p$, does not exceed a constant multiple of M. In the remaining terms it is to be noted that $|\sin\,(p+\frac{1}{2})\,(x_i-x)|\leqq 1$, and that $\sin\frac{1}{2}(x_i-x)$ is not less in absolute value than $2/\pi$ times the difference between $\frac{1}{2}(x_i-x)$ and the nearer of the numbers 0, π, since $\sin y/y\geqq 2/\pi$ for $0<y\leqq\pi/2$, and $\sin y/(\pi-y)$ $=\sin\,(\pi-y)/(\pi-y)\geqq 2/\pi$ for $\pi/2\leqq y<\pi$. Hence the sum of these terms, with the factor $1/m_p$ included, is not greater than a constant multiple of M/m_p times the sum of the reciprocals of a set of numbers succeeding each other at intervals of π/m_p in the range from $1/p$ to $\pi/2$. And this sum of reciprocals lacks only the factor π/m_p to make it an approximation to the value of the integral $\int dt/t$, extended from $1/p$ to $\pi/2$, and so does not exceed a constant multiple of $m_p\log p$. Thus the assertion at the beginning of the paragraph is justified. It follows immediately that

$$|f(x)-\phi_p(x)|\leqq M(C\log p+1)\,.$$

Thirdly, suppose $f(x)$ can be approximately represented by a trigonometric sum $T_p(x)$ of order p so that $|f(x)-T_p(x)|$ does not exceed ϵ_p. The difference between $f(x)$ and $\phi_p(x)$ is the sum of the corresponding differences formed for $f(x)-T_p(x)$ and for $T_p(x)$ respectively. The first of these differences does not exceed $\epsilon_p(C\log p+1)$, by the preceding paragraph, and the second is identically zero, by the paragraph before that. Hence

$$|f(x)-\phi_p(x)|\leqq \epsilon_p(C\log p+1)\,.$$

Finally, if $f(x)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta=0}\omega(\delta)\log\delta=0$, it is known[1] that trigonometric sums $T_p(x)$ can be constructed so that

$$\lim_{p=\infty}\epsilon_p\log p = 0.$$

From this follows the desired conclusion that $\phi_p(x)$ converges uniformly toward $f(x)$.

[1] Lebesgue, *Sur les intégrales singulières*, Annales de la Faculté de Toulouse, (3), vol. 1 (1910), pp. 25–117; pp. 115–116; D. Jackson, *On the approximate representation of an indefinite integral . . . .*, Transactions of the American Mathematical Society, vol. 14 (1913), pp. 343–364; p. 350.

FUNCTIONS OF CLOSEST APPROXIMATION ON AN INFINITE RANGE[1]

By W. D. CAIRNS, Oberlin College

The purpose of this paper is to extend to the interval $(-\infty, \infty)$ a theorem of Dunham Jackson[2] on the existence of the polynomial of specified degree, or of a linear combination of n given functions, which gives the best approximation to a given continuous function in a given interval (a, b). This extension is accomplished by the expedient of introducing the factor $e^{-x^2/4}$. The general argument is quite similar to that of the paper referred to; the analogy is shown more fully by the rather free use of this exponential factor.

Let $p_1(x), p_2(x), \cdots, p_n(x)$ be n functions, continuous for finite x, and such that one can find a positive constant h so that

$$|p_i(x)| \leqq |x|^h \qquad (i = 1, 2, \cdots, n) \text{ for } |x| > 1.$$

Let $$\phi(x)/e^{-x^2/4} = c_1p(x)e^{-x^2/4} + c_2p_2(x)e^{-x^2/4} + \cdots + c_np_n(x)e^{-x^2/4}$$

be an arbitrary linear combination of these functions with constant coefficients and multiplied by $e^{-x^2/4}$. It will then, by hypothesis, be true that

$$\left|\frac{\phi(x)}{e^{-x^2/4}}\right| \leqq C|x|^h e^{-x^2/4} \qquad C = |c_1| + \cdots + |c_n| \text{ for } |x| > 1.$$

LEMMA I. *There exists a constant K, completely determined by the system of functions $p_i(x)$, such that $|c_i| \leqq K$ for all functions $\phi(x)$, $i = 1, 2, \cdots, n$.*

For each value of k let the coefficients in

$$\Phi_k(x)/e^{-x^2/4} = c_{1k}p_1(x)e^{-x^2/4} + \cdots + c_{nk}p_n(x)e^{-x^2/4}$$

be determined so that

$$\int_{-\infty}^{\infty} p_i(x)\Phi_k(x)\,dx = 0, \qquad i \neq k\,; \qquad \int_{-\infty}^{\infty} p_k(x)\Phi_k(x)\,dx \quad 1.$$

This is possible because of the fact that the functions $p_i(x)$ are linearly independent, just as in Professor Jackson's proof. Since for every $\Phi_k(x)$ it is true, as in the case of $\phi(x)$, that

$$\left|\frac{\Phi_k(x)}{e^{-x^2/4}}\right| \leqq C_k|x|^h e^{-x^2/4}, \qquad C_k = |c_{1k}| + \cdots + |c_{nk}|,$$

it follows that

---

[1] Read before the Southern California Section of the Mathematical Association of America, Los Angeles, Calif. March 12, 1927.

[2] Transactions of the American Mathematical Society, vol. 22 (1921), pages 117–128.

$$\left| \int_{-\infty}^{\infty} \frac{\phi(x)}{e^{-x^2/4}} \frac{\Phi_k(x)}{e^{-x^2/4}} dx \right| \leqq CC_k \int_{-\infty}^{\infty} x^{2h} e^{-x^2/2} dx = K,$$

a finite constant, it being easy to show that the last (improper) integral fulfills the test for convergence. But by definition of $\Phi_k(x)$ it is also true that

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{e^{-x^2/4}} \frac{\Phi_k(x)}{e^{-x^2/4}} dx = \int_{-\infty}^{\infty} \sum_{i=1}^{n} c_i p_i(x) e^{-x^2/4} \cdot \frac{\Phi_k(x)}{e^{-x^2/4}} dx = c_k \int_{-\infty}^{\infty} p_k(x) \Phi_k(x) dx = c_k.$$

Hence the c_k are bounded in absolute value by K, this constant depending on the coefficients in $\phi(x)$ inasmuch as the constant factor in the upper bound for $|\phi(x)|e^{x^2/4}$ depends on them.

Let m be a positive constant greater than 1. Set

$$\Delta_m = \int_{-\infty}^{\infty} e^{-x^2/2} \left| \frac{\phi(x)}{e^{-x^2/4}} \right|^m dx\,; \qquad \Delta = \int_{-\infty}^{\infty} e^{-x^2/2} dx = (2\pi)^{1/2}.$$

LEMMA II. *There exists a positive constant K_1, completely determined by the system of functions $p_i(x)$ and independent of m, such that*

$$|c_i| \leqq K_1(\Delta + \Delta_m), \qquad (i = 1, 2, \cdots, n) \textit{ for all functions } \phi(x).$$

As in Professor Jackson's paper, since $m > 1$,

$$|\phi(x)/e^{-x^2/4}| \leqq |\phi(x)/e^{-x^2/4}|^m$$

unless the left member of the inequality is <1; hence in any case

$$|\phi(x)/e^{-x^2/4}| \leqq 1 + |\phi(x)/e^{-x^2/4}|^m$$

for any fixed $m > 1$, and for all values of x. Hence

$$\int_{-\infty}^{\infty} e^{-x^2/2} |\phi(x)/e^{-x^2/4}| dx \leqq \Delta + \Delta_m,$$

and for any finite value of x

$$\left| \int_{-\infty}^{x} e^{-x^2/2} [\phi(x)/e^{-x^2/4}] dx \right| \leqq \int_{-\infty}^{\infty} e^{-x^2/2} |\phi(x)/e^{-x^2/4}| dx \leqq \Delta + \Delta_m.$$

The n functions $\displaystyle\int_{-\infty}^{x} e^{-x^2/2} p_i(x) e^{-x^2/4} dx \qquad (i = 1, 2, \cdots, n)$

are linearly independent, since a linear relation between them would give by differentiation a linear relation connecting the functions $p_i(x)$. And since the last inequality may be written

$$\left| c_1 \int_{-\infty}^{x} e^{-x^2/2} p_1(x) e^{-x^2/4} dx + \cdots + c_n \int_{-\infty}^{x} e^{-x^2/2} p_n(x) e^{-x^2/4} dx \right| \leqq \Delta + \Delta_m,$$

the hypothesis of Lemma I is fulfilled, and by the analysis used in the latter part of the proof of Lemma I,

$$|c_i| \leqq C_k \int_{-\infty}^{\infty} (\Delta + \Delta_m) |x|^h e^{-x^2/4} dx \leqq K_1(\Delta + \Delta_m).$$

Let $f(x)$ be a function continuous for finite x, $f(x)/e^{-x^2/4}$ vanishing at $\pm\infty$ like $|x|^{h'} e^{-x^2/4}$; i.e., when $|x| > 1$,

$$|f(x)/e^{-x^2/4}| \leqq |x|^{h'} e^{-x^2/4} \qquad (h' \text{ a positive constant}).$$

Also let

$$\Delta' = \int_{-\infty}^{\infty} e^{-x^2/2} |x|^{h'} e^{-x^2/4} dx\,; \qquad \delta_m = \int_{-\infty}^{\infty} e^{-x^2/2} \left| \{f(x) - \phi(x)\}/e^{-x^2/4} \right|^m dx.$$

LEMMA III. *For all functions* $\phi(x)$,

$$|c_i| \leqq K_1(\Delta' + \Delta + \delta_m) \qquad (i = 1, 2, \cdots, n),$$

where K_1 *is the positive constant of Lemma II.*

Since, as in the last argument,

$$\left| \{f(x) - \phi(x)\}/e^{-x^2/4} \right| \leqq 1 + \left| \{f(x) - \phi(x)\}/e^{-x^2/4} \right|^m,$$

we have for every $\phi(x)$

$$|\phi(x)/e^{-x^2/4}| \leqq |x|^{h'} e^{-x^2/4} + 1 + \left| \{f(x) - \phi(x)\}/e^{-x^2/4} \right|^m;$$

whence

$$\int_{-\infty}^{\infty} e^{-x^2/2} |\phi(x)/e^{-x^2/4}| dx \leqq \Delta' + \Delta + \delta_m.$$

Hence, by the argument in the last part of Lemma II, $|c_i| \leqq K_1(\Delta' + \Delta + \delta_m)$.

Existence of an approximating function. If the function $f(x)$, the system $p_i(x)$, and the exponent m are given, and the coefficients c_i are regarded as undetermined, the value of δ_m, which is a function of the c_i's, has a lower limit γ_m which is necessarily positive or zero. Since for a given function $f(x)$ there is an infinite number of ways of choosing the coefficients $c_i^{(j)}$, $j = 1, 2, \cdots$, and these by Lemma III are in a bounded region in an n space (regarding these as the coordinates of a point for each value of j), since

$$|c_i^{(j)}| \leqq K_1(\Delta' + \Delta + \delta_m),$$

there will be a set of limiting values for these which will yield a $\phi(x)$ for which δ_m attains its lower limit. In analogy with Professor Jackson's work, this may be called an approximating function for the exponent m.

REMARK: For $m=2$, this means that there exists a set of constants c_i for which

$$\delta_m = \int_{-\infty}^{\infty} [f(x) - \phi(x)]^2 dx$$

has a minimum, actually attained.

The writer is indebted to Professor Jackson for suggestions on details of the proof as well as for a communication on a briefer proof of the final theorem, when the squares of $f(x)$ and $p_i(x)$ are integrable from $-\infty$ to ∞, covering at least the case $m=2$.

THE ELDER ĀRYABHAṬA AND THE MODERN ARITHMETICAL NOTATION

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

Kaye writes[1] that the system of numeral notation by words, which is still popular in India, was introduced into India in the ninth century A.D. But any investigator who takes the trouble of reading the *original* works of Varāhamihira and his successors is sure to be convinced of the incorrectness of this statement. Burgess, Burnell, and Bühler are of the opinion that the system presupposes the existence of the modern arithmetical notation. On the authority of Barth, Burnell admits that the modern arithmetical notation was known to Āryabhaṭa. He also asks:[2] "May not Āryabhaṭa be the discoverer of the decimal notation in India?" In this paper it is proposed to give an answer to this question.

In India there lived two mathematicians and astronomers bearing the name Āryabhaṭa.[3] The elder Āryabhaṭa was born in 476 A.D. and is known as Āryabhaṭa of Kusumapura (modern Pāṭnā). He devised an alphabetic scheme of expressing numbers, which, although certainly based on the modern notation, does not recognise the value of position. The younger Āryabhaṭa lived in the tenth century A.D. and is the author of another alphabetic scheme which has been termed the *kaṭapayādi* system.[4] This latter scheme is nothing but the modern arithmetical notation with the signs replaced by letters. It is Āryabhaṭa of Kusumapura to whom Burnell refers.

---

[1] *Indian Mathematics* (Calcutta, 1915), p. 31.

[2] *South Indian Palæography*, p. 62, foot-note 4.

[3] For the controversy on the two Āryabhaṭas the reader is referred to B. B. Datta's article on the subject published in the Bulletin of the Calcutta Mathematical Society, Vol. 17, pp. 59–74.

[4] Burnell and Bühler have described the later *kaṭapayādi* system and not the one devised by the younger Āryabhaṭa.

On an examination of the elder Āryabhaṭa's rules for the extraction of square and cube roots, Rodet writes:[1] "Donc, non seulement Āryabhaṭa opérait sur des nombres écrits *en chiffres avec valeur de position et zéro*, mais la pratique de ces sortes d'opérations était déjà familière à l'époque où il écrivait, ce qui suppose qu'elle existait déjà depuis un certain temps." But Kaye holds a diametrically opposite view and remarks that "there is not in any part of Āryabhaṭa's work the remotest indication of a notation with 'place values' ."[2] Kaye's writings have modified the opinion of foreign scholars who now accept the 9th century A.D. as the earliest period when the modern notation was in use in India. I have elsewhere[3] shown that the above remark of Kaye is not supported by Āryabhaṭa's rules for the extraction of square and cube roots which Kaye has not been able to interpret correctly. If we assume, for the sake of argument only, that the words *vargāt*, *avargāt*, *ghanāt*, *aghanāt* which occur in the rules refer, as has been supposed by Kaye, not to places but to parts of the number whose root is to be found, the classification of the elements of a number into square (*varga*) and non-square (*avarga*) or into cubic (*ghana*) and non-cubic (*aghana*) is intelligible; but we fail to understand how the elements of a number can be classified as *cubic*, *second non-cubic*, and *first non-cubic*. Again, why should the *second non-cubic part* be taken before the *first* in the extraction of the cube root?[4]

Āryabhaṭa's rules for the extraction of square and cube roots indirectly prove that he was acquainted with the modern arithmetical notation. Direct proof, too, of this point is not lacking. The elder Āryabhaṭa has devoted the second chapter of his work, known as the *Āryabhaṭīyaṃ*, to mathematics. The second verse of this chapter contains an enunciation of the modern arithmetical notation. The verse runs thus:

Ekaṃ ca[5] daśa ca śatañca
sahasramayutaniyute tathā prayutam
Koṭyarbudañca vṛndam
sthānat sthānaṃ daśaguṇaṃ syāt.

Rodet translates this verse as follows:

---

[1] Journal Asiatique, May and June, 1879, p. 408.

[2] Journal of the Asiatic Society of Bengal, July, 1907, p. 494.

[3] Journal of the Bihar & Orissa Research Society, March, 1926, pp. 78–82.

[4] Kaye's translation of Āryabhaṭa's rule for the extraction of the cube root is as follows:—"Multiply the square of the root of the cubic quantity by three and divide the second non-cubic part by the product. Multiply the square of this by three times the preceding and subtract the product from the first non-cubic. Then the cube is to be subtracted from the cube." (Journal of the Asiatic Society of Bengal, March, 1908, p. 120).

[5] I have introduced this word of a single syllable for the sake of the metre. The reading 'ekaṃ daśātha tu śataṃ sahasram etc.' suggested in Kern's edition of the *Āryabhaṭīyam*, page 18, footnote, involves more changes.

"*Eka, daçan, çata, sahasra, ayuta, niyuta, prayuta, koṭi, arbuda, vṛnda,* sont, de place en place, décuples l'un de l'autre" (Journal Asiatique, May & June, 1879, p. 397).

The words 'de place en place' suggest that Rodet takes *eka, daśa, śata,* &c., as the names of places, in which case he makes a near approach to the real meaning of the verse. This suggestion seems to be confirmed by his correct translation[1] of the corresponding verses of Bhāskara which he introduces by way of explanation. Here he says that *eka, daśa, śata,* &c., are the successive places ("les PLACES successives"). But the very significant omission on his part to urge this verse in support of his view that Āryabhaṭa was acquainted with the modern notation shows that he has failed to hit upon the right meaning of the verse and taken *eka, daśa, śata,* &c., as numerical units. As Rodet thinks that the modern notation existed in India even before the time of Āryabhaṭa, the introduction of the words 'de place en place' after a number of numerical units does not lead to any inconsistency as it does in the case of those who hold the opposite view. But his translation, we are bound to observe, is not correct. The words *sthānāt sthānaṃ* do not mean 'from place to place' (de place en place). The verb *syāt* (lit., 'should be') is in the singular number and has for its nominative *sthānaṃ* (place) and not *eka, daśa,* &c.; and the adjective *daśaguṇaṃ* is the subjective complement to the verb *syāt.*

Kaye translates the verse as follows:

"Units, tens, hundreds, thousands, tens of thousands, hundreds of thousands, millions, tens of millions, hundreds of millions, thousands of millions. In these each succeeding place is ten times the preceding place." (Journal of the Asiatic Society of Bengal, March, 1908, p. 117).

This translation consists of two parts. The first part simply names ten numerical units. The second part states the relation between the values of any two consecutive places. Is there any difference between the second part of the above translation and the principle of place-value which is the basis of our modern notation? Certainly not. Yet Kaye sticks to his opinion that the principle of place-value was unknown to Āryabhaṭa. To establish a connection between the two parts Kaye has, according to the usual custom which is to be followed in explaining Sanskrit verses, introduced the words "In these" which mean 'in the numerical units just mentioned'. But, how can one find *places* in those numerical units unless places have been assigned to them?

This defect in Kaye's translation seems to have attracted Fleet's notice. Accordingly, Fleet, who agrees with Kaye so far as the first part is concerned, translates[2] the second part, viz., *sthānāt sthānaṃ daśaguṇaṃ syāt,* as "from

[1] Journal Asiatique, May and June, 1879, p. 404.

[2] Journal of the Royal Asiatic Society, 1911, p. 116.

place to place each is a multiple by ten." Hence, Fleet's interpretation of the verse is practically the same as that given by Rodet. The remarks which have been made on the latter are, therefore, equally applicable to the former.

That none of the above three translations gives Āryabhaṭa's meaning of the verse follows from another consideration also. Āryabhaṭa has omitted the first four rules. It is, therefore, very unlikely that he should give the names of the numerical units which were too well-known in his time.

Let us now try to find the true interpretation of the verse. Let us first take the expression *sthānāt sthānaṃ daśaguṇaṃ syāt.* Here the verb *syāt* is the *vidhiliṅ* form of the verbal root *as* (to be). Hence it implies a *vidhi* or rule and should, therefore, be translated as 'should be, as a rule.' The translation of the expression should, therefore, be: 'each place should, as a rule, be ten times the next (lower) place.' The remaining part of the verse fits in with this expression only when *eka, daśa*, &c., are taken for the names of the places (*sthānānāṃ saṃjñāḥ*) after Bhāskara. Mahāvīra, who lived nearly three centuries and a half after the elder Āryabhaṭa, puts the same interpretation on the words *eka, daśa, śata*, &c. He writes:

Ekaṃ tu prathamaṃ sthānaṃ dvitīyaṃ daśasaṃjñikaṃ,
Tṛtīyaṃ śatamityāhuḥ caturthaṃ tu sahasrakaṃ,
etc. etc. etc.

(*Gaṇita-sāra-saṃgrahaḥ*, Rangācārya's edition, text, page 7).

Eka is the first place; the second (place) has *daśa* for its name (saṃjñā); (they) call the third (place) as *śata* and the fourth (place) as *sahasraka* (i.e. *sahasra*), etc.

This interpretation of the words *eka, daśa, śata*, &c. has passed on from Mahāvīra to Bhāskara through generations of teachers and must have been handed down to Mahāvīra in the same way. It should be used also in understanding Āryabhaṭa's verse under consideration. The meaning of the verse, therefore, seems at first sight to be as follows:

Eka, daśa, śata, sahasra, ayuta, niyuta, prayuta, koṭi, arbuda, and vṛnda are the names of the first ten places of which each place is ten times the next lower place.

This interpretation, like those given by Rodet, Kaye, and Fleet, is open to the following objections:

(a) Like the other Indian *sūtrakāras* (or framers of rules or formulæ) Āryabhaṭa is always brief to a fault. He is never guilty of a redundant statement. And it goes without saying that, whether *eka, daśa, śata*, &c., stand for numerical units or for places, the terms beginning with *daśa* are each ten times the next lower one. Is it then likely that Āryabhaṭa should, for once only, be so verbose, which is contrary to his nature as a *sūtrakāra*, and should add, by way of explanation, a statement which is only too obvious?

(b) According to the above interpretations *eka* is ten times the next lower place or unit. This would go to prove that Āryabhaṭa contemplated decimal fractions also. But there is no evidence in support of this conclusion.

(c) No one can deny that Āryabhaṭa speaks of at least nineteen places in the verse giving his alphabetic scheme.[1] Why, then, does he apply the rule 'each place is ten times the next lower place' to ten places or numerical units only? Is not the rule applicable to higher places or numerical units?

None of the above interpretations is, therefore, correct. I would, therefore, state the meaning of the verse thus:

The names of the first ten places are *eka, daśa, śata, sahasra, ayuta, niyuta prayuta, koṭi, arbuda, vṛnda;* the names of the higher places are to be found by applying the rule, viz., 'each place should be ten times the next lower place,' and by remembering that each place is named after the unit to which it has been assigned, as shown by the first ten names.

Āryabhaṭa names the first ten places only, probably because they are quite sufficient for all practical purposes.[2] It may be noted here that the names of decimal units higher than *vṛnda* have existed in India from before the time of Āryabhaṭa so that his verse under consideration enables us to find the names of as many places as there are decimal units with special names.

It will thus be seen that Āryabhaṭa's verse quoted above contains an exposition of a place-value decimal notation. It has been shown elsewhere[3] that Āryabhaṭa's rule for the extraction of the cube root necessitates the left-to-right order of arrangement of the component parts of a number in descending order of magnitude. We, therefore, conclude that the modern arithmetical notation was given by Āryabhaṭa in the above verse. This conclusion follows also from a comparison of the verse with the corresponding verses of Mahāvīra and Bhāskara who were undoubtedly too familiar with the modern notation.

We have assumed that the elder Āryabhaṭa is the author of the above verse. And we are justified in doing so. Both the Āryabhaṭas have composed their works in the *Āryā* metre and used its various forms. Hence an examination

---

[1] The second half of the verse giving Āryabhaṭa's alphabetic scheme is as follows:—

'Khadvinavake svarā nava varge-avarge navāntyavarge vā.' Here 'khadvinavake' means 'in 18 places.' Fleet takes 'navāntyavarge' to mean 'in the nineteenth place.' But how can 9 vowels be used in the nineteenth place while Āryabhaṭa assigns each vowel to two places? Hence Fleet's interpretation seems to be wrong. The expression 'navantyavarge' is formed of the two words 'nava' and 'antyavarge.' Here 'antyavarge' can only mean 'in the following group (of 18 places)' so that the number of places contemplated by Āryabhaṭa knows no limit (See Bulletin of the Calcutta Mathematical Society, Vol. 17 (1926), pp. 195–196).

[2] Fleet thinks the reason to be that "none of the practical and fundamental elements, which it was absolutely necessary to state in the *Dasagītikāsūtra* (i. e. Chapter I of the *Āryabhaṭīyam*) runs beyond ten places of figures." (Journal of the Royal Asiatic Society, 1911, p. 116).

[3] Journal of the Bihar and Orissa Research Society, March, 1926, pp. 78–82

of the metre will not help us. Let us first consider the style. Leaving out of consideration those verses which contain numbers expressed in alphabetic notations, the style of the elder Āryabhaṭa is more lucid and graceful than that of the younger. And the style of the verse under consideration differs from that of the author of the *kaṭapayādi* system and is quite in keeping with the style of the astronomer who gives 476 A.D. as the date of his birth. Secondly, if the younger Āryabhaṭa had been the author of the above verse, its place would have been between the first and the second verse of the first chapter as given in Sudhākara Dvivedī's edition (Benares, 1910) of his work *Mahāsiddhānta*. For, the second verse gives the *kaṭapayādi* system which, as stated before, is the modern notation with the signs replaced by consonants. Unlike the *Āryabhaṭīyaṃ* the *Mahāsiddhānta* abounds in numbers having more than ten places of figures.[1] Why, then, should its author give the names of ten places only? The reason cannot be brevity, as he has not been brief in other cases. The rule for the extraction of the cube root has been stated by him in two verses and a half whereas his senior namesake has compressed it in a single similar verse. Again, as the modern notation and the names of places were too well-known when the younger Āryabhaṭa wrote, it would have been quite sufficient to name a few places only if the consideration of brevity induced him not to mention the names of places frequently used in his work. For example, brevity and clarity could be simultaneously secured by formulating the rule in the *Upagītikā* formt of he *Āryā* metre as follows:

Ekadaśaśatādināṃ sthānāt sthānaṃ
daśaguṇaṃ syāt

One might ask, 'Why did not the elder Āryabhaṭa simplify the verse in this way?' Our answer is that he was the first to record the principle of place-value which does not seem to have gained sufficient popularity when the *Āryabhaṭīyaṃ* was composed. Thirdly, a comparison of the names of places contained in the verse under consideration with those given by Mahāvīra (850 c.), Śrīdhara (1020 c.), and Bhāskara (1150 c.) lends additional support to our conclusion as to the authorship of the verse. The fact that the above three mathematicians working respectively in Mysore, Bengal, and Rājputanā—provinces of India widely separated from one another—in different centuries use the term *lakṣa* instead of the term *niyuta*, points strongly to the great extent of popularity which the term *lakṣa* gained throughout India in those days and which it still enjoys. If the younger Āryabhaṭa (10th century) had been the author of the

[1] For 11 places of figures see I, 7, 8; II, 5, 6; XIV, 19, 33, 41.
For 12 places of figures see II, 17; XIV, 25, 26, 30, 32; XVII, 8, 13, 15, 27.
For 13 places of figures see I, 13, XIV, 20, 21, 22, 23, 24, 34, 35, 36, 37, 39.
For 16 places of figures see I, 35.
For 17 places of figures see I, 32.

verse, he would have, in all probability, replaced the term *niyuta* by *lakṣa* like his predecessor Mahāvīra and his successors Śrīdhara and Bhāskara.

We are now in a position to return to Burnell's query quoted above. Both Āryabhaṭa and Brahmagupta have omitted the easier topics from the chapters of their works dealing with mathematics. But while the enunciation of the modern arithmetical notation finds a place in the elder Āryabhaṭa's work, Brahmagupta considers it too common-place for inclusion in his work. Āryabhaṭa would have certainly omitted it on account of its simplicity if it were popular when the *Āryabhaṭīyaṃ* was written or if it had occurred in a previous work. This leads us to conclude that the *Āryabhaṭīyaṃ* is the first work in which the modern arithmetical notation has been explained. So we are inclined to answer Burnell's query in the affirmative.

THE EVOLUTES OF A CERTAIN TYPE OF SYMMETRIC PLANE CURVES

By J. B. REYNOLDS, Lehigh University

The following ten theorems have been proved regarding the parabola and its evolutes.[1]

I. All evolutes of the parabola are symmetrical with respect to the axis of the parabola.

II. Every evolute of the parabola of odd order has one cusp only and it lies upon the axis of the parabola. Evolutes of even order have no cusps.

III. The curvature of the evolutes of the parabola at corresponding points decreases as the order of the evolutes increases.

IV. The curvature of each evolute is greatest where the evolute crosses the axis of the parabola.

V. Each evolute of the parabola of odd order has a point in common at its cusp with the evolute of next higher order at its vertex.

VI. The cusps on all evolutes of the parabola of order $4n+1$ lie within the parabola and those of order $4n-1$ lie without the parabola.

VII. The vertices of all evolutes of the parabola of order $4n+2$ lie within the parabola and those of order $4n$ lie without the parabola.

VIII. The vertices of the successive evolutes lie at increasingly greater distances from the vertex of the parabola.

IX. All evolutes of the parabola of odd order are concave towards the tangent at the vertex of the parabola.

---

[1] See J. B. Reynolds, *Evolutes of the Parabola*. Proceedings of Pennsylvania Academy of Science, vol. 1.

X. Evolutes of the parabola of even order beginning with the second have concavity alternately opposite to and in agreement with that of the parabola.

The question arises as to how many of these or similar theorems may be proved to hold for a general type of real curves of which the parabola is a particular case. The parabola and its evolutes of even order are found to have many common properties and may be said to belong to type C as defined below. The evolutes of odd order are of a different sort and do not fall under the definition of type C curves. A close study has led to defining the curves under consideration as those having the following characteristics:

(a) They are plane analytic symmetric curves.

(b) They have continuous evolutes.

(c) Each one consists of one real branch extending to infinity.

(d) They have no real point singularities.

Such a curve is here called a curve of type C. The definition of type C curves is designed to be such that a number of important theorems may be proved for the curve of this type and its evolutes. Furthermore, every evolute of even order satisfies the same conditions. Examples of type C curves are the parabola, its evolutes of even order, one branch of the hyperbola or one branch of $x = \sec y$.

The characteristic (b) might seem unfamiliar and hard to test for elementary curves but may, in fact, be readily applied as will become evident presently. Several of the theorems have to do, also, with the evolutes of a curve of type C.

Any plane curve $y=f(x)$, except the straight line, for which dy/dx is defined, may be represented by the vector equation $r=F(\theta)i+G(\theta)j$ in which i and j are unit vectors in a direction parallel respectively to a pair of mutually perpendicular axes and θ is the angle which the tangent to the curve makes with a fixed line of the plane.[1] For example, let i and j be parallel to the x- and y-axes respectively and the fixed line be the x-axis. This representation then corresponds to the parametric equations $x=F(\theta)$, $y=G(\theta)$, in which $\tan\theta = dy/dx$. To illustrate: For the parabola $y^2=4ax$, taking as parametric equations $x=at^2$ and $y=2at$, we find $dy/dx=\tan\theta=1/t$, whence $t=\cot\theta$ and the corresponding vector equation is $r=ai\cot^2\theta+2aj\cot\theta$.

THEOREM I. *The necessary and sufficient condition for the plane curve $r=F(\theta)i+G(\theta)j$ and all its evolutes to have finite coordinates corresponding to every finite point on r is that $F, F', \cdots, F^{(n)}$ and $G, G', \cdots, G^{(n)}$ shall be finite, the superscripts referring to derivatives with respect to the angle θ which the tangent to the curve makes with a fixed line of the plane.*

For, if $r_1=F_1i+G_1j$, $r_2=F_2i+G_2j$, $\cdots$, $r_n=F_ni+G_nj$ represent the first n evolutes of r we have[2]

[1] See, for example, W. Blaschke, *Vorlesungen über Differential-geometrie* I.

[2] See Scheffer's *Anwendung der Differential und integral Rechnung auf Geometrie* vol. 1. page 68.

$$
\begin{aligned}
F_1 &= F - G' & G_1 &= G + F' \\
F_2 &= F - 2G' - F'' & G_2 &= G + 2F' - G'' \\
F_3 &= F - 3G' - 3F'' + G''' & G_3 &= G + 3F' - 3G'' - F''' \\
&\cdots & &\cdots \\
&\cdots & &\cdots \\
F_{n+1} &= F_n - G_n' & G_{n+1} &= G_n + F_n'.
\end{aligned}
$$

It is readily seen that $F_1, F_2, \cdots, F_n$ and $G_1, G_2, \cdots, G_n$ will be finite if $F, F', \cdots, F^{(n)}$ and $G, G', \cdots, G^{(n)}$ are finite, and further, if any of the latter be infinite, at least one of the former will be infinite. The condition stated in the theorem is therefore necessary and sufficient.[1]

THEOREM II. *If for every point on a curve there corresponds a finite point on each of its evolutes, neither the curve nor any of its evolutes can have a point of inflection.*

At a point of inflection on $r=F(\theta)i+G(\theta)j$, there must exist the condition that

$$\frac{d\theta}{dF} = \frac{d}{d\theta}\,\frac{(\arctan G'/F')}{F'} = \frac{(F'G'' - G'F'')}{(F')^3} = 0.$$

But since $\theta = \arctan G'/F'$, $(F')^2+(G')^2 = F'G''-G'F''$, so that for a point of inflection to exist we must have $1/F'+(G'/F')^2(1/F')=0$. By Theorem I, however, F' cannot be infinite; therefore there can be no point of inflection on r.

Since F_1 and G_1 and all their derivatives with respect to θ are finite, the same method of reasoning shows that r_1 cannot have a point of inflection; likewise for all the other evolutes of r.

THEOREM III. *A plane curve with one infinite branch, no point singularities and continuous evolutes, cannot be symmetrical with respect to a point.*

Since there is only one branch to the curve and no point singularities, if there be a point of symmetry it must lie on the curve. Let the point of symmetry be $P(a, b)$ on the curve $y=f(x)$; let y^* be the value of y at the point $(a+h, b+k)$ in which h is positive and real and k of either sign, and $^*y$ the value of y at the point $(a-h, b-k)$. Then, since y is an analytic function of x,

$$
\begin{aligned}
y^* &= b + h\, dy/dx + \tfrac{1}{2}h^2\, d^2y/dx^2 + (1/6)h^3\, d^3y/dx^3 + \cdots \\
{}^*y &= b - h\, dy/dx + \tfrac{1}{2}h^2\, d^2y/dx^2 - (1/6)h^3\, d^3y/dx^3 + \cdots
\end{aligned}
$$

[1] If one should inquire, for example, whether $4a^3x=y^4$ is of type C, he would find from $x=at^4/4$, $y=at$ that $dy/dx=1/t^3$, whence $G=a\cot^{1/3}\theta$, $G'=-(1/3)\cot^{-2/3}\theta\csc^2\theta$. Therefore G' is infinite for $\theta=\pi/2$. Hence this is not a curve of type C.

For symmetry with respect to P, $b-{}^*y=y^*-b$, whence $d^2y/dx^2=0$. But this makes P a point of inflection; hence there can be no such point on the curve and a curve of type C cannot be symmetrical with respect to a point.

THEOREM IV. *A curve of type C crosses an axis of symmetry once and only once.*

For $r=Fi+Gj$ to be symmetrical with respect to the i-axis, F must be an even function of θ and G an odd function of θ. The curve must cross the axis at least once or it will have two infinite branches. Where the curve crosses the axis of symmetry it must cross at right angles, else there will be a singular point at the point of crossing. Suppose the curve crosses the axis of symmetry at A. If it again crosses at B, it must cross there at an angle of 90° with the axis and the arc AB will correspond to a symmetrical arc on the opposite side of the axis, and the curve will be entire and finite or else discontinuous. It therefore crosses the axis of symmetry once and only once.

THEOREM V. *A curve of type C and all its evolutes are symmetrical with respect to the same axis.*

Since F is an even function of θ and G an odd one, G', F'', G''', F'''' etc. are even functions while F', G'', F''', G'''' etc. are odd functions of θ. An inspection of the equations given in the proof of Theorem I shows that F_1, F_2, F_3, F_4, etc. are even functions of θ and G_1, G_2, G_3, G_4, etc. are odd ones, whence it follows directly that all the evolutes of a plane curve symmetrical with respect to an axis are symmetrical with respect to the same axis.

THEOREM VI. *A curve of type C cannot at any point be parallel to the axis of symmetry but may approach parallelism as it recedes towards infinity.*

As θ decreases from 90° at the point where the curve crosses the axis, if the curve should at any point become parallel to the axis of symmetry, it must at some succeeding point either cross the axis (a second time) or pass through a point of inflection, the result in either case being contrary to the previous theorems. Since θ may vary from 90° to 0° and since $d\theta/dF<0$, the curve may approach parallelism with the axis as θ approaches 0°. We shall suppose that θ varies from 90° to A and from $-90°$ to $-A$ where $0<A<90°$.

THEOREM VII. *The odd ordered evolutes of a curve of type C have cusps on the axis of symmetry and the even ordered evolutes cross the axis of symmetry at right angles at points of greatest absolute curvature.*

We assume the arcs to increase on all curves in either direction from the point on the axis of symmetry and the tangent directed in the sense of increase of the arc s. This means that, starting from the point on the axis of symmetry,

s is measured positively for r into both the first and fourth quadrants as θ varies from 90° to A and from $-90°$ to $-A$. Now we have

$$\tan\theta = G'/F' = -F_1'/G_1' = G_2'/F_2' = \cdots = G_{2k}'/F_{2k}' = -F_{2k+1}'/G_{2k+1}'$$

whence $\theta=\theta_1\pm 90°=\theta_2\pm 180°=\theta_3\pm 270°$, the upper signs going with the range 90° to A and the lower signs with the range $-90°$ to $-A$. In general

$$\theta_{4n} = \theta_{4n+1} \pm 90° = \theta_{4n+2} \pm 180° = \theta_{4n+3} \pm 270° \quad (n = 0,1,2,\cdots).$$

As θ varies from 90° to A, θ_{4n} varies from 90° to A; θ_{4n+1} varies from 0° to $A-90°$; θ_{4n+2} varies from $-90°$ to $A-180°$; and θ_{4n+3} varies from 180° to $A+90°$.

As θ varies from $-90°$ to $-A$, θ_{4n} varies from $-90°$ to $-A$; θ_{4n+1} varies from 0° to $90°-A$; θ_{4n+2} varies from 90° to $180°-A$; and θ_{4n+3} varies $-180°$ to $-90°+A$.

An inspection of these relations shows that, starting from the point on the axis of symmetry, even ordered evolutes extend perpendicularly to the axis in opposite directions, that is, cross it at right angles. Odd ordered evolutes, starting from the point on the axis of symmetry, extend in the same direction (parallel to the axis) and therefore have cusps at this point. It follows, therefore, that corresponding to the cusps on the odd ordered evolutes are points of greatest absolute curvature (vertices) on the even ordered evolutes.

THEOREM VIII. *Each evolute of a curve of type C of odd order has a point in common at its cusp with the evolute of next higher order at its vertex.*

Since $F_{2k}=F_{2k-1}-G'_{2k-1}$ and $G_{2k}=G_{2k-1}-F'_{2k-1}$ $(k=1,\ 2,\ \cdots)$, and since at a cusp on r_{2k-1}, F'_{2k-1} and G'_{2k-1} are each zero, it follows for this point on the axis of symmetry that $F_{2k}=F_{2k-1}$ and $G_{2k}=G_{2k-1}$; that is the curves r_{2k-1} and r_{2k} have a point in common on the axis of symmetry. Since the absolute value of the curvature is a maximum at this point on r_{2k}, it corresponds to a vertex on the curve.

THEOREM IX. *Evolutes of order* $4n$ *and* $4n+3$ *are concave in the same sense as the given curve of type C while evolutes of order* $4n+1$ *and* $4n+2$ *are concave oppositely to the given curve.*

An inspection of the variations in θ_{4n} $(n=0,\ 1,\ 2,\ \cdots)$ and in θ_{4n+3} as given in Theorem VII shows that for these evolutes the concavity is in the same direction as for θ, while the variations in θ_{4n+1} and in θ_{4n+2} are in the opposite direction. Taking the axis of symmetry as a half line directed positively from the vertex of the given curve as initial line for θ, we may say that the concavity of evolutes of order $4n$ and $4n+3$ is positive while that of evolutes of order $4n+1$ and $4n+2$ is negative.

ON *MÛLA*, THE HINDU TERM FOR "ROOT"

By BIBHUTIBHUSAN DATTA, University of Calcutta

The May (1926) issue of this Monthly contains an interesting article by Mr. Solomon Gandz on the origin of the term "root." It is interesting, no doubt, to follow the steps by which most of the civilized nations of the world came to use in the theory of numbers a term which essentially belongs to the vegetable kingdom. But certain statements in that article require correction. The learned writer has said: "The Chinese, indeed, do use the word *kun* to mean root, grass, and shrub, and the Hindus also use the word *mûla* for the root of a plant, but this was very likely due to the Arabic influence, which is so often seen in China and which may have spread into India by way of China." The latter part of this observation is certainly far from the truth.

Mr. Gandz together with Professor Smith[1] holds the opinion that the term "root" has its origin in the Arabic term *jadhr*. Originally *jadhr* was a concrete concept meaning "square basis." The earliest Arabic mathematician to give a clear definition of *jadhr* was Mohammed ibn Mûsâ al-Khowârizmî (*c.* 825). He "was possibly influenced by his Hebrew predecessors," who used the term *iqqar* probably in the same concrete sense. In later times, the original significance of *jadhr* was, however, forgotten by the Arabic scholars themselves. About 1600, they understood by it only an abstract concept. To express this misunderstood concept the medieval Latin translators of the Arabic works used the term *radix* ("root").[2] All these have been well pointed out by the learned writer. But he has fallen into an error in speaking of the Hindu mathematics.

So far as our present knowledge goes, the Hindus have used the term *mûla*[3] in connection with the theory of numbers, from more than a millenium before

[1] David Eugene Smith, *History of Mathematics*, vol. 2, p. 150.

[2] Some early Latin translators of Al-Khowârizmî's works such as John of Sevile (1140), Robert of Chester (1145), and Gerard of Cremona (1150) used *radix* for *shai*, the Arabic term for the first power of the unknown quantity in an algebraic equation. As has been already remarked by Professor Karpinski, this use is unique. It seems that Al-Khowârizmî's definition is somewhat responsible for this usage. "Of these then the root is anything composed of units which can be multiplied by itself, or any number greater than unity multiplied by itself: or that which is found to be diminished below unity when multiplied by itself. The square is that which results from the multiplication of a root by itself." (See Robert of Chester's Latin translation of the *Algebra of Al-Khowârizmî*, edited with English translation by Louis Charles Karpinski, New York, 1915, p. 66). It is not easy to say why Al-Khowârizmî defined the term for the unknown quantity by the property that it can be multiplied by itself. But such a definition is bound to lead others to regard the unknown quantity as the root. At any rate it discloses an abstract concept of the Arabic term for root.

[3] The term *mûla* is also used with a different significance in the Hindu mathematics. It means the capital of a loan as opposed to the interest. But our observations in this article will always remain confined to its use in connection with the square root and the cube root. It may be pointed out that the former use of the word *mûla* is also an old one.

the Arabic *jadhr* became transformed into the Latin *radix*. It is found in the works of the earliest known Hindu mathematician, Âryabhaṭa who wrote in 499 A.D., that is, three centuries and a quarter before Mohammed ibn Mûsâ. It was also used by Varāhamihira (505), Lalla (c. 578), Brahmagupta (628), Śrîdhara (c. 750), Mahâvîra (c. 853), Âryabhaṭa the younger (c. 950), and Bhâskara(1150). It is noteworthy, indeed, that none of the Hindu scholars has ever attempted to define the term *mûla*. They began with clear definitions of the terms *varga* (square) and *ghana* (cube). Here they used more or less identical words: "the product of two equal (quantities) is a square," "the product of three equal (quantities) is a cube."[1] Immediately after the definitions, the Hindu mathematicians proceeded to expound the rules for extracting the *mûla* of a square and of a cube. So for them the term *mûla* must have been derived directly from the reverse process. This will be further corroborated by the following statement of Brahmagupta which was made on a different occasion: "The root of a square is that of which it is the square."[2] Thus we find that the Hindus used the term *mûla*, in the theory of numbers, as an abstract concept from the fifth century A.D. Hence Mr. Gandz's conjecture that the Hindus did so under the Arabic influence must be wrong.

Synonymous with the term *mûla*, the Hindus use another term in the theory of numbers, namely *pada*.[3] The word *pada* has been used with this significance by almost all the Hindu mathematicians with the exception of Âryabhaṭa the elder. Hence the use of *mûla* is older than the use of *pada* in this respect. Now *pada* is not certainly a botanical term. In fact, it has very little to do with the vegetable kingdom.

In Sanskrit—as also in Pali—the word *pada* means 'step,' 'footing,' 'part, portion,' 'side,' 'place,' 'cause,' 'a square on a chess-board,' 'a plot of ground,' etc. The word *mûla* has various and wider meanings. Among others it means 'root' (of a plant or tree; but also figuratively the foot or lowest part or bottom of anything), 'basis,' 'foundation,' 'cause,' 'origin,' 'the edge' (of the horizon),

---

[1] Brahmagupta has also terms for higher powers of a number. His term for the continued product of four or more like quantities is *tad-gata* literally, 'raised to that'; thus *pañca-gata* means fifth power—literally, 'raised to the fifth (power)'; *ṣaḍ-gata* means sixth power. The higher powers of a number are also expressed by a combination of *varga* and *ghana*. Evidently the latter mode can have only limited applications. See *Brâhma-sphuṭa-siddhânta*, ed. Sudhakara Dvivedi, Benares (1902), Ch. xviii, verse 42.

[2] *Padam kṛtir-yat tat. Brâhma-sphuṭa-siddhânta*, Ch. xviii, verse 35.

[3] In the Hindu mathematical treatises, the word *pada* has also been used to denote the number of terms in a series in arithmetical progression. It is very likely that this application arose out of the analogy with the particular result, $1+3+5+\cdots$ to n terms $=n^2$; so that $n=\sqrt{\text{sum}}$. Cf. Śrîdhara's *Triśatikâ*, ed. Sudhakara Dvivedi, Benares, verse 11, where the term *varga* has also been defined as the sum of this particular series in arithmetical progression.

and 'immediate neighborhood.'[1] Certain meanings of either of the two words must be figurative. The import common to the two are 'cause,' 'origin,' 'foundation,' or 'basis.' These are therefore the concepts embodied in the terms *mûla* and *pada* in the theory of numbers. Again it is very difficult to say which meaning of the word *mûla*, botanical or otherwise as 'cause,' etc., is the older. The word was used with each significance in the earliest Vedic literature of the Hindus; and, indeed, by the time it was introduced into the Hindu mathematics, it had acquired various other meanings. But it is also a fact that the use of the word *mûla* in the abstract sense of 'cause,' 'origin,' 'foundation,' 'basis' has still wider and more copious applications in the Sanskrit and Pali literatures as well as in the Hindu theological and philosophical terminologies. So it is natural to assume as we have already done from other considerations, that the Hindus introduced the word *mûla*—like the word *pada*—into the theory of numbers in the abstract sense of 'cause,' 'origin,' 'foundation,' or 'basis.'

There is, however, one Hindu mathematician, namely Âryabhaṭa the elder, who seems to have in view also a concrete concept for the term *mûla*, for he gives geometrical definitions for the terms *varga* and *ghana* side by side with the arithmetical ones. He says: "*Varga* is the equi-four-sided figure (*samacaturasra*), the area (*phala*) and the product of two equal (quantities); similarly *ghana* is the (equi-) twelve-sided figure (*dvâdaśâsra*), the volume (*phala*) and the product of three equal (quantities)."[2] Hence the *mûla* of the *varga* must mean the cause or basis of the square figure, that is, the side; the cause or basis of the square area, that is, square unit; and the cause or basis of the square number, that is, a pure number. Similarly the *mûla* of the *ghana* must mean the side, the unit cube, as also a pure number. From what has been said before, it will appear that these interpretations as the side or the edge, the square unit, or the unit cube, are not foreign to the words *mûla* and *pada*. It is to be noticed, however, that the geometrical significance contained in the above definitions of Âryabhaṭa appears to have been overlooked by the later Hindu mathematicians. For we find nothing of the kind in the works of Brahmagupta, Śrîdhara, Mahâvîra and Bhâskara. Certain recognition was given to it in Prithudakaswâmi's commentary on *Brâhma-sphuṭa-siddhânta* (975).[3]

A clearer and fuller exposition is that of Parameśvara (c. 1430), a scholiast of the elder Âryabhaṭa. In commenting upon the above definition of *varga*, he

[1] See Monier-Williams's *Sanskrit-English Dictionary,* new edition by Leumann and Cappeller; Rhys Davids and Steele's *Pali-English Dictionary*. Cf. also the St. Petersburg Sanskrit dictionary.

[2] *Âryabhaṭîya, Gaṇitapâda,* verse 3. The term *samacaturasra* is a very old one; literally it means equi-four-angled figure, for *asra* = angle. But it is used to denote a square figure. Similarly *dvâdaśâsra* is used to denote a cube.

[3] See Henry T. Colebrooke, *Algebra with Arithmetic and Mensuration from The Sanscrit of Brahmegupta and Bhascara,* London (1817), pp. 280–1, footnotes.

remarks: "The summation of the *kṣetraphalas* (meaning the elementary areas) is termed the *varga*. In the case of a given square, the *kṣetraphalas* are the squares formed by four sides measuring one cubit each, that are contained inside it. Similarly, in the case of a triangle, a circle, or any (non-square) figure also, the *kṣetraphalas* are the square pieces formed by four sides measuring one cubit each, that can be assumed to be contained inside it." Another explicit statement is that of Kamalākara, the author of the *Siddhānta-tattva-viveka*. According to him "the square root (*mûla*) is of two kinds, a line and a number." His further statements deserve more than a passing notice. He says that the root of a perfect square number is a line as well as a number but the root of an imperfect square number, that is, the value of a surd is only a line. He has explained his ideas with the help of illustrative examples and by evaluating certain surds geometrically.[1]

It may be remarked in conclusion that what is found in the geometry section of the works of Mohammed ibn Mûsâ is an explicit statement and fuller treatment of one of the above mentioned geometrical ideas of Âryabhaṭa, viz., that of the unit square. Though Âryabhaṭa's ideas appear less explicit,[2] they are certainly wider and more logical. For his ideas about the *mûla* include the Greek conception of 'side,' the so-called Arabic conception of 'area' as well as arithmetical conception of pure number. Further it includes the conception of a unit cube which is found nowhere else.[3] As Al-Khowârizmî was acquainted with the Hindu mathematics, particularly with the *Gaṇita* of Âryabhaṭa, is it possible that he was influenced by his Hindu predecessor.[4]

---

[1] *Siddhânta-tattva-viveka*, ed. Sudhakara Dvivedi, Benares (1885), Ch. iii, verse 21 (*vâsanâbhâṣya*); cf. also Ch. ii, verse 158.

[2] The extreme conciseness of Âryabhaṭa's book is probably responsible for it.

[3] It may be noted in passing that Bhâskara's measures include definitions of unit cube (*ghana-hasta*, literally solid cubit) and also unit area (*nivartana*).

[4] Mohammed ibn Mûsâ al-Khowârizmî wrote a book, *On the Hindu Art of Reckoning*. His algebra contains a reference to Âryabhaṭa's value of π.

The definition of *iqqar*, the Hebrew word for *jadhr*, quoted by Mr. Gandz from an old Hebrew geometry by an anonymous writer, is vague and almost meaningless—as has also been admitted by him. He had to amend it in the light of the concrete meaning of Al-Khowârizmî, so as to bring it into line with the definition of the latter writer. Moreover, the date of this anonymous Hebrew writer is so uncertain that it is difficult to determine the chronological sequence between him and Mohammed ibn Mûsâ (Smith, *History*, vol. 1, p. 174). So it cannot be stated definitely who was influenced by whom. Dr. Gandz thinks "perhaps the two were influenced by some common source which is now unknown." It is not impossible that the works of Âryabhaṭa were this common source. For we know it on the authority of the Bishop Sêverus Sebokht of Nisibus (in Northern Mesopotamia) that the knowledge of the Hindu science of arithmetic had spread into Syria as early as the beginning of the seventh century A. D.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. CONCERNING A THEOREM IN DETERMINANTS

By J. J. NASSAU, Case School of Applied Science

The following is a proof of a theorem stated in this Monthly by Professor H. S. Uhler:[1]

The result of compounding the matrix

$$M_b = \begin{pmatrix} b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_k & a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix} ; \; k \leqq (n-1), \; rank = k$$

with its conjugate is a square matrix. The determinant D_b of this matrix is[2]

$$\begin{vmatrix} b_1^2 + c_{11} & b_1b_2 + c_{12} & \cdots & b_1b_k + c_{1k} \\ b_1b_2 + c_{21} & b_2^2 + c_{22} & \cdots & b_2b_k + c_{2k} \\ \vdots & \vdots & & \vdots \\ b_1b_k + c_{k1} & b_2b_k + c_{k2} & \cdots & b_k^2 + c_{kk} \end{vmatrix} = \begin{vmatrix} b_1 & a_{11} & a_{12} & \cdots & a_{1,k-1} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2,k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ b_k & a_{k1} & a_{k2} & \cdots & a_{k,k-1} \end{vmatrix}^2$$

---

[1] Let $-N$ denote the bordered symmetrical determinant

$$\begin{vmatrix} 0 & b_1 & b_2 & \cdots & b_k \\ b_1 & c_{11} & c_{12} & \cdots & c_{1k} \\ b_2 & c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ b_k & c_{k1} & c_{k2} & \cdots & c_{kk} \end{vmatrix} \text{ where } c_{r,s} \equiv \sum_{j=1}^{j=n} (a_{r,j}\, a_{s,j}) = c_{s,r} \text{ and the } a\text{'s belong}$$

$$\text{to the matrix } M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}; \; k \lessgtr (n-1), \text{ rank} = k.$$

Is there a simple proof for the following theorem? *N equals the arithmetical sum of the squares of $\binom{n}{k-1}$ determinants of order k in every one of which the first column read downward consists of $b_1, b_2, \cdots, b_k$ and the remaining $k-1$ monomial columns constitute collectively one of the combinations of the n columns of the matrix M taken $k-1$ at a time.*

[2] Scott and Mathews *Theory o. Determinants*, p. 54.

$$
+\begin{vmatrix} b_1 & a_{11} & \cdots & a_{1,k-2} & a_{1k} \\ b_2 & a_{21} & \cdots & a_{2,k-2} & a_{2k} \\ \vdots & \vdots & & \vdots & \vdots \\ b_k & a_{k1} & \cdots & a_{k,k-2} & a_{kk} \end{vmatrix}^2 + \cdots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix}^2
$$

$$
+\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1,k+1} \\ a_{21} & a_{23} & \cdots & a_{2,k+1} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k3} & \cdots & a_{k,k+1} \end{vmatrix}^2 + \cdots .
$$

The left-hand side can be written as the sum of 2^k determinants of which $\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k}$[1] determinants have two or more columns alike. Hence the left-hand side composed of the $k+1$ remaining determinants is

$$
\begin{vmatrix} b_1^2 & c_{12} & \cdots & c_{1k} \\ b_1b_2 & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & & \vdots \\ b_1b_k & c_{k2} & \cdots & c_{kk} \end{vmatrix} + \begin{vmatrix} c_{11} & b_1b_2 & c_{13} & \cdots & c_{1k} \\ c_{21} & b_2^2 & c_{23} & \cdots & c_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{k1} & b_2b_k & c_{k3} & \cdots & c_{kk} \end{vmatrix} + \cdots
$$

$$
+\begin{vmatrix} c_{11} & \cdots & c_{1,k-1} & b_1b_k \\ c_{21} & \cdots & c_{2,k-1} & b_2b_k \\ \vdots & & \vdots & \vdots \\ c_{k1} & \cdots & c_{k,k-1} & b_k^2 \end{vmatrix} + \left|c_{11} \quad c_{22} \quad \cdots \quad c_{kk}\right| =
$$

$$
-\begin{vmatrix} 0 & b_1 & b_2 & \cdots & b_k \\ b_1 & c_{11} & c_{12} & \cdots & c_{1k} \\ \vdots & \vdots & \vdots & & \vdots \\ b_k & c_{k1} & c_{k2} & \cdots & c_{kk} \end{vmatrix} + \left|c_{11} \quad c_{22} \quad \cdots \quad c_{kk}\right|.
$$

The right-hand side is made up of the sum of $\binom{n}{k-1}$ determinants of the type $|b_1a_{21} \cdots a_{kk-1}|^2$ and $\binom{n}{k}$ determinants of the type $|a_{11}a_{22} \cdots a_{kk}|^2$.

This last sum of $\binom{n}{k}$ determinants is equivalent to the determinant of the square matrix obtained by compounding the matrix

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix} \text{ with its conjugate, and hence is}
$$

equal to $|c_{11}c_{22} \cdots c_{kk}|$.

[1] The symbol $\binom{k}{l}$ means the number of combinations of k things l at a time. It is well known that

$$\binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k}=2^k.$$

From the above, the proof of the theorem is evident.

REMARKS: 1. The above proof suggests the relation

$$D_b = -\begin{vmatrix} -1 & b_1 & \cdots & b_k \\ b_1 & c_{11} & \cdots & c_{1k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_k & c_{k1} & \cdots & c_{kk} \end{vmatrix}$$

2. If in the determinant $\begin{vmatrix} 0 & b_1 & \cdots & b_k \\ b_1 & c_{11} & \cdots & c_{1k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_k & c_{k1} & \cdots & c_{kk} \end{vmatrix}$ we let $b_1 = b_2 = \cdots = 1$,

we have the following theorem:

THEOREM: *The sum of the signed primary minors*[1] *of the symmetrical determinant* $|c_{11}c_{22} \cdots c_{kk}|$ *is equal to the sum of the squares of* $\binom{n}{k-1}$ *determinants of order k in each of which the first column is composed of ones and the remaining* $k-1$ *monomial columns are composed collectively of one of the combinations of the n columns of the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix} \text{ taken } k-1 \text{ at a time.}$$

II. A SIMPLE WAY TO DISCUSS POINTS OF INFLECTION ON PLANE CUBIC CURVES

ALAN D. CAMPBELL, Syracuse University

In this paper we show by means of illustrations a simple but very effective way to discuss points of inflection on plane cubic curves. We used this method with excellent results in a paper[2] entitled *Plane cubic curves in the Galois fields of order* 2^n and in another paper (not yet published) on plane cubic curves in the Galois fields of order p^n, $p>2$. [We abbreviate the name of these fields to $GF(p^n)$, $p>2$]. The method can be used also in the real and the complex domains and it avoids the (often difficult) task of solving simultaneously a cubic and its Hessian. In the $GF(2^n)$ the Hessian of a cubic is the cubic itself multiplied by the coefficient of the xyz-term in the cubic (loc. cit p. 396); so here the type of discussion given below seems to be the only feasible one.

[1] Muir, Proceedings of the Royal Society of Edinburgh, vol. 24, Part IV, p. 390.

[2] Annals of Mathematics, vol. 27 (1926), pp. 395–406.

Our plan consists mainly in seeking the tangents at points of inflection and not the points of inflection themselves. (We call such points more briefly "inflections"). Let us consider the cubic

$$y^2z = x^2(x+z), \tag{1}$$

which has an inflection at (0, 1, 0) with $z=0$ as tangent and a crunode at (0, 0, 1) with $y=\pm x$ as tangents. The tangent at any other inflection must have an equation of the form $l \equiv z-ax-by=0$. Solving this simultaneously with (1) we get the equation

$$x^3(1+a)+bx^2y-axy^2-by^3=0. \tag{2}$$

If l is to be a tangent at an inflection and (1) lies in the complex or the real domain or the $GF(p^n)$ with $p>3$, then (2) must be of the form

$$y^3+3cxy^2+3c^2x^2y+c^3x^3=0. \tag{3}$$

Note that these $GF(p^n)$ are the finite algebras corresponding to the finite geometries that have just p^n+1 points on each line.[1] In these algebras every number is reduced modulo p, and if $n>1$ a second modulus called a Galois imaginary is used. Thus in $GF(2^2)$ the five points on the line $x=0$ have the coordinates (0, 0, 1), (0, 1, 0), (0, 1, 1), (0, i, 1), (0, $i+1$, 1) where i is a root of the equation $x^2+x+1=0$ which has neither 0 nor 1 as a root. Also in $GF(2^n)$ we have $(\alpha x+\beta y+\gamma z)^2=\alpha^2x^2+\beta^2y^2+\gamma^2z^2$ and every number is a perfect square with just one square root. In $GF(p^n)$, $p>2$ there are non-squares. If $p=3$ we have $(\alpha x+\beta y+\gamma z)^3=\alpha^3x^3+\beta^3y^3+\gamma^3z^3$. In $GF(p^n)$, where $p^n=3m+1$, there are non-cubes, but in the other Galois fields every number has at least one cube root (and in some fields three cube roots). There are irreducible cubic equations in every $GF(p^n)$.

From (3) we see that we must have $(1+a)/(-b)=c^3$, $-1=3c^2$, $a/b=3c$; hence $c=\pm\sqrt{(-1/3)}$, $a=-3/(3+c^2)$, $b=-1/(3c+c^3)$. [The value $b=0$ gives $a=0$ and the tangent $z=0$ at (0, 1, 0)]. Therefore in the real domain the two other inflections on (1) are imaginary, but in the $GF(p^n)$, $p>3$, where $-1/3$ is a square, these inflections are real. (This method gives also the tangents at possible nodes or cusps).

In the $GF(2^n)$ the equation (3) must be replaced by

$$y^3+cxy^2+c^2x^2y+c^3x^3=0. \tag{3'}$$

From (3′) we get $(1+a)/b=c^3$, $1=c^2$, $a/b=c$, since $-1\equiv+1$ modulo 2. Hence $c=1$, $a=b$, $1+a=b$ which are incompatible equations; so if $p=2$ the cubic (1) has no more inflections.

In the $GF(3^n)$ we must replace (3) by $y^3+c^3x^3=0$; so we must have $(1+a)/(-b)=c^3$, $-1=0$, $a/b=0$ (which are impossible equations); hence for $p=3$ the cubic (1) has just the one inflection (0, 1, 0). The above facts can be obtained easily (except for $p=2$) by solving the Hessian of (1) simultaneously

---

[1] See Veblen and Young *Projective Geometry* vol. 1, pp. 201, 202 and also L. E. Dickson's *Linear Groups*, pp. 1–40.

with (1); but the conditions for the existence or nonexistence of other real inflections on (1) appear more readily by the above process.

Next let us consider the cubic

$$(4) \qquad y^2z = x^3 + (\alpha + 1)x^2z + \alpha xz^2, \qquad \alpha \neq 0 \text{ or } 1,$$

which has an inflection at (0, 1, 0) with $z=0$ as tangent. If we solve $y=ax+bz$ simultaneously with (4) we get

$$x^3 + (-a^2 + \alpha + 1)x^2z + (\alpha - 2ab)xz^2 - b^2z^3 = 0.$$

In the real or the complex domain and in the $GF(p^n)$, $p>3$ this equation must be of the form (3) with x and z replacing y and x respectively. Hence we have $-a^2+\alpha+1=3c$, $\alpha-2ab=3c^2$, $-b^2=c^3$, giving us the quartic

$$(5) \qquad 3c^4 - 4(\alpha + 1)c^3 + 6\alpha c^2 - \alpha^2 = 0$$

with resolvent cubic[1]

$$(6) \qquad \theta^3 + (1/108)(\alpha^4 - 2\alpha^3 + \alpha^2) = 0.$$

Dickson shows[2] that, if (6) has one and only one root in a $GF(p^n)$, $p>3$, then (5) is irreducible and so (4) has no more real inflections; and if (6) is irreducible then (5) has just one real root and so (4) has two more real inflections (if $-a^2+\alpha+1=3c$ and $b^2=c^3$ have real roots in a and b).

In the $GF(3^n)$ we have for the cubic (4) $-a^2+\alpha+1=0$, $\alpha-2ab=0$, $-b^2=c^3$; hence $a=\pm(-\alpha-1)^{1/2}$, $b=\alpha/\pm 2(-\alpha-1)^{1/2}$; so that according as $-\alpha-1$ is (or is not) a perfect square the cubic has two more real (or imaginary) inflections. In the $GF(2^n)$ we have for the cubic (4) $-a^2+\alpha+1=c$, $\alpha=c^2$, $-b^2=c^3$; hence $a=\alpha^{1/2}+\alpha^{1/4}+1$, $b=\alpha^{3/4}$ give us the tangent at one more real inflection.

Finally let us consider the cubic

$$(7) \qquad y^3 + \alpha z^3 + xyz = 0, \qquad \alpha \neq 0,$$

which has a crunode at (1, 0, 0) with $y=0$ as one of the tangents. If we solve (7) simultaneously with $x=ay+bz$, we get $y^3+ay^2z+byz^2+\alpha z^3=0$. If α is not a cube, we have no real inflections on (7). If $p=3$ the cubic has just one real inflection with tangent $x=0$. In the real domain we have one real and two imaginary inflections on (7).

In the above mentioned finite geometries we find plane cubic curves with numerous peculiarities, such as cubics with no node or cusp or real inflection,[3]

---

[1] See Burnside and Panton *Theory of Equations*, 2nd. edition, p. 127.

[2] Bulletin of American Mathematical Society, vol. 13 (1906), p. 5.

[3] Annals of Mathematics, loc. cit., pp. 398–400.

cubics with nine real inflections (such as $z^3+x^2y+xy^2=0$ when $p=2$, $n>1$), cubics with just one real inflection (see above), and so forth. These peculiarities are well brought out by the method (discussed in this paper) of finding the tangents at inflections.

III. A Note on the Function $y=x^x$

By RAYMOND GARVER, University of Rochester

I consider the function $y=a^x$, $a<0$, mentioned by Professor Campbell in the April Monthly, particularly useful because it illustrates that not all functions can be represented graphically. This is a novel idea to most students who have had analytic geometry and a beginning course in calculus, and one which I believe is worth-while bringing out. It might also be well to list other conditions (besides discontinuity at an infinite number of points) which render a function incapable of graphical representation.

A somewhat similar and equally interesting function is $y=x^x$. For $x>0$, the function is continuous and differentiable, and if we define the function as 1 for $x=0$, it is also continuous on the right for this value. The right-hand derivative does not exist, however, for $x=0$; at least it does not remain finite. The function has a minimum value for $x=1/e$, and the corresponding functional value is approximately .69. The area bounded by the curve, two positive ordinates, and the x-axis is given by $\int_a^b x^x dx$. I have not been able to evaluate the indefinite integral, but the definite integral can of course be approximated. On the other hand, for $x<0$, the function is completely discontinuous, much as the function a^x, $a<0$.

More generally, we might consider functions of the type $y=f(x)^{g(x)}$. Almost every special case will be found to give interesting results.

IV. On the Approximate Division of the Circle[1]

By ROGER A. JOHNSON, Hunter College

A recent note in this Monthly[2] discusses a construction for the approximate division of the circle into any number of equal parts, said to be used by draftsmen. The purpose of the present note is to call attention to a very similar construction, no more complicated but a much closer approximation.

The striking aspect of these constructions is the fact that one and the same method is applicable to all cases and yields a fairly accurate result. In the second

---

[1] EDITOR'S NOTE: The construction given in this paper is credited to the German Duke Karl Bernhard of Sachsen-Weimar. See K. T. Vahlen, *Konstruktionen und Approximationen*, (Teubner, 1911), p. 305.

[2] E. J. McShane, *An approximate division of the circle*, this Monthly, vol. 34 (1927), p. 140.

construction introduced below, the theoretical error is less than the inaccuracy of ordinary drawing.

The construction discussed by Messrs. Gwinner and McShane is as follows:

Let O be the center, AB a diameter, of a circle whose circumference is to be divided into n equal parts. Let ABP be an equilateral triangle; divide AB into n equal parts, and let C be the second point of division from A. Let PC produced meet the circle at D; then arc AD is nearly the nth part of the circumference.

The following construction is given in the *Exercise Manual in Geometry of Wentworth and Hill* (Ginn & Co., 1883) without references or comments.

With the same data as before, let AB be divided into n equal parts; and extend OA to E, so that AE equals one of these parts. Draw OF perpendicular to AB and equal to OE; let the first intersection of EF with the circle (i.e., that which is nearer to E) be G. Let H be the third point of division from A; then GH is approximately equal to a side of a regular polygon of n sides.

It will be seen that neither of these approximations is very close when n is 5, and that both are exact when $n=6$. For values of n greater than 6, the second method is uniformly a much closer approximation than the first. Mr. McShane finds that in the first method the error remains less than the value 10.265%, which it approaches as n increases; it would seem that this persistent high percentage of error renders the method less interesting from a theoretic standpoint. We can show that in the second method the error tends toward a limiting value of approximately 2/3 of 1% as the number of sides increases.

Without much difficulty we find that in the first solution the result, taking the radius OA as unity, is

$$\sin\theta = \frac{3^{1/2}[(1+16n^{-1}-32n^{-2})^{1/2}-(1-4n^{-1})^2]}{4-8n^{-1}+16n^{-2}}$$

and in the second

$$GH = [1-8n^{-1}+48n^{-2}-(1-6n^{-1})(1-4n^{-1}-4n^{-2})^{1/2}].$$

It is not easy to compare these results with the true values $\sin(2\pi/n)$ and $2\sin(\pi/n)$ respectively, except by actual computation. The following numerical values may be of interest:

n	8	10	16	24	50
True central angle	45°	36°	22°30′	15°	7°12′
Angle by first method	45°11′	36°21′	23°6′	15°38′	7°40′
Angle by second method	44°57.2′	35°56′	22°28.4′	15°0.3′	7°13.3′

If we expand the formulas according to increasing powers of $1/n$, we have in the respective cases

$$\sin\theta = 3^{1/2}\left(\frac{4}{n}-\frac{8}{n^2}\cdots\right); \qquad \overline{GH}^2 = 40^{1/2}\left(\frac{1}{n}-\frac{1}{5n^2}\cdots\right).$$

For sufficiently large values of n we may compare the accuracy of the solutions by comparison of the first terms of these expansions with sin $(2\pi/n)$ or merely with $(2\pi/n)$. In the first method, as obtained by McShane, the ratio of the solution to the true value approaches as limit the ratio

$$4\sqrt{3}/2\pi = 2\sqrt{3}/\pi = 1.10265$$

and in the second case,

$$\sqrt{40}/2\pi = \sqrt{10}/\pi = 1.0067 \quad .$$

Thus there is a residual error in the first method of over 10% and in the second method of less than 1%. It would be interesting to know whether a method can be devised in which the error approaches zero with increasing values of n.

RECENT PUBLICATIONS

Edited by W. B. Carver, Cornell University, to whom books and communications should be sent.

NEW BOOKS RECEIVED

Editor's Note: Hereafter there will be published each month in this department a list of all new books received. The first of these lists, which appears below, includes books received for some months past for which reviews have not yet been published. Not all books received are suitable for review in this periodical; and it is not always possible to find some one competent and willing to write a review. For these reasons we can not guarantee to publish reviews of all books sent to us for that purpose, but titles of all books received will be printed monthly.

Barnett, I. A. *Plane Analytic Geometry.* New York, John Wiley and Sons, 1926. ix+269 pages. $2.00.

Beck, H. *Einführung in die Axiomatik der Algebra.* Berlin, Walter de Gruyter, 1926. x+197 pages.

Bernard, D. M. *Plane Geometry.* Richmond, Johnson Publishing Company, 1927. xv+334 pages. $1.24.

Berwick, W. E. H. *Integral Bases.* (Cambridge tracts, No. 22) Cambridge, University Press, 1927. 95 pages. 6½ shillings.

Bridgman, P. W. *The Logic of Modern Physics.* New York, The Macmillan Company, 1927. xiv+228 pages. $2.50.

Buchanan, H. E. and Sperry, P. *Plane Trigonometry and Tables.* Richmond, Johnson Publishing Company, 1926. xi+116+112 pages. $1.64.

Carmichael, R. D., Davis, H. T., MacMillan, W. D. and Hufford, M. E. *A Debate on the Theory of Relativity.* Chicago, Open Court Publishing Company, 1927. viii+154 pages. $2.00.

REYMOND, ARNOLD. *History of the Sciences in Greco-Roman Antiquity*. Translated by Ruth Gheury de Bray. New York, E. P. Dutton and Company, 1927. x+245 pages. $2.50.

RIETZ, H. L. *Mathematical Statistics* (The third Carus Mathematical Monograph). Chicago, Open Court Publishing Company, 1927. xi+181 pages.

SOMMERFELD, A. *Three Lectures on Atomic Physics*. (Trans. by A. L. Brose.) New York, E. P. Dutton and Company, 1926. 70 pages.

STEFFENSEN, J. F. *Interpolation*. Baltimore, The Williams and Wilkins Company, 1927. ix+248 pages. $8.00.

TREVOR, J. E. *The General Theory of Thermodynamics*. Boston, Ginn and Company, 1927. x+104 pages. $1.60.

WALMSLEY, C. *Mathematical Analysis*. Cambridge, University Press, 1926. x+293 pages.

WASSERLOOS, EWALD and WOLFF, GEORG; Editors. *Mathematisch-Naturwissenschaftlich-Technische Bucherei*. Band 1, 2, 3, 5, 6. Berlin, Otto Salle, 1927. 142, 120, 75, 62, 122 pages.

WEATHERBURN, C. E. *Differential Geometry of Three Dimensions*. Cambridge, University Press, 1927. xii+268 pages.

WHITEHEAD, A. N. and RUSSELL, BERTRAND. *Principia Mathematica*, Vols. 1, 2, 3. Second edition. Cambridge, University Press, 1925–1927. 45 shillings, 45 shillings, and 25 shillings.

REVIEWS

Statistical Method in Educational Measurement. By ARTHUR S. OTIS. Yonkers, N.Y., World Book Company, 1925. xi+337 pages.

Serious difficulties face the scientific workers in the field of Education who have introduced *measurement* as the basis for improving methods of teaching. In the first place, while the statistical theory essential for correct interpretation of measurements is as yet somewhat imperfect, it is nevertheless necessary to apply the theory in its present state to meet immediate needs. In the second place, useful application of the statistical theory of measurement must be made by large numbers of teachers and administrators of slight mathematical and statistical training. They must therefore be provided with simple working rules, charts, and "common-sense" explanations of principles. In the third place, the considerable volume of experience which has been built up by the use of the simple approximate methods constitutes somewhat of an obstacle to the refinements which ought now to be inaugurated, both in order to make possible further progress in the fields in which the new point of view has been accepted, and to facilitate the entrance into new fields (e.g., college grading) in which the scientific inadequacy of the approximate methods is one cause of the opposition encountered.

In this new text-book, Dr. Otis attacks very successfully the difficulties of the second type. As he says in his preface:

"The purpose of the book is to present the subject in such a way that it can be understood by those who know nothing whatever about statistical method. · · · Several new charts for practical use are introduced. · · · Indeed, throughout the book the practical application of methods has been kept in mind, and only such discussion of theory is given as is thought necessary to make the use of the methods and devices intelligent rather than rule-of-thumb."

Nevertheless, while stressing the simple working procedure, Dr. Otis avoids the third difficulty mentioned above, by admitting frankly the approximate nature of many of the methods recommended, and even advocating improved methods in certain cases, as for instance in the computation of the "I.Q." The research worker who is primarily concerned with improvement in methods will find in many cases that a phrase or a paragraph of Dr. Otis opens the way for what he has in mind. For this reason the book may be expected to prove helpful to students of mathematical statistics who wish a practical background in the field of education for the mathematical discussion of frequency curves and correlation.

From the mathematical point of view, however, there are several points in which the treatment might well be modified somewhat. In his introductory discussion of "percentile curves," for instance, he would lead the mathematician more easily along familiar paths if these percentile graphs were related more directly to the areas of frequency curves, and the titles were stated more specifically—for example, "Per cent of cases in which the test score exceeded that indicated on base line." It should be noted that if the diagram implied by the idea of "area" or by the title just stated is turned through 90°, so that the base line becomes the right hand boundary, the result is exactly the form of the diagram recommended by Dr. Otis.

The use of the phrase "reliability coefficient" by Dr. Otis and other educational statisticians seems objectionable. He says (p. 256):

"The reliability of a test is the degree to which it is consistent in measuring that which it measures, but the validity of a test is the degree to which it measures that which it purports to measure."

If educational statisticians would only use the word "consistency" for what they call "reliability," they could then use "reliability" in its long established sense, without having to adopt a less familiar word, "validity."

A third point to which a mathematician might take exception concerns the "correction for attenuation"—i.e., the formula by which it is claimed that one may infer what the coefficient of correlation between two tests would be if there were no errors of measurement. Until the reviewer learns of a careful study of the probable error of such a corrected coefficient for small samples, he will feel that

the correction can be used only with much more caution than Dr. Otis suggests. Most students of statistics, probability, and philosophy regard the determination of the real that underlies the stream of appearance as a task calling for prayer and fasting. Yet this magic formula professes to remove the effect of errors of measurement on the calculation of one of the most tricky inventions of the statistician—and to do it reliably even for rather small samples.

In spite of a few points such as these mentioned, Dr. Otis's text seems to the reviewer acceptable to the mathematician or mathematical student who is making his first contact with educational statistical methods. Indeed, it may be recommended as a sort of antidote to a point of view too purely mathematical—as a glimpse of a world where ease in application is a more important characteristic of a proposed method than rigorous accuracy, and where informal discussions of a new principle that link it with accepted notions are valued more highly than logical deduction from definitely formulated postulates.

R. W. BURGESS

A Course of Geometrical Analysis. By HARIDAS BAGCHI. Calcutta, Chuckervertty, Chatterji and Co., 1926. 8+562 pages..

The author, Haridas Bagchi, M.A., Ph. D., Premchand Roychand Scholar and Lecturer in Pure Mathematics, Calcutta University, here makes an "attempt to harmonize as far as practicable Geometry and Analysis." This statement is of course further explained. A scrutiny of the book shows at once that the subject matter can be described as conventional differential geometry with an occasional timely excursus into some other mathematical field. Numerous references are made to standard British texts in algebra, calculus, theory of functions, vector analysis, and differential geometry. A few references are made also to Eisenhart's work on differential geometry and to some continental writers, e.g. "Wierstrauss" and "Lami."

The book is unusual and interesting. It shows clearly an inquisitive mind not content to lie bound among formulas and theorems. Apparently (although internal evidence may be misleading) the author has struggled through—and mastered—largely through his own efforts, Forsyth's Differential Geometry, keeping his mind open to possible applications, and has met with frequent difficulties. Some of these were cleared up by his own further reading of this same excellent text, but the pursuit of others has led him far afield. Attempting to teach the subject to others who may have shown less enthusiasm and less natural inquisitiveness has convinced the author that an isolated hierarchy of formal symbols may prove almost meaningless, while the fault lies not in the subject but in the exposition. This text taken by itself is hardly satisfying as a fundamental basis of instruction. Its many references to standard works will not prove too numerous to a critical reader who is accustomed to formal

proofs. But its detailed explanations, its suggestive and original comments, its interesting applications, its frequent cautions, and its relative freedom from specialized symbols and technical notations, make it a delightful contrast to most treatises on differential geometry and a particularly serviceable aid in self instruction. Perhaps a more severe author would have relegated much of the discussion to exercises, by relying upon experienced class room instruction to cover the technical difficulties. The book has no index, and the table of contents, even aided by the page of summary preceding each chapter, is not sufficient to make easy reference practicable. One misses the flavor of Euclidean precision in the details of proofs and this casts some doubt upon the validity of many of the statements, except where more ample proofs are found in the references. There appears to be not a little new material, and at least much here found is not readily accessible in other standard texts. Occasional remarks reveal however sad gaps in the author's scholarship, as might be expected in a work touching upon so many fields. Without intending any restrictions upon the functions employed, the author refers as follows to the solutions of Cauchy's familiar functional equations: "If a function conforms to the Law of Addition it must conform also to that of Multiplication, and vice versa. In either case, the function must have need to be linear and homogeneous. Further functions of this stamp constitute the only class to be thought of in connection with the afore-mentioned Laws." His proof uses differentiation. In an alternative arithmetic "proof," that $f(u+v)=f(u)+f(v)$, implies, $f(uv)=vf(u)$, he remarks after discussing the case of v rational, "There is no denying the fact that, while the Multiplicative Law, taken by itself, counts as a handy functional equation, its deduction from the Law of Addition is by no means an easy one, when v happens to be a transcendental, or even an incommensurable number. In all likelihood, modern speculations on the Theory of Numbers may lead to a simple solution of the thorny problem referred to."

The diction is frequently quaint although rarely ungrammatical, and is certainly excellent for one whose native tongue is not even European. One may be excused a goodnatured smile at reading, "The needment in the next stage can easily be met with by the students themselves, who should put the last hand to the problem," or "The constants a and b are as arbitrary as anything can be," or again "Where x and y behave like parameters, and as such flatly refuse to be struck out by elimination."

ALBERT A. BENNETT

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Annals of Mathematics, volume 28, no. 2, April 1927: "Some relations between a continuous curve and its subsets" by H. M. Gehman, 103–111; "Transformations of one principal equation into another" by R. Garver, 112–116; "On related maxima and minima" by N. Miller, 117–126; "On Fredholm's integral equations, whose kernels are analytic in a parameter" by J. D. Tamarkin, 127–152; "On the existence of the absolute minimum in space problems of the calculus variations" by L. M. Graves, 153–170; "The mixed mean function" by C. A. Shook, 171–179; "The focal point for the problem of Lagrange with one variable end point" by L. H. McFarlan, 180–195; "Differential invariants of affinely connected manifolds" by T. Y. Thomas and F. D. Michal, 196–236; "Linear congruences in a general arithmetic" by H. L. Olson, 237–239; "Note on a differential equation" by J. M. Thomas, 240–244; "On the class of a covariant tensor" by J. W. Alexander, 245–250.

American Journal of Mathematics, volume 49, no. 2, April 1927: "A class of functions harmonic within the sphere" by G. C. Evans, 153–180; "Notes on formal modular protomorphs" by O. C. Hazlett, 181–188; "Irreducible continuous curves" by H. M. Gehman, 189–196; "The plane quintic with five cusps" by M. Lehr, 197–214; "The convergence of general means and the invariance of form of certain frequency functions" by E. L. Dodd, 215–220; "On the interpolatory properties of a linear combination of continuous functions" by D. V. Widder, 221–234; "Note on Tchebycheff approximation" by D. V. Widder, 235–240; "Generalizations of Waring's theorem on fourth, sixth, and eighth powers" by L. E. Dickson, 241–250; "On complete systems of irrational invariants of associated point sets" by C. M. Huber, 251–267; "Regular maps and their groups" by H. R. Brahana; "The groups belonging to a linear associative algebra" by A. Ranum, 285–308.

Bulletin of the American Mathematical Society, volume 33, no. 4, July-August 1927: "A class of transcendental numbers" by S. Mandelbrojt, 413–415; "Cauchy's cyclotomic function and functional powers" by E. T. Bell, 416–422; "A point set which has no true quasi-components, and which becomes connected upon the addition of a single point" by R. L. Wilder, 423–426; "A simple method for normalizing Tchebycheff polynomials and evaluating the elements of the allied continued fractions" by J. A. Shohat, 427–432; "On Hilbert's thirteenth Paris problem" by H. W. Raudenbush, Jr., 433–434; "Generalization of the Beltrami equations to curved n-space" by G. E. Raynor, 435–438; "The non existence of a certain type of regular point set" by R. L. Wilder, 439–446; "A theorem of Frobenius on quadratic forms" by P. Franklin, 447–452; "On the partitions of a group and the resulting classification" by J. W. Young, 453–460; "Analytic functions with assigned values" by P. Franklin, 461–466; "An application of analysis situs to statistics" by H. Hotelling, 467–476; "A generalization of recurrents" by M. Ward, 477–490.

Journal of the London Mathematical Society, volume 2, no. 6, April 1927: "The zeros of certain integral functions" by J. D. Tamarkin, 66–69; "On a theorem of Bochner and Hardy" by N. Wiener, 118–123.

Journal of Mathematics and Physics, Mass. Inst. of Technology, volume 6, no. 4, June 1927: "A geometric characterization of equipotential and stream lines" by P. Franklin, 191–208; "On the conditions of validity of macromechanics" by M. S. Vallanta, 209–222; "Contact transformations in intrinsic geometry" by S. D. Zeldin, 223–239; "Convergent interpolation coefficients with convergent sum" by G. Rutledge, 240–249.

Mathematische Annalen, volume 97, nos. 4–5, June 1927: "On the reality of singularities of plane curves" by T. R. Hollcroft, 775–787.

Mathematische Zeitschrift, volume 26, no. 5, June 1927: "Concerning the sum of a countable infinity of mutually exclusive continua" by J. R. Kline, 687–690.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 4, April 1927: "On the expansion of harmonic functions in terms of harmonic polynomials" by J. L. Walsh, 175–179. No. 5, May 1927: "Concerning points on a continuous curve which are not accessible from each other" by C. M. Cleveland, 275; "Recent progress of investigations by symbolical methods of the invariants of biternary quantics" by O. E. Glenn, 276–279; "A dynamical theory of economic equilibrium" by C. F. Roos, 280–285; "On the foundations of the theory of discontinuous groups of linear transformations by combination" by L. R. Ford, 289–290. No. 6, June 1927: "Groups of collineations in a space of paths" by M. S. Knebelman, 396–399.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3280. Proposed by Philip Fitch, Denver Public Schools.

If a flexible chain, suspended at the ends from points in a horizontal line, is loaded so that the load varies directly as the square of the distance along the horizontal from the mid-point of the chain, find the equation of the curve made by the chain.

3281. Proposed by Emma M. Gibson, High School, Springfield, Mo.

Two particles of masses m_1 and m_2 are tied by fine inextensible strings to a third particle of mass m_3. They lie on a rough plane, whose inclination to the horizon (θ) is less than the angle of friction (ϵ), with the strings stretched and making angles α and β with the line of intersection of the plane with the horizontal plane. Find the magnitude and direction of the least horizontal force which, on being applied to the third particle, will move all three.

3282. Proposed by H. Halperin, A. & M. College of Texas, College Station, Texas.

Justify the following method of solving an exact differential equation of the first order, $Mdx+Ndy=0$: Replace y in M and N by ux and dy by udx; integrate the resulting expression with regard to x, considering u as constant; replace u by y/x, equate to a constant, C, and obtain the solution.

3283. Proposed by W. L. Ayres, University of Pennsylvania.

Let C_1 and C_2 be the circles $x^2+y^2=1$ and $x^2+(y-b)^2=r^2$, where $r+1>b>r>0$. Let L_1 be the upper arc of C_1 and L_2 the lower arc of C_2 between the points of intersection of the two circles. If the volume of the circular ring formed by revolving L_1+L_2 about the x-axis is a constant V, find b and r such that the area of the portion of the surface of the ring generated by L_2 may be a minimum.

3284. Proposed by Burrell Morgan, Krollitz, W. Va. and Norman Anning, University of Michigan.

In a given sphere is inscribed the maximum right prism whose bases are regular polygons of a given number of sides. Show that the altitude of the prism is independent of the number of its lateral faces.

3285. Proposed by R. H. Sciobereti, University of California.

Find the most general function $f(x)$ such that the integral $\int_a^b \phi(x)\ f(x-h)dx$ should have a value independent of h, where $\phi(x)$ is a given continuous function. Examine the cases where one of the limits or both become infinite.

3286. Proposed by V. M. Spunar, Chicago, Illinois.

A doubly infinite system of similar conics in parallel planes have their centers collinear and their corresponding axes parallel. Show that they can be cut orthogonally by a family of surfaces only if the line of centers is perpendicular to their planes.

3287. Proposed by D. M. Yost, California Institute of Technology.

Evaluate the definite integral $\int_0^\infty (x^n/x^x)dx$.

SOLUTIONS

3176[3172; 1926, 105]. Proposed by J. B. Reynolds, Lehigh University.

A linkage consisting of four equal uniform rods each of length $2a$ and weight w, loosely jointed in the form of a rhombus in a vertical plane, carries a weight P at the lowest vertex and is supported by the two upper rods resting against a smooth horizontal cylinder of radius r; find the time of a small vibration of the system in a vertical plane.

Solution by the Proposer.

Assume the x-axis horizontal in the plane of motion, the y-axis vertically upward, and the origin at the center of the cylinder. Let (x, y) be the coordinates of the center of the right upper rod, (x', y') of the center of the right lower rod, and (X, Y) of the weight P. Suppose that in equilibrium the radius of the cylinder to the point of contact of the upper rod makes an angle A with the vertical. Let the system be slightly displaced in a vertical plane so that the radius to the point of contact makes an angle $A-\theta$ with the vertical. Then we shall have

$$\begin{aligned} x &= a\cos(A-\theta) & y &= r\sec(A-\theta) - a\sin(A-\theta)\\ x' &= x & y' &= y - 2a\sin(A-\theta)\\ X &= 0 & Y &= y' - a\sin(A-\theta). \end{aligned}$$

Let K be the kinetic energy of the oscillating system and Q its potential energy with reference to the position of equilibrium. We shall then have

$$gK = w(\dot{x}^2+\dot{y}^2+\dot{x}'^2+\dot{y}'^2) + (P/2)(\dot{X}^2+\dot{Y}^2) + (2w/3)a^2\dot{\theta}^2$$

in which the dots indicate derivatives with respect to time; and

$$Q = 2w(y-y_0) + 2w(y'-y_0') + P(Y-Y_0)$$

in which the subscripts refer to the values of the coordinates in the position of equilibrium.

Expanding to the squares of small terms in θ we find

$$(1)\quad gK = \{(1/2)(4w+P)r^2\sec^2 A\tan^2 A - 4(2w+P)ar\tan A + 2[(4/3)w + 4(w+P)\cos^2 A]a^2\}\dot{\theta}^2$$

and

$$(2)\qquad Q = [(8w+4P)a\cos A - (4w+P)r\sec A\tan A]\theta + (1/2)[(4w+P)r\sec A(2\sec^2 A - 1) + (8w+4P)a\sin A]\theta^2.$$

Since Q is a minimum for $\theta=0$ we must have

$$(3)\qquad (4w+P)r\sin A = 4(2w+P)a\cos^3 A,$$

which determines A for the position of equilibrium. This result could have been obtained from the equations of equilibrium for the system.

By means of (3) we may now write (1) and (2) in the forms

$$gK = (4aw/3)(2a - 6a\cos^2 A + 3r\tan A)\dot{\theta}^2 = H\dot{\theta}^2,$$

and

$$Q = 2a(2w+P)(2\sin A + \operatorname{cosec} A)\,\theta^2 = G\theta^2.$$

Now $K+Q=(H/g)\dot{\theta}^2+G\theta^2=$ constant and, upon differentiating, we find $H\ddot{\theta}+gG\theta=0$ whence, if T is the time of a complete oscillation,

$$T = 2\pi(H/Gg)^{1/2} = (2\pi/3)[6w\sin A(2a - 6a\cos^2 A + 3r\tan A)/(2w+P)(2\sin^2 A+1)g]^{1/2},$$

in which A is in the first quadrant and is determined by equation (3).

Also solved by William Hoover.

3217[1926; 480]. Proposed by R. H. Sciobereti, Berkeley, Calif.

Find a curve such that the radius of curvature at any point is proportional to the reciprocal of the normal.

SOLUTION BY EUGENE M. BERRY, Lynchburg College, Lynchburg, Va.

Putting the curvature equal to a times the normal, we have the differential equation

$$(d^2y/dx^2)/[1+(dy/dx)^2]^{3/2} = ay[1+(dy/dx)^2]^{1/2}. \tag{1}$$

If we put $p=dy/dx$ and $p(dp/dy)=dy^2/dx^2$ this becomes $pdp/(1+p^2)^2=aydy$.

If we integrate and solve for p, we get $p=\pm[(1-c+ay^2)/(c-ay^2)]^{1/2}$, whence

$$x = \pm\int[(c-ay^2)/(1-c+ay^2)]^{1/2}dy. \tag{2}$$

If $c=1$ in equation (2), we have

$$x = \pm\frac{1}{\sqrt{a}}\left\{\sqrt{(1-ay^2)}-\log\left(\frac{1+\sqrt{(1-ay^2)}}{y\sqrt{a}}\right)\right\}+C \tag{3}$$

which is the equation of a tractrix.

If $c=0$ in equation (2), we have $x=\pm[(1+ay^2)/(-a)]^{1/2}+C$, which is the equation of a circle.

With these two exceptions the integrals in the above equations are elliptic integrals. If the curves are to be real, we must have either $a>0$, $c>0$ or $a<0$, $c<1$. We can put these in type forms as follows.

When $a>0$, $0<c<1$, put $y=(c/a)^{1/2}\cos\theta$; equation (2) then becomes

$$x = \pm a^{-1/2}\int[(1-c\sin^2\theta)^{1/2}-(1-c\sin^2\theta)^{-1/2}]d\theta. \tag{5}$$

When $a>0$, $c>1$, put $y=[(c-\sin^2\theta)/a]^{1/2}$; equation (2) then becomes

$$x = \pm(c/a)^{1/2}\int[(1-c^{-1}\sin^2\theta)^{1/2}-(1-c^{-1}\sin^2\theta)^{-1/2}]d\theta. \tag{6}$$

When $a<0$, $0<c<1$, put $y=[(1-c)/(-a)]^{1/2}\cos\theta$; equation (2) becomes

$$x = \mp(-a)^{-1/2}\int[1-(1-c)\sin^2\theta]^{1/2}d\theta. \tag{7}$$

When $a<0$, $c<0$, put $y=[(\cos^2\theta-c)/(-a)]^{1/2}$; equation (2) becomes

$$x = \mp(-a)^{-1/2}\int[(1-c)^{1/2}\{1-(1-c)^{-1}\sin^2\theta\}^{1/2}+c(1-c)^{-1/2}\{1-(1-c)^{-1}\sin^2\theta\}^{-1/2}]d\theta. \tag{8}$$

Also solved by J. B. REYNOLDS and the PROPOSER.

3219[1926, 480]. Proposed by Philip Fitch, Denver, Colorado.

Construct a polygon similar to a given polygon and having the reciprocal of its area equal to the sum of the reciprocals of the areas of a certain number of given polygons.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

To find graphically a line of length H such that for two given lines of lengths h_1 and h_2 there shall exist the relation $1/H=1/h_1+1/h_2$, draw rays OA, OB, OC, and OD in a plane making successive angles of 60° each. On OA lay off $OA=h_1$, on OC lay off $OC=h_2$. Let AC cut OB at B. Let $OB=H$. Then Area AOC = Area AOB + Area BOC, or $h_1h_2\sin 60°=h_1H\sin 60°+h_2H\sin 60°$; and therefore $1/H=1/h_1+1/h_2$.

On OD lay off $OD=h_3$. Draw BD cutting OC at E. Let $OE=H'$.

Then $1/H'=1/h_3+1/H=1/h_1+1/h_2+1/h_3$.

This process can be continued to any number of reciprocals. Now let there be n given polygons of areas $A_1, A_2, \cdots, A_n$; to find A such that $1/A=1/A_1+1/A_2+\cdots+1/A_n$.

Construct n right triangles each of unit base and altitudes $h_1, h_2, \cdots, h_n$ whose areas are equal to $A_1, A_2, \cdots, A_n$, respectively. Find h by the relation $1/h=1/h_1+1/h_2+\cdots+1/h_n$. Construct a right triangle of unit base and altitude h; then $h=2A$, $h_1=2A_1$, $h_2=2A_2, \cdots, h_n=2A_n$; whence $1/A=1/A_1+1/A_2+\cdots+1/A_n$, giving A, the area of the required polygon.

Construct two isosceles right triangles of areas A and A' and legs a and a', where A' is the area of the given polygon. Construct the required similar polygon with its sides in the ratio a/a' to those of the given polygon.

NOTE BY THE EDITORS

One could say: Construct n rectangles each of unit base with altitudes $h_1, h_2, \cdots$, whose areas are $A_1, A_2, \cdots$. Then $h_1 = A_1$, $h_2 = A_2, \cdots$.

Also solved by MICHAEL GOLDBERG and the PROPOSER.

3221[1926, 480]. Proposed by H. E. Trefethen, Colby College.

A variable rectangle has a diagonal of constant length and two sides lying upon two fixed perpendicular straight lines. Determine geometrically the locus of the foot of the perpendicular from the vertex opposite the fixed vertex upon the diagonal which does not pass through that vertex.

SOLUTION BY ROSCOE WOODS, University of Iowa.

Let the vertices of the variable rectangle be O, A, B, C, where O is the fixed vertex, A and C are vertices on the two fixed perpendicular straight lines, and B the remaining vertex whose locus is evidently a circle, K, whose center is O and whose radius is r which is the length of the constant diagonal of this variable rectangle. Consider the rectangle in one of its possible positions and draw the diagonals OB and AC and let M be their point of intersection. From B draw a perpendicular to AC, cutting AC in P. The triangle MPB is a right triangle so that a circle K' on BM as diameter always passes through P. Since BM is a constant the radius of K' is one-fourth the radius OB of K. Since BM, the diameter of K', lies along the radius OB of K, K and K' are tangent at B. Let S be the center of K'. From a figure, it is easily seen that the angle PSB is four times the angle AOB, which shows that the arc subtending the angle PSB in K' is equal in length to the arc subtending the angle AOB in K. This proves that P is a fixed point of the circumference of K' if it rolls on K; therefore the locus of P is a hypocycloid of four cusps.

Also solved by J. M. BARBOUR, THEODORE BENNETT, ALICE M. GRANT, MICHAEL GOLDBERG, J. B. REYNOLDS, and the PROPOSER.

3223[1926, 481]. Proposed by Paul Capron, U. S. Naval Academy.

A circle of radius b and a straight line at a distance a from the center of the circle, lie in the same plane; the circle is revolved about the line generating a torus. A plane Π is passed through the axis l, intersecting the torus in two circles S_1 and S_2; a plane Σ is passed perpendicular to Π and containing the common interior tangent of S_1 and S_2. Show that Σ intersects the torus in two circles.

SOLUTION BY R. H. SCIOBERETI, University of California.

Let Oz be the axis of the torus; OX and Ox, the traces of the bitangent plane Σ and of any meridian on the equator, respectively. Let us denote by M one of the points of intersection of the plane Σ with the meridian circle S' of center C so that $OC=a$, $CM=b$. Let P and Q be the orthogonal projections of M on Ox and OX respectively and R the projection of point O on CM. Then we shall have angle $PQM=\theta$, where θ denotes the angle between the bitangent and the equatorial planes; hence,

$$\sin PQM = \sin\theta = b/a = PM/QM = CM/OC. \tag{1}$$

On the other hand the area of the triangle OCM evaluated in two different ways gives

$$OC \cdot PM = CM \cdot OR, \qquad \text{or} \qquad CM/OC = PM/OR,$$

and from a comparison with (1) it follows that $QM=OR$; hence $MR=OQ$, since the two right triangles MRO and OQM are congruent.

Let us now consider a point D on OX at a distance b from O, and such that the three points O, Q, D will be in the same order as the points M, R, C, so that $QD=OD-OQ=CM-MR=RC$; hence, the two right triangles DQM and CRO are congruent and consequently $DM=CO=a$. When the meridian circle

S' is rotated about Oz, point M will describe a circle of center D and of radius a. A similar reasoning shows that the locus of the second point of intersection M' of the plane Σ with the meridian circle S' is a circle equal to the first one, but whose center is the symmetrical point of D with respect to O.

REMARK I. These two circles, known as the Villarceau circles, intersect the parallels of the torus at a constant angle. This property may be shown by means of elementary geometry as follows: Let MT be the tangent to the Villarceau circle passing through M, and let MV be the tangent to the parallel through M; MT lies in the plane ODM and is orthogonal to DM, whereas MV is perpendicular to the plane OCM. Since the skew quadrilateral $ODMC$ has its opposite sides equal to each other, it follows that the two triangles ODM and MCO are congruent; hence $DH=CH$, where H denotes the point bisecting OM. In a similar manner it may be shown that if K is the mid-point of DC, $OK=MK$. Hence the line HK is the common perpendicular to OM and DC at their mid-points. This line HK may, therefore, be considered as an axis of symmetry for the skew quadrilateral. Now the symmetrical line of the tangent MT is the perpendicular to CO at the point O in the plane COM; hence it is Oz. The symmetrical line of the tangent MV is a line OU through O orthogonal to the plane DOM; then since angle $VMT=$angle $UOz=\theta$, it follows that the Villarceau circles intersect the parallels of the torus at a constant angle θ.

REMARK II. The projection of the Villarceau circle on the equator is an ellipse with one focus at point O. In fact its center is at D and its major axis, which is along OX, is $2a$; the minor axis is $2a\cos\theta$; hence, the focal distance is $a\sin\theta=b$ which proves the proposition.

The solver sent in also an analytical solution.

Also solved by THEODORE BENNETT, PAUL CAPRON, RUFUS CRANE, MICHAEL GOLDBERG, HARRY LANGMAN, and J. B. REYNOLDS.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The second Madison Colloquium of the American Mathematical Society was held during the week of September 5, in connection with the meetings of the Society and the Association. The lecturers were Professor E. T. Bell of the California Institute of Technology and Professor Anna Pell Wheeler of Bryn Mawr College. They gave five lectures each, the former on "Algebraic Arithmetic" and the latter on "The Theory of Quadratic Forms in Infinitely many Variables and Applications."

The long delay in publishing the Rhind Mathematical Papyrus is due to unforeseen difficulties in connection with the correction of the 140 plates by the hieroglyphic expert. It is now confidently expected that the two volumes will be ready for delivery in December. Personal explanation will be made by letter to all advance subscribers already filed. Further subscriptions will be filed in the order of their receipt. In the November Monthly there will appear a complete formal description of this great work on the eve of its appearance. The description will be written by Professor R. C. Archibald.

A $1000 fellowship in mathematics has recently been established at Brown University by Mr. H. D. SHARPE.

At the Armour Institute of Technology, Professor DONALD F. CAMPBELL has resigned; and Associate Professor C. I. PALMER has been promoted to a full professorship of mathematics. Professor Palmer is also acting dean of students.

Mr. HERBERT E. ARNOLD has been promoted to an assistant professorship of mathematics at Wesleyan University.

Assistant Professor L. C. BAGBY of the University of South Dakota has been appointed professor of mathematics at the Linsly Institute of Technology.

Mr. M. A. BASOCO, who was reported last month as appointed to an instructorship at the University of California, Los Angeles, has accepted a graduate fellowship at the California Institute of Technology.

Mr. A. H. BLUE of the University of Iowa has been elected professor of mathematics at West Union College.

Assistant Professor EVELYN T. CARROLL of Wells College has been granted leave of absence for the year 1927–28.

Mr. T. F. COPE, instructor last year at Western Reserve University, has been appointed instructor in mathematics at the University of Chicago for 1927–28.

Professor L. S. DEDERICK of St. Stephen's College has been appointed mathematician at the Aberdeen Proving Ground.

Associate Professor H. J. ETTLINGER of the University of Texas has been promoted to a full professorship of mathematics.

Assistant Professor M. C. FOSTER of Williams College has been appointed associate professor of mathematics at Wesleyan University.

Mr. D. E. HARKIN, who has just received the doctorate at the University of Chicago, has been appointed professor of mathematics at Alabama Polytechnic Institute.

Miss MARIE JOHNSON, who was reported last month as appointed to an instructorship at Pennsylvania State College, has been released to accept an assistant professorship at Oberlin College.

Assistant Professor C. G. LATIMER of Tulane University has been appointed professor of mathematics at the University of Kentucky.

Professor P. H. LINEHAN of the College of the City of New York has been made director of the evening session and of the division of vocational studies.

Mr. W. T. MacCreadie of Cornell University has been appointed assistant professor of mathematics at Bucknell University.

Mr. T. H. Milne has been appointed assistant professor of mathematics at the University of Manitoba.

Miss Ethel I. Moody, M.A., Cornell, 1927, has been appointed instructor in mathematics at Wells College.

Dr. J. H. Neelley of Yale University has been appointed associate professor of mathematics at the Carnegie Institute of Technology.

Mr. J. C. Nixon, S.M., University of Chicago, March, 1927, has accepted an instructorship at the University of Alabama.

Associate Professor R. S. Underwood of Alabama Polytechnic Institute has been appointed associate professor of mathematics at Texas Technological College.

Assistant Professor E. E. Whitford of the College of the City of New York has been promoted to an associate professorship of mathematics.

Dr. W. H. Wilson of the University of Iowa has been appointed associate professor of mathematics at the University of Florida.

Dr. H. A. Zinszer of the Mississippi State College for Women has been appointed professor of mathematics at Hanover College.

The following appointments to instructorships in mathematics are announced:

Cornell University, Mr. C. C. Torrance;
University of Florida, Mr. C. A. Messick of the University of Iowa;
Rutgers College, Mr. C. R. Wilson of the University of Iowa;
Northwestern University, Mr. H. L. Garabedian;
Yale University, Mr. T. C. Benton.
Junior College at Jefferson City, Miss Frances Baker of Tabor College.

Professor W. G. Bullard of Syracuse University died February 16, 1927.

Professor W. W. Johnson of the United States Naval Academy died May 14, 1927, at the age of eighty-three.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.
Twelfth Annual Meeting, Nashville, Tenn., December, 1927.
The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 14.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7 and Georgetown University, December 3.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 6.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November 26.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 9, NOVEMBER

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE RHIND MATHEMATICAL PAPYRUS

The following description of the forthcoming publication of the oldest known mathematical document in the world will be of interest to readers of the Monthly.

British Museum 10057 and 10058. Photographic Facsimile of the Rhind Mathematical Papyrus, Hieroglyphic Transcription, Transliteration, Literal Translation, Free Translation, Mathematical Commentary and Bibliography. In two sumptuous volumes.

Volume I, $11\frac{1}{4}\times 8$ inches: Free Translation and Commentary by ARNOLD BUFFUM CHACE, Chancellor of Brown University, with the assistance of HENRY PARKER MANNING: Bibliography of Egyptian Mathematics by RAYMOND CLARE ARCHIBALD. $8+210$ pp.

Volume II, $11\frac{1}{4}\times 14\frac{1}{4}$ inches: Photographic Facsimile, Hieroglyphic Transcription, Transliteration, Literal Translation, by ARNOLD BUFFUM CHACE, with the assistance of LUDLOW BULL, Metropolitan Museum of Fine Arts, New York, and HENRY PARKER MANNING; also Leather Roll, British Museum 10250, facsimile with description by S. R. K. GLANVILLE, British Museum; in all 140 plates with text and introductions, all the plates and text of the Rhind Papyrus being in the two colors (red and black) of the original.

To be published by the Mathematical Association of America, Oberlin, Ohio, U. S. A., in December 1927. Price to individual and institutional members of the Association for personal use, $15.00 for the set. To all others $20.00.

This forthcoming work on the Rhind Papyrus is a publication of great interest and value not only to the student of mathematics but also to almost anyone interested in a civilization of nearly four thousand years ago. Through the courtesy of the British Museum authorities, Chancellor Chace is able to give for the first time a complete photographic reproduction of this very important mathematical work written about 1650 B.C. in the hieratic language, but copied from a work considerably older. Other plates give, under the hieratic writing, which is from right to left, the corresponding hieroglyphic transcription and transliteration, while on the opposite page is the transliteration from left to right and the literal translation. The first volume contains the free translation with detailed commentary and Professor Archibald's elaborate bibliography of Egyptian mathematics which with its indexes, occupies 86 pages.

It was not till the summer of 1927 that a distinguished English chemist succeeded in unrolling another mathematical document in the British Museum,

consisting of a leather roll of the 17th century B.C. The Chancellor was fortunate in securing almost immediate permission to publish a photographic facsimile of this roll together with a descriptive note by a member of the Museum's staff. This was accomplished just in time to be included in the second volume.

Subscriptions for the set by members of the Association should be sent to the Secretary Treasurer, W. D. CAIRNS, Oberlin, Ohio. All other orders should be addressed to the Open Court Publishing Company, 339 East Chicago Avenue, Chicago, Illinois.

THE TWELFTH SUMMER MEETING OF THE ASSOCIATION

The twelfth summer meeting of the Mathematical Association of America was held, by invitation, at the University of Wisconsin on Monday and Tuesday, September 5–6, 1927, in conjunction with the summer meeting and the colloquium of the American Mathematical Society. 201 persons were in attendance at the meetings, including the following 118 members of the Association:

L. K. ADKINS, LaCrosse State Teachers College.
FLORENCE E. ALLEN, University of Wisconsin.
NATHAN ALTSHILLER-COURT, University of Oklahoma.
W. E. ANDERSON, Miami University.
R. C. ARCHIBALD, Brown University.

I. A. BARNETT, University of Cincinnati.
P. M. BATCHELDER, University of Texas.
LEON BATTIG, Oberlin College.
W. O. BEAL, University of Minnesota.
Mrs. W. E. BECKWITH, Milwaukee-Downer College.
E. T. BELL, California Institute of Technology.
G. D. BIRKHOFF, Harvard University.
H. F. BLICHFELDT, Stanford University.
H. E. BRAY, Rice Institute.
W. C. BRENKE, University of Nebraska.
R. W. BRINK, University of Minnesota.
E. W. BROWN, Yale University.
H. E. BUCHANAN, Tulane University.
L. H. BUNYAN, University of Wisconsin.
W. H. BUSSEY, University of Minnesota.

W. D. CAIRNS, Oberlin College.
C. C. CAMP, University of Nebraska.
ELIZABETH CARLSON, University of Minnesota.
R. D. CARMICHAEL, University of Illinois.
E. W. CHITTENDEN, University of Iowa.
JULIA T. COLPITTS, Iowa State College.
C. E. COMSTOCK, Bradley Polytechnic Institute.
N. B. CONKWRIGHT, University of Iowa.
A. R. CRATHORNE, University of Illinois.

H. H. DALAKER, University of Minnesota.
C. W. DANCER, University of Wisconsin.
H. T. DAVIS, Indiana University.
ALICE C. DEAN, Rice Institute.
L. A. V. DE CLEENE, St. Norbert's College.
L. E. DICKSON, University of Chicago.
L. W. DOWLING, University of Wisconsin.
ARNOLD DRESDEN, Swarthmore College.
O. L. DUSTHEIMER, Baldwin-Wallace College.

J. M. EARL, University of Minnesota.
P. D. EDWARDS, Ball Teachers College.
ARNOLD EMCH, University of Illinois.
G. C. EVANS, Rice Institute.

W. B. FORD, University of Michigan.
T. C. FRY, Bell Telephone Labs., New York, N.Y.

M. G. GABA, University of Nebraska.
W. H. GARRETT, Baker University.
B. P. GILL, College of the City of New York.
D. C. GILLESPIE, Cornell University.
W. C. GRAUSTEIN, Harvard University.
L. M. GRAVES, University of Chicago.
V. G. GROVE, Michigan State College.

LAURENCE HAMPTON, University of Oklahoma
W. L. HART, University of Minnesota.
W. W. HART, University of Wisconsin.
E. R. HEDRICK, University of California at Los Angeles.
T. H. HILDEBRANDT, University of Michigan.
T. R. HOLLCROFT, Wells College.
I. O. HORSFALL, L. D. S. College.
H. M. HOSFORD, Southern Methodist University
R. C. HUFFER, Beloit College.
JEWELL C. HUGHES, University of Arkansas.

LOUIS INGOLD, University of Missouri.
M. H. INGRAHAM, University of Wisconsin.

DUNHAM JACKSON, University of Minnesota.
C. M. JENSEN, Macalester College

DORA E. KEARNEY, Cedar Falls State Teachers College.
A. J. KEMPNER, University of Colorado.
E. C. KIEFER, James Millikin University.
W. C. KRATHWOHL, Armour Institute.

E. P. LANE, University of Chicago.
R. E. LANGER, University of Wisconsin.
C. G. LATIMER, University of Kentucky.

KURT LAVES, University of Chicago.
MAYME I. LOGSDON, University of Chicago.
W. D. MACMILLAN, University of Chicago.
J. V. MCKELVEY, Iowa State College.
A. S. MERRILL, University of Montana.
A. D. MICHAL, Ohio State University.
E. L. MICKLESON, University of Wisconsin.
C. N. MILLS, Illinois Normal University.
W. L. MISER, Vanderbilt University.
E. J. MOULTON, Northwestern University.

H. L. OLSON, Michigan State College.
F. W. OWENS, Pennsylvania State College.
HELEN B. OWENS, State College, Pa.

C. I. PALMER, Armour Institute.
G. A. PARKINSON, University of Wisconsin.
H. P. PETTIT, Marquette University.
T. A. PIERCE, University of Nebraska.

R. G. D. RICHARDSON, Brown University.
H. L. RIETZ, University of Iowa.
W. C. RISSELMAN, University of Minnesota.
E. D. ROE, Jr., Syracuse University.
W. H. ROEVER, Washington University.
W. E. ROTH, University of Wisconsin.
LULU RUNGE, University of Nebraska.

R. S. SHAW, Graduate School, University of Chicago.
W. A. SHEWHART, Bell Telephone Labs., New York, N. Y.
J. A. SHOHAT, University of Michigan.
MARY EMILY SINCLAIR, Oberlin College.
E. B. SKINNER, University of Wisconsin.
H. E. SLAUGHT, University of Chicago.
C. S. SLICHTER, University of Wisconsin.
G. W. SNEDECOR, Iowa State College.
VIRGIL SNYDER, Cornell University.
E. B. STOUFFER, University of Kansas.
K. D. SWARTZEL, University of Pittsburgh.

MARION TORREY, Goucher College.

E. B. VAN VLECK, University of Wisconsin.

I. N. WARNER, Platteville State Teachers College.
J. H. WEAVER, Ohio State University.
WARREN WEAVER, University of Wisconsin.
W. D. A. WESTFALL, University of Missouri.
ANNA PELL WHEELER, Bryn Mawr College.
W. R. WOODMANSEE, Ripon College.
EUPHEMIA R. WORTHINGTON, University of California at Los Angeles.

B. F. YANNEY, College of Wooster.
C. H. YEATON, Oberlin College.

The meetings were unusually pleasant and successful by reason of the conveniences afforded in the living arrangements and the manifold ways of entertaining those in attendance. The mathematicians and their friends for the most part had their living quarters and the meals in Chadbourne Hall, a central location with adequate social facilities and ready access to the lecture halls. On Wednesday afternoon two launches took a large party around Lake Mendota and to an afternoon tea at the summer home of Professor and Mrs. Slichter. Automobiles furnished by the Madison group took the visitors on Thursday afternoon about the many beautiful parts of Madison and vicinity and on Friday to the unique region of Devil's Lake. Smaller groups took fre-

quent advantage of the opportunities for golf, tennis, bathing, and canoeing. The many courtesies were recognized by a joint resolution offered at the banquet by Professor Hollcroft and adopted unanimously by a rising vote; this expressed the sincere thanks of the members to the administration, Professors Slichter and Van Vleck and the mathematical staff, the ladies, Professor Babcock and the committee on arrangements, the University Club and golf clubs, and the Press Bureau of the University.

The American Mathematical Society held its eleventh colloquium on Tuesday, Wednesday, Thursday and Saturday mornings and on Thursday evening, with lectures by Professor Anna Pell Wheeler on "The theory of quadratic forms in infinitely many variables and applications," and by Professor E. T. Bell on "Algebraic arithmetic." A large enrollment for these lectures assured the continued success of the Society's colloquium. The Society held a business session on Friday morning and sessions for the reading of papers on Thursday and Friday mornings.

160 persons attended the joint banquet held Wednesday evening at the Maple Bluff Golf Club under the eloquent toastmastership of Professor Slichter. A telegram of affectionate greeting was sent to Professor E. H. Moore by the secretaries on vote of those present; a reply was received the next morning expressing Professor Moore's grateful appreciation of the greetings and deep regret at his inability to attend the meetings. The banquet was unique in being addressed by three presidents. President Snyder of the Society told of the plans for the forthcoming meetings at Nashville, Amherst, and Bologna. President Ford of the Association spoke of the special field of mathematics characterized as the domain in which we lay down definitions of things; he emphasized the value of our passing over to society the means of clarifying social relations through clear adequate definitions. President Glenn Frank of the University of Wisconsin welcomed the mathematicians, expressing a high regard for their position in developing accuracy, clarity, and the creative imagination. The development of the sciences and the treatment of social problems are at heart, he said, one problem,—how can we make men outside and inside the laboratory? Can we find means to instill in our students the scientific spirit? It cannot be done simply as a by-product of our teaching. Shorter speeches were made by Professor Birkhoff on the new visiting mathematical professorships of the Society, by Professor Bell on the social contacts made in connection with these meetings, and by Professor Graves on the hope that the future holds in the field of mathematics.

The Mathematical Association held sessions on Monday and Tuesay afternoons with President Ford in the chair, Vice-President Kempner presiding for a part of the Tuesday session. The program was arranged by a committee consisting of Professors E. J. Moulton (chairman), Arnold Dresden, W. L. Hart,

W. B. Ford, and H. E. Buchanan. Abstracts of the papers are given, numbered in accordance with the numbers of the papers.

FIRST SESSION OF THE ASSOCIATION

(1) "Conditions for mathematical study in Italy" by Professor MAYME I. LOGSDON, University of Chicago.

(2) "A geometrical method for solving the biquadratic quation" by Professor W. C. GRAUSTEIN, Harvard University.

(3) "Machines for handling statistics" by Mr. F. A. HARPER, Tabulating Machine Company, Milwaukee, Wis., assisted by Mr. C. W. VAUGHN, chief disbursing officer of the University of Wisconsin.

1. The first part of the paper discussed informally reasons for going to Italy for mathematics, the curriculum and organization of the mathematical departments of the Italian universities, what the professors at Rome were doing in 1925–26, the libraries and Rome a Mecca for mathematicians. The second part outlined a portion of a theory of maxima and minima as developed by Steiner, Caratheodory, Schwarz, Zenadoro, Minkowski, and others, using instead of integration Cavalieri's method of indivisibles, a method "intermediate between the method of exhaustion of the Greeks and the later methods of the calculus."

Professors Stouffer and Lane, who have also visited and studied recently in Italy, agreed in saying that it is wise for Americans to have had their doctor's degree before going to Italy and to know quite clearly at what they wish to work, as the Italians are not so thoroughly acquainted with what has been accomplished in American mathematics and are accustomed to completing for themselves promising lines of research instead of giving these out to younger investigators as is done in America and other countries. The Italians nevertheless are most cordial to the Americans.

2. This paper will appear in an early issue of the Monthly.

Professor Snyder called special attention to the value of the last modification made in Professor Graustein's treatment, wherein the templet for $y^2-x=0$ can be turned through 90° and used as $x'^2-y'=0$; this he called a real contribution to the solution of the biquadratic.

3. After referring to the development of tabulating machines as described by Mr. Victor Johns in the issue of the Monthly for December, 1926, Mr.Harper said that mechanical accounting machines have unlimited uses: Insurance companies in general use them in the compilation of mortality and associated statistics, the United States and foreign governments apply them for compiling population, agricultural, manufactures, and vital statistics and for preparation of numerous exhibits shown in the survey of business and in sectional record books of business statistics prepared by the Department of Commerce. In

1900 the record of the capital, wages, and value of products of manufactures rose to figures almost beyond comprehension. The capital invested was \$9,846,628,564; the salaries and wages paid amounted to \$2,735,430,848; and the value of the products was \$13,039,279,566. In agriculture the figures are almost equally impressive. The total value of farms in 1900 was \$16,674,690,247 and that of agricultural products in 1899 was \$4,739,118,752. To gather and collate such stupendous figures, not only with accuracy, but so swiftly that the record of population in 1900 appeared as quickly as did the little report of the first census, was a task of the first magnitude. Such a report could be prepared only with the help of modern tabulating machines.

Mechanical accounting equipment may be found, in a number of our largest colleges, for every-day use and for instructional purposes. The Universities of Wisconsin and Michigan, the La Salle Extension, Johns Hopkins, Cornell, and Columbia Universities are some of the users. The machines are used to prepare payrolls, to analyze direct labor charges, accounts payable and receivable, for research analysis work and for demonstration purposes in accounting and commercial courses.

Mr. Vaughn showed the actual use of the sorting and tabulating machines by means of a set of cards representing the classification and the totalling of individual expenditures of university funds among several special funds according to which classifications are desired.

Second Session of the Association

(4) "Mathematics for students of chemistry" by Professor Farrington Daniels, University of Wisconsin.

(5) "Mathematics for commerce students" by Professor E. B. Skinner, University of Wisconsin. Discussion by Professor W. L. Hart, University of Minnesota.

(6) "The Rhind mathematical papyrus" by Professor R. C. Archibald, Brown University.

4. This paper will appear in an early issue of the Monthly.

5. After a brief account of the way in which the course in investment theory came into existence at the University of Wisconsin, the author outlined what he considered a reasonable first course in the subject giving at the same tine what he deemed to be the necessary prerequisites. The foundations of the subject are the theory of interest and the theory of the annuity certain. Following these subjects the important problems that present themselves in the business world are three in number: (1) Amortization problems, (2) Bond value problems, (3) Depreciation and sinking fund problems.

The problems relating to building and loan association work are of less importance, mainly for the reason that they are less amenable to mathematical treatment.

Consideration was given to later work for the commerce student and to a number of problems which confront the teacher of the subject. The paper closed with a plea for the recognition of courses in mathematics for commerce students even though they promise little in the way of leading students toward advanced courses in mathematics.

Professor Hart said that Professor Skinner developed for the first time in the United States a course in the mathematics of investment, and mentioned his success in the selection of material for the detailed work of the course. The mathematics of this subject is essentially simple, the theory of annuities being the basis of the course; this simplicity should be emphasized by the teacher, along with the important applications of the theory.

6. Professor Archibald told of the discovery of the Rhind mathematical papyrus now in the British Museum and of certain of its fragments discovered in 1922 in the library of the New York Historical Society. He showed photographs of portions of the papyrus including a photograph, which had just been made of 36 of the 40 fragments put in place with reference to the ends of the two rolls of the papyrus in the British Museum. He referred also to seven mathematical documents besides the Rhind papyrus, which all belonged to the golden period of Egyptian science, 1900–1650 B.C. The processes in Egyptian mathematics and the contents of the papyrus were described in detail, with examples. Indications were given of the contents of Chancellor Chace's work on the Rhind papyrus to be published by the Association in December, 1927. Certain facts were mentioned with regard to the mathematical leather roll, B.M. 10250, which with the aid of a chemist was unrolled in the summer of 1927. The interesting announcement was made that this is to be reproduced in facsimile in the Chancellor's work.

Meeting of the Board of Trustees

Eleven trustees were present at the two sessions on Monday and Tuesday.

The following thirty-seven persons and one institution were elected to membership, on application duly certified.

To Individual Membership.

W. N. Barnes, B.S. (Chicago). Teacher, San Antonio, Texas.

A. K. Bettinger, A.M. (Wisconsin). Instr., Creighton Univ., Omaha, Nebr.

Sam Byrd, M.S. (Arkansas). Teacher, Central High, Tulsa, Okla.

A. S. Croom. Asst. Prof., Washburn College, Topeka, Kans.

Elinor B. Flagg, M.S. (Illinois). Asst. Prof., Illinois State Normal Univ., Normal, Ill.

W. H. Gage, M.A. (Brit. Columbia). Instr., Victoria Coll., Victoria, B. C.

B. P. Gill, A.M. (Columbia). Instr., Coll. of City of New York, N. Y.

Charles Hatfield, A.M. (Tennessee). Head of Dept., Lincoln Memorial Univ., Harrogate, Tenn.

Evelyn Hesseltine, A.M. (Nebraska). Teacher, Spearfish Normal School, Spearfish, S. Dak.

H. K. Holt, A.M. (Ohio State). Asst. Prof., Pennsylvania Military Coll., Chester, Pa.

C. M. Jensen, Ph.D. (Minnesota). Asst. Prof., Macalester Coll., St. Paul, Minn.

Letterio Laboccetta, C.E. (Naples). Engineer, Rome, Italy.

R. E. Langer, Ph.D. (Harvard). Prof., Univ. of Wisconsin, Madison, Wis.

A. J. Lewis, A.M. (Denver). Instr., University of Denver, Denver, Colo.

E. L. Mackie, Ph.D. (Chicago). Asso. Prof., Univ. of North Carolina, Chapel Hill, N. C.

Nellie P. Myser (Mrs. W. L.), A.B. (Huron). Teacher, College Math., Ward-Belmont, Nashville, Tenn.

Mary L. Newton, A.M. (Columbia). Dean, All Saints' College, Vicksburg, Miss.

Bess Patton, M.S. (Chicago). Teacher, Girls High School, Atlanta, Ga.

R. E. Peterson, B.S. (Grove City). Instr., Culver Military Acad., Culver, Ind.

E. J. Purcell. Instr. in Math. and Physics, Evans School, Tucson, Ariz.

Margaret Ramsey, B.S. (Linfield). Instr., Linfield Coll., McMinnville, Ore.

D. M. Rasel, B.S. (Washington and Jefferson). Instr., Applied Math., Washington and Jefferson College, Washington, Pa.

W. J. Reid, A.M. (Texas). Austin, Texas.

W. A. Richards, A.M. (Chicago). Morton Jr. Coll., Cicero, Ill.

W. C. Risselman, A.M. (Minnesota). Teaching Asst., Univ. of Minn., Minneapolis, Minn.

R. S. Shaw, B.S. (Bates). 5461 Univ. Ave., Chicago, Ill.

R. D. Smink, M.S. (Bucknell). Teacher, High School, Lock Haven, Pa.

E. H. Snyder, B.S. (Iowa). Industrial Engineer, Wis. Pearl Button Co., La Crosse, Wis.

J. R. K. Stauffer, A.B. (Franklin and Marshal). Instr., Purdue Univ., LaFayette, Ind.

W. R. Stevens. Meteorologist, U. S. Weather Bureau, Washington, D. C.

Guy Stevenson, Ph.D. (Illinois). Asst. Prof., Univ. of Louisville, Louisville, Ky.

Josephine Stone, A.M. (Peabody). Teacher, Athens Coll., Athens, Ala.

F. E. Ulrich, B.S. in E.E. (Union). Instr., Union Coll., Schenectady, N. Y.

B. B. Vance, B.S. (Univ. of Louisville). Instr., Univ. of Louisville, Louisville, Ky.

Mildred Watt, A.M. (Cornell). Teacher, Buffalo Seminary, Buffalo, N. Y.

S. W. Williams, A.B., B.E. (Vanderbilt). Prof., Arkansas Coll., Batesville, Ark.

F. L. Wren, A.M. (Peabody). Asso. Prof., Teaching of Math., George Peabody Coll. for Teachers, Nashville, Tenn.

To Institutional Membership.

Rhode Island College of Education, Providence, R. I.

Professor T. M. Focke and the Secretary-Treasurer were appointed as representatives of the Association on the Council of the American Association for the Advancement of Science.

The Trustees voted that the President and two others to be appointed by him be nominated as representatives of the Association on the American Section of the International Mathematical Union, in connection with the Congress to be held at Bologna, September 1–10, 1928. President Ford subsequently named Professors Mayme I. Logsdon and R. C. Archibald.

It was voted to have the finance committee transfer an additional thousand dollars to the endowment fund and to purchase a bond for that amount. This committee was empowered to make any changes in the investments of the Association funds which they may think wise, it being understood that the investments should at all times be so made as to afford adequate protection for the Association.

A report on mathematical symbols was approved as made by the subcommittee on mathematical symbols of the American Engineering Standards

Committee, dated April 25, 1927. This report incorporates the present day procedure in elementary and higher mathematics and has been printed in several of the journals of the sponsoring engineering societies.

Brief discussion took place on the future meetings of the Association, progress made in the Carus Monograph plan and the mutual activities and co-operation of the sections and the parent body.

W. D. CAIRNS, *Secretary-Treasurer.*

THE CLASSIFICATION OF QUADRICS IN EUCLIDEAN N-SPACE BY MEANS OF COVARIANTS[1]

By PHILIP FRANKLIN, Massachusetts Institute of Technology

1. Introduction. The first *complete* classification in terms of invariants and covariants for the Cartesian equations of conics was recently given by Professor MacDuffee.[2] While his results leave nothing to be added, the method used does not seem to be that most natural for the problem. The treatment is from the standpoint of Lie, whereas the problem is purely algebraic. Also the method gives no suggestion as to what the covariants would be for higher dimensions,[3] and would be almost inapplicable if these covariants were not already known from other considerations. Consequently it seems desirable to give a purely algebraic treatment of the question, which generalizes at once to n-space. This is done in the present paper.

2. Invariants of a conic. For the sake of completeness, as well as to pave the way for generalization, we give here a derivation of the well-known invariants of a second degree polynomial

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = \phi(x,y) \tag{1}$$

under the Euclidean transformations

$$x = b_{11}x' + b_{12}y' + b_{13}, \quad y = b_{21}x' + b_{22}y' + b_{23}, \tag{2}$$

where

$$\left\| \begin{matrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{matrix} \right\| \cdot \left\| \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\| . \tag{3}$$

---

[1] Presented to the American Mathematical Society, October 30, 1926.

[2] C. C. MacDuffee, *Euclidean invariants of second degree curves*, this Monthly, vol. 33 (1926), pp. 243–252.

[3] Thus the application to 3-space requires a new investigation, equal in length to the old. Cf. L. J. Paradiso, *A classification of second degree loci of space*, this Monthly, vol. 33 (1926), pp. 406–418.

The matrix form of (1) is

$$(4)\qquad \begin{Vmatrix} x & y & 1 \end{Vmatrix} \cdot \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ 1 \end{Vmatrix} = \phi(x,y)\ ; \qquad (a_{ij} = a_{ji}).$$

We may write (2) in either of the forms

$$(5)\qquad \begin{Vmatrix} x & y & 1 \end{Vmatrix} = \begin{Vmatrix} x' & y' & 1 \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & 0 \\ b_{13} & b_{23} & 1 \end{Vmatrix}$$

or

$$(6)\qquad \begin{Vmatrix} x \\ y \\ 1 \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x' \\ y' \\ 1 \end{Vmatrix}.$$

Using these last two equations, we find as the transform of (4)

$$(7)\qquad \begin{Vmatrix} x' & y' & 1 \end{Vmatrix} \cdot \begin{Vmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{Vmatrix} \cdot \begin{Vmatrix} x' \\ y' \\ 1 \end{Vmatrix} = \phi'(x',y')\ ,$$

where

$$(8)\qquad \begin{Vmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & 0 \\ b_{13} & b_{23} & 1 \end{Vmatrix} \cdot \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{Vmatrix}.$$

As this matrix equation implies the corresponding determinant equation, and as the product of the two determinants in the b's is unity from our orthogonality requirement (3), we have

$$(9)\qquad D_3 = |a_{ij}| = |a'_{ij}|, \text{ as the first invariants.}$$

Also, from the way the zeros occur in (8), that equation implies

$$(10)\qquad \begin{Vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{Vmatrix} \cdot \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

But, from (3), we have for all values of k,

$$(11)\qquad \begin{Vmatrix} k & 0 \\ 0 & k \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{Vmatrix} \cdot \begin{Vmatrix} k & 0 \\ 0 & k \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

On subtracting the terms of (11) from those of (10), we obtain :

$$(12)\quad \begin{Vmatrix} a'_{11}-k & a'_{12} \\ a'_{21} & a'_{22}-k \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{Vmatrix} \cdot \begin{Vmatrix} a_{11}-k & a_{12} \\ a_{21} & a_{22}-k \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

As this matrix equation implies the corresponding determinant equation, and since the product of the two determinants in the b's is unity by (3), we have

$$(13)\quad \begin{Vmatrix} a_{11}-k & a_{12} \\ a_{21} & a_{22}-k \end{Vmatrix} = k^2 - D_1k + D_2$$

is invariant. As this is so for all values of k, the coefficients must separately be invariant.

We thus have as the *invariants* of the polynomial (1), D_3, D_2 and D_1, whose *weights* or degrees in the coefficients are given by their subscripts. Their ratios are invariants for the locus of points for which $\phi(x, y) = 0$, and have a geometric significance which we shall give later.

3. Covariants of a conic. To obtain the covariants of the second degree polynomial (1), we shall find it convenient to introduce line coördinates. We first transform to homogeneous point coördinates, z_1, z_2, z_3, where $x = z_1/z_3$, $y = z_2/z_3$, so that the equation of our conic, $\phi(x, y) = 0$ becomes :

$$(14)\quad \sum_i \sum_j a_{ij}z_iz_j = 0.$$

Let us temporarily assume $D_3 = |a_{ij}| \neq 0$; we shall remove this restriction later. We find as the equation of the tangent to (14) at z_1, z_2, z_3,

$$(15)\quad \sum_i \sum_j a_{ij}Z_iz_j = 0,$$

where Z_1, Z_2, Z_3 denote current coördinates. The coördinates of this line in homogeneous line coördinates are

$$(16)\quad u_i = \sum_j a_{ij}z_j,$$

since the equation in point coördinates Z_i, of a line with line coördinates u_i is

$$(17)\quad \sum_i u_iZ_i = 0.$$

Solving equations (16) for z_j, and using A_{ij} to denote the co-factor of a_{ij} in $A = D_3 = |a_{ij}|$, we find :

$$(18)\quad z_j = \sum_i u_iA_{ij}/A,$$

and on inserting this in (14) we get

$$(19)\qquad \sum_i \sum_p \sum_q \sum_j \frac{a_{ij}u_pA_{pi}u_qA_{qj}}{A^2} = \sum_q \sum_j \frac{A_{qj}u_qu_j}{A} = 0.$$

The relation between (14) and (19) is reciprocal, as was to be expected from the duality relation of point to line coördinates. We note in passing that if (19) is used, multiplying the left member of the point equation by a constant is equivalent to dividing the left member of the line equation by this constant. If we start with the line equation:

$$(20)\qquad \sum_i \sum_j s_{ij}u_iu_j = \sum_i \sum_j A_{ij}u_iu_j/A = 0,$$

we may write the point equation as[1,2]

$$(21)\qquad \sum_i \sum_j a_{ij}z_iz_j = \sum_i \sum_j S_{ij}z_iz_j/S = 1/S \begin{vmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & s_{11} & s_{12} & s_{13} \\ z_2 & s_{21} & s_{22} & s_{23} \\ z_3 & s_{31} & s_{32} & s_{33} \end{vmatrix}.$$

Here $S = |s_{ij}|$ and S_{ij} is the cofactor of s_{ij} in S.

We now wish to consider the equations of transformation for the z_i and u_i. In view of (2) and (6), we have for the z_i:

$$(22)\qquad \left\|\begin{matrix} z_1 \\ z_2 \\ z_3 \end{matrix}\right\| = n \left\|\begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} z_1' \\ z_2' \\ z_3' \end{matrix}\right\|.$$

As the u_i are defined by (17), we must have

$$(23)\qquad \sum_i u_i'z_i' = m \sum_i u_iz_i,$$

where m is a constant multiplier or in matrix form

$$(24)\qquad \|u_1' \quad u_2' \quad u_3'\| \cdot \left\|\begin{matrix} z_1' \\ z_2' \\ z_3' \end{matrix}\right\| = m\|u_1 \quad u_2 \quad u_3\| \cdot \left\|\begin{matrix} z_1 \\ z_2 \\ z_3 \end{matrix}\right\|.$$

By combining this with (22), we see that the law of transformation of the u_i is

$$(25)\qquad \|u_1' \quad u_2' \quad u_3'\| = mn\|u_1 \quad u_2 \quad u_3\| \cdot \left\|\begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{matrix}\right\|.$$

[1] For the properties of minors here used, cf. Bôcher, *Higher Algebra*, p. 30.

[2] For the dual coördinates, cf. G. Salmon, *Analytic Geometry of Three Dimensions*, revised by A. P. Rogers, 5th ed., 1912, vol. 1, p. 65, p. 126.

We note that, in consequence of this,

$$(26)\qquad \|u_1' \quad u_2'\| = mn\|u_1 \quad u_2\| \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

From this, in view of (3), we derive

$$(27)\qquad u_1'^2 + u_2'^2 = m^2n^2(u_1^2 + u_2^2).$$

Now consider the equation

$$(28)\qquad \sum_i \sum_j s_{ij}u_iu_j - k(u_1^2 + u_2^2) = 0,$$

where k is any number. On transforming the coördinates by (25), this equation becomes

$$(29)\qquad (1/m^2n^2)\left[\sum_i \sum_j s_{ij}'u_i'u_j' - k(u_1'^2 + u_2'^2)\right] = 0.$$

By (21), the point form of (28) is :

$$(30)\qquad \frac{1}{S}\begin{vmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & s_{11}-k & s_{12} & s_{13} \\ z_2 & s_{21} & s_{22}-k & s_{23} \\ z_3 & s_{31} & s_{32} & s_{33} \end{vmatrix} = A\begin{vmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & \dfrac{A_{11}}{A}-k & \dfrac{A_{12}}{A} & \dfrac{A_{13}}{A} \\ z_2 & \dfrac{A_{21}}{A} & \dfrac{A_{22}}{A}-k & \dfrac{A_{23}}{A} \\ z_3 & \dfrac{A_{31}}{A} & \dfrac{A_{32}}{A} & \dfrac{A_{33}}{A} \end{vmatrix} = 0.$$

The point form of (29), after removal of the factor m^2n^2, will only differ from this in having primes on the A's. As these two point equations represent the same locus, their left members will differ at most by a factor. If we form the corresponding Cartesian equations by replacing z_1, z_2, z_3 by x, y, 1, respectively, the highest power of k in each expression will be the same, since $A = A' = D_3$, so that the factor is unity, and

$$(31)\qquad -A\begin{vmatrix} 0 & x & y & 1 \\ x & \dfrac{A_{11}}{A}-k & \dfrac{A_{12}}{A} & \dfrac{A_{13}}{A} \\ y & \dfrac{A_{21}}{A} & \dfrac{A_{22}}{A}-k & \dfrac{A_{23}}{A} \\ 1 & \dfrac{A_{31}}{A} & \dfrac{A_{32}}{A} & \dfrac{A_{33}}{A} \end{vmatrix} = Ak^2 - C_2k + C_1$$

will be equal to the corresponding expression in the primed variables. As k is arbitrary, the coefficients may be separately equated, and hence are covariants.

Using known properties of determinants, (see p. 456, ftn. 1) we may write them explicitly as

$$\begin{aligned} C_2 &= (x^2 + y^2)A_{33} - 2xA_{13} - 2yA_{23} + A_{11} + A_{22}, \\ C_1 &= \phi(x,y). \end{aligned} \tag{32}$$

Thus the apparent denominator cancels, and they are polynomials in the x, y and coefficients, of weight in the latter equal to their subscripts. The degree of C_2 in x, y may fall for particular polynomials, but is independent of coördinates owing to the linear character of (2).

While the proof of the covariance of C_1 and C_2 just given only applies if $A \neq 0$, since these expressions are polynomials in the coefficients, continuity considerations show that they must also be covariants when $A = 0$.

The equation $C_1 = 0$ represents the original locus; the geometrical significance of $C_2 = 0$ will be given later.

4. Classification of conics. We are now in a position to give a complete classification of the loci given by $\phi(x, y) = 0$. We begin by recalling the theorem that a real quadratic form in n variables of rank r can be reduced by a real orthogonal transformation in the n variables to the form[1]

$$c_1x_1^2 + c_2x_2^2 + \cdots + c_rx_r^2. \tag{33}$$

This shows that, by taking $b_{13} = b_{23} = 0$, and suitable values of the remaining coefficients in (2), i.e. by a rotation, we may reduce (1) to the form:

$$a_{11}x^2 + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33}. \tag{34}$$

If $a_{11} \neq 0$, and $a_{22} \neq 0$, a translation of axes gives the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}. \tag{35}$$

If in (34) a_{11} and a_{22} were both zero, we would not have a conic, but a first degree equation, so that this case can not occur. If either one is zero, we take it as a_{22} and, by a translation, reduce (34) to the form

$$a_{11}x^2 + 2a_{23}y + a_{33}. \tag{36}$$

Unless $a_{23} = 0$, we may by a further translation make $a_{33} = 0$.

Thus we are led to the following cases:

I. $D_2 \neq 0$. Hence $a_{11} \neq 0$, $a_{22} \neq 0$, and $\phi(x, y)$ may be reduced to

$$C_1 = a_{11}x^2 + a_{22}y^2 + a_{33}. \tag{37}$$

---

[1] Cf. Bôcher, *Higher Algebra*, pp. 171–173.

For this case (13) becomes

$$(a_{11} - k)(a_{22} - k) = k^2 - D_1 k + D_2, \tag{38}$$

so that a_{11} and a_{12} are determined from our invariants as the zeros of this expression in k, while from (9), $a_{33} = D_3/D_2$. We may also solve for x^2 and y^2 in terms of our invariants and covariants, unless $a_{11} = a_{22}$. For, in this case, (31) becomes

$$-a_{11}a_{22}a_{33}\begin{vmatrix} 0 & x & y & 1 \\ x & \dfrac{1}{a_{11}} - k & 0 & 0 \\ y & 0 & \dfrac{1}{a_{22}} - k & 0 \\ 1 & 0 & 0 & \dfrac{1}{a_{33}} \end{vmatrix} = Ak^2 - C_2 k + C_1. \tag{39}$$

On putting $k = 1/a_{11}$, this gives

$$x^2(a_{11} - a_{22}) = A(1/a_{11})^2 - C_2(1/a_{11}) + C_1, \tag{40}$$

which determines x^2, and on putting $k = 1/a_{22}$ it gives

$$y^2(a_{22} - a_{11}) = A(1/a_{22})^2 - C_2(1/a_{22}) + C_1, \tag{41}$$

which determines y^2, in terms of the a_{ii}, which have already been shown to be given by the invariants, and the covariants.

If $a_{11} = a_{22}$, we can only solve for $x^2 + y^2$. We do this by differentiating (39) with respect to k, and then putting $k = 1/a_{11} = 1/a_{22}$, obtaining:

$$- a_{11}a_{22}(x^2 + y^2) = 2A(1/a_{11}) - C_2. \tag{42}$$

II. $D_2 = 0$, $D_3 \neq 0$. Hence $a_{22} = 0$, $a_{23} \neq 0$, and $\phi(x, y)$ may be reduced to

$$C_1 = a_{11}x^2 + 2a_{23}y. \tag{43}$$

For this case (13) becomes

$$- k(a_{11} - k) = k^2 - D_1 k, \tag{43}$$

which determines a_{11}, while $a_{23}^2 = -D_3/D_1$. We may solve for x^2 in terms of our invariants and covariants by evaluating (31) for this case, and putting $k = 1/a_{11}$. There results

$$a_{11}x^2 = A(1/a_{11})^2 - C_2(1/a_{11}) + C_1. \tag{44}$$

We obtain y by combining this with (43).

III. $D_2 = 0$, $D_3 = 0$. Hence $a_{22} = a_{23} = 0$, and $\phi(x, y)$ may be reduced to

$$C_1 = a_{11}x^2 + a_{33}. \tag{45}$$

Here (13) again takes the form (43), from which $a_{11}=D_1$, while (32) shows that C_2 is here independent of x and y, and $a_{33}=C_2/D_1$. We may solve for x^2 in terms of our invariants and covariants by using (45).

From the discussion just given, we may draw the following conclusions. If the invariants D_1, D_2 and D_3 are known, we may at once determine which of the three cases is applicable, and hence the appropriate canonical form. The coefficients of this canonical form are given, in cases I and II in terms of these invariants; in case III, in terms of these invariants and C_2. As any invariant of the given polynomial may be calculated from the coefficients of the canonical form, it must be a function of such invariants as determine these coefficients. Thus, in cases I and II, D_1, D_2, D_3 form a *complete system* of algebraic invariants. In case III, D_1, D_2, D_3 and C_2 form a complete system.

We further note that any covariant of the given polynomial would, for the canonical form, reduce to a function of the coefficients and variables in this canonical form, and such a function of the variables as to be unchanged by any transformation of coördinates which left the canonical form unchanged. Thus in case I, the covariant could only involve x^2 and y^2, as (37) is unchanged by $x=-x'$, or $y=-y'$. If $a_{11}=a_{22}$, (37) would be unchanged by the rotations which leave x^2+y^2 unchanged, so that the covariant would only involve x and y in this combination. Similarly in case II, a covariant could only involve x^2, and y. In case III, the covariant could only involve x^2, and must be independent of y, since $y=y'+b_{23}$ leaves (45) unchanged. Since each of these combinations of the variables has been calculated in terms of the invariants and covariants for the case in question, it follows that D_1, D_2, D_3, C_1, C_2 form a *complete system* of algebraic invariants and covariants for the given ploynomial. That the system is complete in the sense of having the minimum number follows from the number of coefficients and variables in the canonical terms, which are all independent.

It might be supposed that the discussion here used to verify that we have the right number of invariants and covariants was unnecessary since an application of the Lie theory, apparently merely requires us to count the number of variables in our form and parameters in the group and subtract. However, to justify this process we would have actually to set up the generators of the group explicitly in order to verify that the differential equations obtained by equating them to zero are independent.[1]

[1] The system of differential equations obtained by equating to zero the generators of an r-parameter group if independent, form a complete system, but are not always independent. (cf. E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 4.). The fact that equations (6), MacDuffee, l.c. p. 246 and (2) Paradiso, l.c. p. 407 are independent does not follow from the Lie theory, a priori, but must be proved, either by setting up and inspecting the equations, as these authors have done, or by arguing as we have done here.

5. Geometric considerations. To connect a second degree polynomial in x and y with a geometric locus, we equate the polynomial to zero. As the locus is unchanged when the polynomial is multiplied by a non-vanishing constant, only invariants and covariants of weight zero in the coefficients are geometric invariants or covariants. From the five invariants and covariants already found, by taking ratios, we may form four invariants and covariants of weight zero, which form a complete system of geometric invariants and covariants. The significance of the invariants varies with the case, but may be found from the canonical form.

Thus for central conics, case I, $D_3 \neq 0$, the reciprocals of the squares of the semi-axes are a_{11}/a_{33} and a_{22}/a_{33} and hence by (38) the ratios D_1D_2/D_3 and D_2^3/D_3^2 are the symmetric functions of these reciprocals. For degenerate central conics, case I, $D_3=0$, the ratios of the reciprocals of the squares of the semi-axes to the product of these reciprocal semi-axes for central conics having the degenerate one as a limit have significance, and the symmetric functions of these ratios are given by $D_1/D_2^{1/2}$ and 1.

For parabolas, case II, the reciprocal of the square of the semi-chord at unit distance from the vertex is $a_{11}/2a_{23}$, and hence by (43) is $\sqrt{(-D_1^3/4D_3)}$.

While only the ratios of the invariants have geometric significance the vanishing of any one of them has a direct geometric meaning. Thus from the classification it follows that $D_3=0$ means the locus degenerates into two straight lines. $D_2=0$ means that the locus at infinity degenerates giving a single point there either for the parabola of case II, or the parallel lines of case III. For $D_1=0$, we must have $D_2 \neq 0$ and from (37) we see that either the locus is degenerate, and consists of two perpendicular lines, or it is a central conic with perpendicular asymptotes.

Since the covariants represent the symmetric functions of the zeros of (31) we must interpret the meaning of these zeros to see the significance of the covariants. Each of these values of k, when inserted in (28) determines a conic passing through x, y, confocal to the given one, since it has the same tangents from the circular points at infinity.[1] These two conics are orthogonal. Let us introduce homogeneous line coördinates, and take the normals to these conics as axes. The first conic, by (28), has as its equation:

$$\sum_i \sum_j s_{ij}u_iu_j - k_1(u_1^2 + u_2^2) = 0. \tag{46}$$

With this choice of coördinates, the x axis, $(1,0,0)$, is a tangent, which necessitates $s_{11}=k_1$, and the equation of its point of contact

$$\sum_i s_{i1}u_1 - k_1u_1 = 0 \tag{47}$$

[1] Salmon-Rogers, l.c., p. 177.

must reduce to the equation of the origin, $u_3=0$. Thus $s_{12}=0$. A similar argument for the second conic shows that $s_{22}=k_2$ and $s_{23}=0$, so that the equation of the given conic is:

$$k_1u_1^2 + k_2u_2^2 + 2s_{13}u_1u_3 + 2s_{23}u_2u_3 + s_{33}u_3^2 = 0. \tag{48}$$

The tangent cone from the origin is the intersection of this with $u_3=0$, and has the equation:

$$k_1u_1^2 + k_2u_2^2 = 0. \tag{49}$$

The point form is

$$x^2/k_1 + y^2/k_2 = 0. \tag{50}$$

This shows that $C_2=0$ is the equation of the locus of points from which two perpendicular tangents can be drawn to the conic, while $C_1=0$, the given locus is that of points from which the two tangents coincide.

6. Second degree loci in n-space. Let us now consider the second degree polynomial in n variables $(i, j=1, 2, \cdots, n)$,

$$\sum_i \sum_j a_{ij}x_ix_j + 2\sum_i a_{i,n+1}x_i + a_{n+1,n+1} = \phi(x_1, x_2, \cdots, x_n). \tag{51}$$

We regard the x_i as coördinates in n-space, transformed by

$$x_i = \sum_j b_{ij}x_j' + b_{i,n+1}, \tag{52}$$

the corresponding homogeneous transformation being orthogonal. The equation $\phi(x_i)=0$ is thus the Cartesian equation of a quadric in Euclidean n-space.

The argument of section 2 at once generalizes and leads us to the invariants $D_1, D_2, \cdots, D_{n+1}$ of weight in the coefficients of (51) given by their subscripts, defined by:

$$D_{n+1} = |a_{ij}| \tag{53}$$

and

$$\begin{vmatrix} a_{11}-k & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-k & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-k \end{vmatrix} = (-k)^n + D_1(-k)^{n-1} + D_2(-k)^{n-2} + \cdots + D_n. \tag{54}$$

The arguments of section 3 may also be applied here, using as the dual coördinates $(n-1)$-space coördinates, and give the covariants $C_1, C_2, \cdots, C_n$, with weights equal to their subscripts, defined by:

$$
(55)\qquad -A\begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n & 1 \\ x_1 & \dfrac{A_{11}}{A}-k & \dfrac{A_{12}}{A} & \cdots & \dfrac{A_{1n}}{A} & \dfrac{A_{1,n+1}}{A} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_n & \dfrac{A_{n1}}{A} & \dfrac{A_{n2}}{A} & \cdots & \dfrac{A_{nn}}{A}-k & \dfrac{A_{n,n+1}}{A} \\ 1 & \dfrac{A_{n+1,1}}{A} & \dfrac{A_{n+2,2}}{A} & \cdots & \dfrac{A_{n+1,n}}{A} & \dfrac{A_{n+1,n+1}}{A} \end{vmatrix}
$$

$$
= A(-k)^n + C_n(-k)^{n-1} + C_{n-1}(-k)^{n-2} + \cdots + C_1.
$$

The apparent denominator cancels (see p. 456. ftn. 1), the C_i being polynomials in x_i and the a_{ij}, of the second degree in the x_i, except for special given polynomials, and of weight in the a_{ij} equal to their subscripts.

To classify the canonical forms to which $\phi(x_i)$ may be reduced, we first note that by the theorem quoted at the beginning of section 4, there exists a homogeneous orthogonal transformation (52) which reduces (51) to the form

$$
(56)\qquad \sum_i a_{ii}x_i^2 + 2\sum_i a_{n+1,i}x_i + a_{n+1,n+1}.
$$

If $a_{ii}\neq 0$, $i=1, 2, \cdots, n$, a translation of the axes reduces this to

$$
(57)\qquad \sum_i a_{ii}x_i^2 + a_{n+1,n+1}.
$$

If $n-r$ of the $a_{ii}=0$, we may take them as the last $n-r$, and by a translation of the axes reduce (56) to

$$
(58)\qquad \sum_{s=1}^{r} a_{ss}x_s^2 + 2\sum_{t=r+1}^{n} a_{n+1,t}x_t + a_{n+1,n+1}.
$$

If the $a_{n+1,t}$ are not all zero, we may take as the new x_{r+1} coördinate $n-1$ space that whose equation is

$$
(59)\qquad 2\sum_{t=r+1}^{n} a_{n+1,t}x_t + a_{n+1,n+1} = 0, \text{ and (58) takes the forms}
$$

$$
(60)\qquad \sum_{s=1}^{r} a_{ss}x_s^2 + 2a_{r+1,n+1}x_{r+1}.
$$

If all the $a_{n+1\ t}=0$, (58) reduces to

$$
(61)\qquad \sum_{s+1}^{r} a_{ss}x_s^2 + a_{n+1,n+1}.
$$

We may now consider the cases which can occur.

I. $D_n \neq 0$. Hence in (56) $a_{ii} \neq 0, i=1, 2, \cdots, n$ and $\phi(x_i)$ may be reduced to

$$C_1 = \sum_i a_{ii}x_i^2 + a_{n+1,n+1}. \tag{62}$$

For this case (54) becomes

$$(a_{11} - k)(a_{22} - k) \cdots (a_{nn} - k) = (-k)^n + D_1(-k)^{n-1} + D_2(-k)^{n-2} + \cdots + D_n, \tag{63}$$

so that the a_{ii} are determined in terms of our invariants, while from (53) $a_{n+1\ n+1} = D_{n+1}/D_n$. To solve for x_i^2 in terms of our invariants and covariants, we use (62) to reduce (55), and put $k=A_{ii}/A=1/a_{ii}$, which reduces (55) to the form

$$\frac{x_i^2}{a_{ii}^{n-2}}(a_{ii} - a_{11}) \cdots (a_{ii} - a_{i-1\ i-1})(a_{ii} - a_{i+1\ i+1}) \cdots (a_{ii} - a_{nn}) = A(-1/a_{ii})^n + C_n(-1/a_{ii})^{n-1} + C_{n-2}(-1/a_{ii})^{n-2} + \cdots + C_1. \tag{64}$$

If p of the a_{ii} are equal, the corresponding x_i^2 can no longer be determined by (64).
The sum of these p x_i^2 may be determined, however, by differentiating (55) $p-1$ times with respect to k before putting $k=1/a_{11}=1/a_{22}= \cdots 1/a_{pp}$, and thus obtaining a formula analogous to (42).

II. $D_n=0$, $D_{n+1}\neq 0$. Hence in (56) $a_{nn}=0$, $a_{n+1,n}\neq 0$, and, by (60), $\phi(x_i)$ may be reduced to

$$\sum_{s=1}^{n-1} a_{ss}x_s^2 + 2a_{n,n+1}x_n. \tag{65}$$

For this case (54) becomes

$$-k(a_{11} - k) \cdots (a_{n-1,n-1} - k) = (-k)^n + D_1(-k)^{n-1} + \cdots + D_{n-1}(-k), \tag{66}$$

which determines the a_{ss}, while $a_{n,n+1}^2 = -D_{n+1}/D_{n-1}$. To solve for x_s^2, we reduce (55) by using (65) and putting $k=1/a_{ss}$. This leads to

$$\frac{x_s}{a_{ss}^{n-3}}(a_{ss} - a_{11}) \cdots (a_{ss} - a_{s-1,s-1})(a_{ss} - a_{s+1,s+1}) \cdots (a_{ss} - a_{n-1,n-1}) = A(-1/a_{ss})^n + C_n(-1/a_{ss})^{n-1} + C_{n-2}(-1/a_{ss})^{n-2} + \cdots + C_1. \tag{66}$$

If p of the a_{ss} are equal, the corresponding x_s^2 can no longer be determined by (66), but by differentiating (55) $p-1$ times with respect to k before substituting $1/a_{ss}$ for k, the sum of these p x_s^2 may be determined.

III. $D_{n+1}=D_n=0$. Hence in (56), by the method used to obtain (60) and (61) we may make $a_{nn}=a_{n,n+1}=0$.

Here the right member of (55) is a determinate polynomial in the a_{ij}, but can no longer be directly evaluated from the left member, owing to the vanishing denominator. One simple way of evaluating this left member in this case, where all the $a_{ni}=0$, is to first put $a_{ni}=0$, $i\neq n$, $a_{nn}\neq 0$, and find the limit when a_{nn} approaches zero. If we put $A=a_{nn}A'$, and $a_{ni}=0$, $i\neq n$, we shall have $A_{ij}=a_{nn}A'_{ij}$, $(i, j\neq n)$, and may write the left member of (55) in the form:

$$(67)\qquad -A'\begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n & 1 \\ x_1 & \dfrac{A'_{11}}{A'}-k & \dfrac{A'_{12}}{A'} & \cdots & 0 & \dfrac{A'_{1,n+1}}{A'} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{nn}x_n & 0 & 0 & \cdots & 1-a_{nn}k & 0 \\ 1 & \dfrac{A'_{n+1,1}}{A'} & \dfrac{A'_{n+1,2}}{A'} & \cdots & 0 & \dfrac{A'_{n+1,n+1}}{A'} \end{vmatrix}.$$

The limit of this when a_{nn} approaches zero is

$$(68)\qquad -A'\begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_{n-1} & 1 \\ x_1 & \dfrac{A'_{11}}{A'}-k & \dfrac{A'_{12}}{A'} & \cdots & \dfrac{A'_{1,n-1}}{A'} & \dfrac{A'_{1,n+1}}{A'} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n-1} & \dfrac{A'_{n-1,1}}{A'} & \dfrac{A'_{n-1,2}}{A'} & \cdots & \dfrac{A'_{n-1,n-1}}{A'}-k & \dfrac{A'_{n-1,n+1}}{A'} \\ 1 & \dfrac{A'_{n+1,1}}{A'} & \dfrac{A'_{n+1,2}}{A'} & \cdots & \dfrac{A'_{n+1,n}}{A'} & \dfrac{A'_{n+1,n+1}}{A'} \end{vmatrix}$$

$$= A'(-k)^{n-1}+C'_{n-1}(-k)^{n-2}+C'_{n-2}(-k)^{n-3}+\cdots+C'_1.$$

This is of essentially the same form as (55) except for the primes and the numbering $1, 2, \cdots, n-1, n+1$ instead of $1, 2, \cdots, n, n+1$. A comparison with (55) shows that $C_j'=C_j$, $j=1, 2, \cdots, n-1$ while $C_n=A'$, so that this particular covariant is constant in this and hence in all coördinate systems and reduces to an invariant.

On putting $a_{ni}=0$ $(i=1, 2, \cdots, n)$ in (54), it becomes

$$(69)\qquad \begin{vmatrix} a_{11}-k & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22}-k & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-2,2} & \cdots & a_{n-1,n-1}-k \end{vmatrix}$$

$$=(-k)^{n-1}+D_1(-k)^{n-2}+D_2(-k)^{n-3}+\cdots+D_{n-1}.$$

This is of the same form as (54) in one less variable.

As we have found for this case coördinates such that one of them, x_n is missing from the given equation, the locus in this case is a "cylinder" in n space. The right section in the space $x_i(i=1, 2, \cdots, n-1)$ is a quadric in this space, which may be classified into two cases, analogous to I and II above. The coefficients and variables in the canonical forms are obtained by the same method previously used, except that equations (54) and (55) are replaced by (69) and (68) respectively.

We have the analogue of case I if $D_{n-1}\neq 0$, and that of case II if $C_n\neq 0$. If $D_{n-1}=0$ and $C_n\equiv 0$, certain of our formulas will contain the vanishing factor A' in the denominator, but in this case we may repeat the process used to obtain (68) and (69). Evidently if our given equation involves essentially $n-r$ variables we need to repeat the process r times. There will be $n-r$ true covariants, the remaining r will reduce to invariants for this case.

By the reasoning used at the end of section 4, we see that the invariants, together with certain of the covariants in special cases, form a complete system of algebraic invariants for the given polynomial. The set of invariants and covariants we have found form a complete system of algebraic invariants and covariants for this polynomial.

7. Geometric considerations. As in section 5, we see that if we form, from our $n+1$ invariants and n covariants, $2n$ ratios of weight zero, we shall have a complete system of geometric invariants and covariants for the locus of $\phi(x_i)=0$. The geometric significance of these ratios is readily obtained from the canonical form.

Thus in case I, we have central n-dimensional quadrics, and the symmetric functions of the reciprocals of the squares of their semi axes, $a_{ii}/a_{n+1,n+1}$, are $D_iD_n^i/(D_{n+1})^i$, by (63). For degenerate central quadrics, n-dimensional cones, the ratios of the reciprocals of the squares of the semi-axes to the nth root of the product of these reciprocal squares for central quadrics having the degenerate ones as a limit have significance, and the symmetrical functions of these ratios are given by $D_i/(D_n)^{i/n}$.

For n-dimensional paraboloids, the $a_{ii}/a_{n,n+1}$ are the reciprocals of the squares of the semiaxes of a section at unit distance from the vertex, and their symmetric functions are by (66) $D_i(-D_{n-1}/D_{n+1})^{i/2}$.

Analogous considerations apply to the subdivisions of case III.

The vanishing of the invariants has a geometric interpretation. Thus $D_{n+1}=0$, if $D_n\neq 0$, means that the locus is an n-dimensional cone, while $D_n=0$, means that the locus at infinity degenerates into a cone. If our locus is an n-dimensional cone, the condition $D_i=0$, $i=1, 2, \cdots, n-1$, means that there exist $n!/i!\,(n-i)!$ mutually perpendicular i-spaces, each tangent to this cone. For, suppose such a set of spaces exist, and take them as the coördinate i-spaces. Since the space $x_{i+1}=x_{i+2}=\cdots=x_n=0$ is tangent to the cone, on putting these

values in the equation, the remaining terms must represent this i-space counted twice, i.e., must factor. Hence, regarding these terms as the equation of a cone in i-space, this cone must cut the $i-1$ space at infinity in a degenerate locus, and for it $D_i' = 0$. That is, a principal i-rowed minor of D_n vanishes and by similar reasoning all such principal i-rowed minors vanish. But, by (54) D_i is the sum of these minors, and hence vanishes, for the particular coördinates used. As it is invariant, it vanishes in all systems. This proves the necessity of the condition $D_i=0$ for the cone's having the maximum number of mutually perpendicular tangent i-spaces. From the algebraic formulation of the problem, it appears that there is only one equation giving the necessary and sufficient condition, so that the condition $D_i=0$ is also sufficient. For a central quadric not a cone, the condition $D_i=0$ means that the asymptotic cone has the above property.

The interpretation of the covariants hinges on the meaning of k in (55). Each value of k which makes the members of (55) vanish, when inserted in the equation

$$(70)\quad \sum_i \sum_j s_{ij}u_iu_j - k\sum_{s=1}^{n} u_s^2 = 0\,; \quad [s_{ij} = A_{ij}/A\,; \quad i,j = 1,2,\cdots,n+1],$$

analogous to (28), gives, in homogeneous dual $n-1$ space coördinates, the equation of a quadric passing through the point x_i confocal to the given one. The normals to these n quadrics at the point x_i are mutually perpendicular, and taking them as coordinate axes, by an argument (see p. 461, ftn. 1) similar to that of section 5, we find for the dual equation of the tangent cone from the origin to our original quadric

$$\sum_{s=1}^{n} k_su_s^2 = 0, \quad \text{and as its point form} \quad \sum_{s=1}^{n} x_i^2/k_i = 0.$$

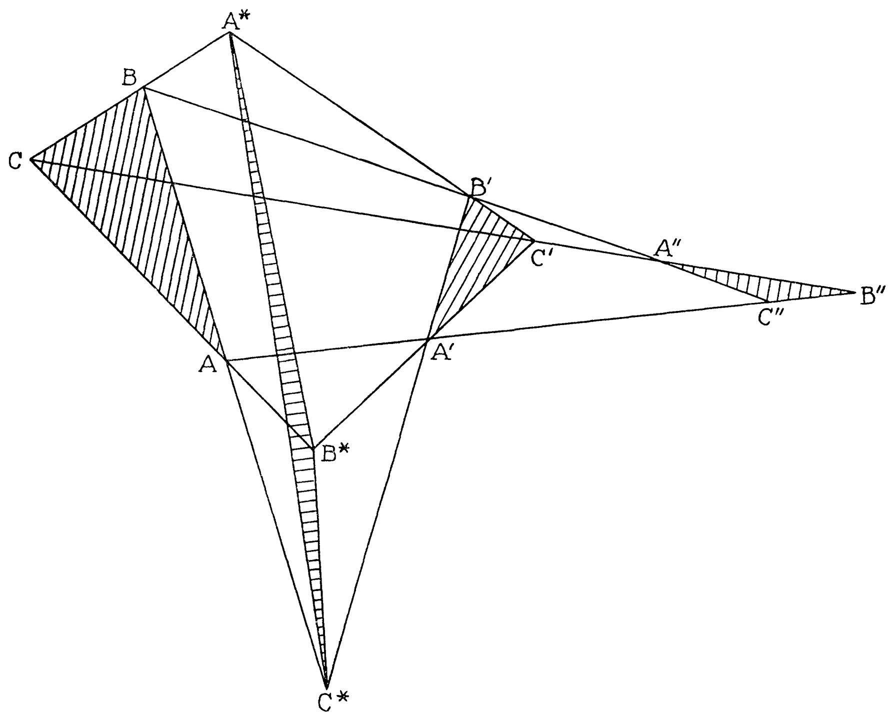
The condition that this cone contain the maximum possible number of mutually perpendicular i spaces, $i=1, 2, \cdots, n-1$ is that the symmetric function of the ith degree in the $1/k_s$ vanish. This is equivalent to the vanishing of the symmetric functions of the $(n-i)$th degree in the k_s, or C_{i+1} by (55). If $C_1=0$, the product of the k_s vanishes, and hence one of them, and by (72) the tangent cone breaks up into two coincident $n-1$ spaces, which checks with the fact that $C_1=0$ is the equation of the given locus. We see from the above that $C_{i+1}=0$ is the locus of points from which the maximum number, $n!/i!(n-i)!$, mutually perpendicular i-spaces can be drawn all tangent to the given quadric.

THE THEOREMS OF CEVA AND MENELAUS AND THEIR EXTENSION[1]

By PAUL WERNICKE, Washington, D. C.

The theorems referred to in the title occur in elementary geometry. That of Ceva states that three concurrent (straight) lines drawn from the vertices A, B, C of a triangle, in its plane, divide the opposite sides BC, CA, AB in ratios the product of which is unity.

Menelaus' theorem, in slightly modernized enunciation, is to the effect that any straight transversal divides the sides BC, CA, AB of a triangle ABC in ratios the product of which is negative unity.



We now consider two triangles, ABC, $A'B'C'$, in a plane S_2, and draw the joins AA', BB', CC' of their vertices. These will in general meet in three points $A''=(BB', CC')$, $B''=(CC', AA')$, $C''=(AA', BB')$. Triangle $A''B''C''$ is the central imperspective triangle of ABC and $A'B'C'$. It reduces to their center of perspective when its area vanishes. The product of the ratios in which the sides BC, CA, AB are divided by the lines AA', BB', CC' will be called the *ceva* of ABC. Similarly for $A'B'C'$.

[1] Read before the Maryland–District of Columbia–Virginia Section of the Association, Dec. 4, 1926.

Finding the intersections $A^* = (BC, B'C')$, $B^* = (CA, C'A')$, $C^* = (AB, A'B')$ of pairs of corresponding sides, we also call the product of the ratios in which they divide the sides of either triangle the *menelaus* of the latter. $A^*B^*C^*$ is the axial imperspective triangle of ABC and $A'B'C'$, since it reduces to the so-called axis of perspective when its area vanishes.

We next prove the theorem: *The ceva of ABC is the negative menelaus of $A'B'C'$ (and vice-versa).* In barycentric coordinates with ABC as triangle of reference we write

$$\begin{array}{llll} A' = \alpha_0 A + \alpha_1 B + \alpha_2 C & \text{or briefly} & = (\alpha_0 A +), \text{ where} & \alpha_0 + \alpha_1 + \alpha_2 \\ B' = \beta_0 A + \beta_1 B + \beta_2 C & \text{“} \quad \text{“} & = (\beta_0 A +), & = \beta_0 + \beta_1 + \beta_2 \\ C' = \gamma_0 A + \gamma_1 B + \gamma_2 C & \text{“} \quad \text{“} & = (\gamma_0 A +), & = \gamma_0 + \gamma_1 + \gamma_2 = 1 \end{array}$$

Two triangles having the bases CA, AB and a common summit on AA' then have their areas in the ratio α_1/α_2. Accordingly, BC is divided by AA' in the ratio α_2/α_1. The ratios of division in CA and AB are similarly found to be β_0/β_2 and γ_1/γ_0. The ceva of ABC is thus $(\alpha_2\beta_0\gamma_1)/(\alpha_1\beta_2\gamma_0)$.

The ceva and menelaus are dualistically formed products. Each of the former's three factors is a ratio of two barycentric coordinates, this ratio being constant on one of the three joins of vertices, AA', BB', CC'. Each factor of the menelaus is a negative ratio of two perpendiculars dropped from the triangle's vertices onto *any* line drawn through one of the intersections A^*, B^*, C^*. Such a ratio is also that of two barycentric coordinates of such a line and is constant for each of the points A^*, B^*, C^*.

Now BC meets $B'C'$ in $A^* = (1-u)B + uC = (1-v)(\beta_0 A +) + v(\gamma_0 A +)$. The coefficients of A must be equal in the last two members: $(1-v)\beta_0 + v\gamma_0 = 0$. This shows that $v = \beta_0/(\beta_0 - \gamma_0)$, $1 - v = -\gamma_0/(\beta_0 - \gamma_0)$ and that $B'C'$ is divided in the ratio $-\beta_0/\gamma_0$. B^* similarly divides $C'A'$ in the ratio $-\gamma_1/\alpha_1$, and C^* divides $A'B'$ in the ratio $-\alpha_2/\beta_2$. For the menelaus of $A'B'C'$ we find $-(\alpha_2\beta_0\gamma_1)/(\alpha_1\beta_2\gamma_0)$. This proves the theorem.

The vertices of the central imperspective triangles are found to be

$$\begin{aligned} A'' &= (\beta_0\gamma_0 A + \beta_0\gamma_1 B + \beta_2\gamma_0 C) : (\beta_0\gamma_0 + \beta_0\gamma_1 + \beta_2\gamma_0) \\ B'' &= (\gamma_0\alpha_1 A + \gamma_1\alpha_1 B + \gamma_1\alpha_2 C) : (\gamma_0\alpha_1 + \gamma_1\alpha_1 + \gamma_1\alpha_2) \\ C'' &= (\alpha_2\beta_0 A + \alpha_1\beta_2 + \alpha_2\beta_2 C) : (\alpha_2\beta_0 + \alpha_1\beta_2 + \alpha_2\beta_2), \end{aligned}$$

where the denominators have been reduced by means of the relations

$$\alpha_0 + \alpha_1 + \alpha_2 = \beta_0 + \beta_1 + \beta_2 = \gamma_0 + \gamma_1 + \gamma_2 = 1.$$

Its area is proportional to the determinant Δ_1 given below, whose elements are found to be the cofactors of the corresponding elements in the determinant Δ_2, of which it is, therefore, the square.

$$\Delta_1 = \begin{vmatrix} \beta_0\gamma_0 & \beta_0\gamma_1 & \beta_2\gamma_0 \\ \gamma_0\alpha_1 & \gamma_1\alpha_1 & \gamma_1\alpha_2 \\ \alpha_2\beta_0 & \alpha_1\beta_2 & \alpha_2\beta_2 \end{vmatrix} ; \quad \Delta_2 = \begin{vmatrix} 0 & \alpha_2 & -\alpha_1 \\ -\beta_2 & 0 & \beta_0 \\ \gamma_1 & -\gamma_0 & 0 \end{vmatrix}$$

Triangles ABC and $A'B'C'$ are in perspective when this determinant Δ_2 vanishes. This is Ceva's theorem.

The vertices of the axial imperspective triangle being

$$\begin{aligned} A^* &= [\qquad\qquad (\beta_0\gamma_1)B + (\beta_0\gamma_2)C]:[(\beta_0\gamma_1) + (\beta_0\gamma_2)] \\ B^* &= [(\gamma_1\alpha_0)A \qquad\qquad + (\gamma_1\alpha_2)C]:[(\gamma_1\alpha_2) + (\gamma_1\alpha_0)] \\ C^* &= [(\alpha_2\beta_0)A + (\alpha_2\beta_1)B \qquad\qquad]:[(\alpha_2\beta_0) + (\alpha_2\beta_1)], \end{aligned}$$

where $(\beta_i\gamma_j)$ stands for $\beta_i\gamma_j - \beta_j\gamma_i$, its area is proportional to

$$\begin{vmatrix} 0 & (\beta_0\gamma_1) & (\beta_0\gamma_2) \\ (\gamma_1\alpha_0) & 0 & (\gamma_1\alpha_2) \\ (\alpha_2\beta_0) & (\alpha_2\beta_1) & 0 \end{vmatrix} = \begin{vmatrix} 0 & -\gamma_0 & \beta_0 \\ \gamma_1 & 0 & -\alpha_1 \\ -\beta_2 & \alpha_2 & 0 \end{vmatrix} \cdot \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \\ \gamma_0 & \gamma_1 & \gamma_2 \end{vmatrix} .$$

The condition for the vanishing of its area is evidently the same, namely $\Delta_2 = 0$.

If ABC and $A'B'C'$ are in perspective, let the lengths of the perpendiculars from A', B', C' to the axis of perspective $(A^*B^*C^*)$ be a, b, c. Then

$$-\beta_0/\gamma_0 = b/c; \quad -\gamma_1/\alpha_1 = c/a; \quad -\alpha_2/\beta_2 = a/b.$$

Designate the center of perspective by $E = \epsilon_0 A + \epsilon_1 B + \epsilon_2 C = (\epsilon_0 A +)$. Then

$$\alpha_2/\alpha_1 = \epsilon_2/\epsilon_1 ; \quad \beta_0/\beta_2 = \epsilon_0/\epsilon_2 ; \quad \gamma_1/\gamma_0 = \epsilon_1/\epsilon_0 .$$

We may write

$$\begin{aligned} (1 - u_0)A + u_0(e_0A +) &= (\alpha_0A +), \\ (1 - u_1)B + u_1(e_0A +) &= (\beta_0A +), \\ (1 - u_2)C + u_2(e_0A +) &= (\gamma_0A +), \end{aligned}$$

whence

$$\begin{aligned} u_0 &= \frac{1-\alpha_0}{1-e_0}, & \alpha_1 &= e_1u_0, & \alpha_2 &= e_2u_0, \\ \beta_0 &= e_0u_1, & u_1 &= \frac{1-\beta_1}{1-e_1}, & \beta_2 &= e_2u_1, \\ \gamma_0 &= e_0u_2, & \gamma_1 &= e_1u_2, & u_2 &= \frac{1-\gamma_2}{1-e_2} . \end{aligned}$$

A fourth point of reference, D, is needed if the triangles ABC, $A'B'C'$ lie in a three dimensional space S_3 but no longer in the same plane. The vertices A', B', C' and the center E will then be

$$
\begin{aligned}
A' &= \alpha_0 A + \alpha_1 B + \alpha_2 C + \alpha_3 D = (\alpha_0 A +), \quad \text{where} & \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \\
B' &= (\beta_0 A +) & = \beta_0 + \beta_1 + \beta_2 + \beta_3 \\
C' &= (\gamma_0 A +) & = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 \\
E &= (e_0 A +) & = e_0 + e_1 + e_2 + e_3 = 1.
\end{aligned}
$$

The above set of equations then becomes

$$
\begin{array}{llll}
u_0 = \dfrac{1-\alpha_0}{1-e_0}, & \alpha_1 = e_1 u_0, & \alpha_2 = e_2 u_0, & \alpha_3 = e_3 u_0 \\
\beta_0 = e_0 u_1, & u_1 = \dfrac{1-\beta_1}{1-e_1}, & \beta_2 = e_2 u_1, & \beta_3 = e_3 u_1 \\
\gamma_0 = e_0 u_2, & \gamma_1 = e_1 u_1, & u_2 = \dfrac{1-\gamma_2}{1-e_2}, & \gamma_3 = e_3 u_2.
\end{array}
$$

These give the equations of complanarity of B, C, B', C' and two point-quadruples obtained therefrom by cyclic permutation:

$$
\begin{array}{l}
(\beta_0\gamma_3) = 0, \text{ which is} \\
(\gamma_1\alpha_3) = 0 \\
(\alpha_2\beta_3) = 0
\end{array}
\quad
\begin{vmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\beta_0 & \beta_1 & \beta_2 & \beta_3 \\
\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix} = 0
$$

and two similar equations indicating complanarity of four points. $B'C'$, $C'A'$, $A'B'$ are again divided in A^*, B^*, C^*, respectively, in the ratios $-\beta_0/\gamma_0$, $-\gamma_1/\alpha_1$, $-\alpha_2/\beta_2$. These by virtue of the conditions of complanarity, may be written $-\beta_3/\gamma_3$, $-\gamma_3/\alpha_3$, $-\alpha_3/\beta_3$. The menelaus of $A'B'C'$ is therefore again -1. The first factor of the ceva, α_2/α_1, is the ratio of the volumes of two tetrahedra whose bases, DBA and DAC, meet in the edge DA and whose common summit is any point of AA'. The interpretation of the two other factors is analogous.

By assuming a fourth point $D' = \delta_0 A + \delta_1 B + \delta_2 C + \delta_3 D$, $\Sigma\delta_i = 1$ on DE, we obtain a tetrahedron $A'B'C'D'$ in perspective with $ABCD$. A fourth row $\delta_0 = \epsilon_0 u_3$, $\delta_1 = \epsilon_1 u_3$, $\delta_2 = \epsilon_2 u_3$, $u_3 = (1-\delta_3)/(1-\epsilon_3)$ is added to our set of equations, leading to three further conditions of complanarity $(\alpha_1\delta_2) = 0$, $(\beta_2\delta_0) = 0$, $(\gamma_0\delta_1) = 0$ and to the relations $\alpha_1\beta_2\gamma_3\delta_0 = \alpha_2\beta_3\gamma_0\delta_1 = \alpha_3\beta_0\gamma_1\delta_2$. They take the place of Ceva's theorem for tetrahedra. From them the corresponding theorems for pairs of corresponding triangular sides are derivable by applying the conditions of complanarity.

The edges of tetrahedron $ABCD$ form three *tetragrams* or quadrilaterals in S_3: $ABCD$, $ACBD$, $ADCB$. Each of these defines the tetrahedron. Ceva's theorem for the tetragram $ABCD$ would read: *Planes laid through a point E and one of the edges AB, BC, CD, DA of a tetragram divide its opposite edges in ratios whose product is unity.*

Pairs of corresponding triangular sides (ABC, $A'B'C'$) of two tetrahedra in perspective, meet in lines containing, each, three intersections of corresponding edges. A complete quadrilateral results, the plane of which may be called, the *base of perspective* of the tetrahedra, or of the tetragrams $ABCD$, $A'B'C'D'$ defining them. Perpendiculars from B', C' to any plane laid through the intersection of ABC with $A'B'C'$ are in the ratio β_0/γ_0, the negative ratio of division of $B'C'$ where it meets BC. Similarly for the remaining edges taken as sides of the triangles ABC, DCB, DAC, DBA, so that each occurs in opposite directions in adjacent triangles.

Menelaus' theorem may then be extended to read: *Any plane meets the sides of a tetragram in points dividing them in ratios the product of which is unity.*

Lastly, we consider two tetragrams $ABCD$, $A'B'C'D'$ in space S_3, but not necessarily in perspective. Lay planes through DA and A', AB and B', etc. These divide the opposite edges BC, CA, etc, in four ratios the product of which may be called the ceva of tetragram $ABCD$. On the other hand, the product of the four ratios in which the planes $a=DCB$, $b=DAC$, $c=DBA$, $d=ABC$ divide the sides $d'a'=B'C'$, $a'b'=C'D'$, $b'c'=D'A'$, $c'd'=A'B'$ would correspondingly be the menelaus of tetragram $A'B'C'D'$. Obviously: *The menelaus of either tetragram equals the other's ceva.* As these products have an even number of factors, there is no difference of sign.

The proof may be indicated by pointing out that BC was found to be divided by plane DAA' in the ratio α_2/α_1, while plane $a=DBC$ divides $b'c'=D'A'$ in the ratio $-\delta_1/\alpha_1$. Both these ratios, each with three similar ones, enter into the expression $(\alpha_2\beta_3\gamma_0\delta_1)/(\alpha_1\beta_2\gamma_3\delta_0)$ for the ceva of $ABCD$ or the menelaus of $A'B'C'D'=a'b'c'd'$.

The theorems here given pave the way for further generalizations, especially in connection with perspective.

REMARKS ON THE PROBABLE ERROR OF A MEAN[1]

By CECIL C. CRAIG, University of Michigan

1. Statement of the problem. The standard deviation of an empirically determined mean based on a sample of N from an infinite parent is well known to be $(\theta_2/N)^{1/2}$ in which θ_2 is the second moment about the mean of the parent distribution. If it be assumed that the distribution of sample means is normal, it is customary to write

$$\theta_1 = m_x \pm .6745(\theta_2/N)^{1/2}$$

[1] Read before the Michigan Section of the Association, April 28, 1927.

in which θ_1 is the mean of the parent distribution and m_x is the mean of the sample of N. The quantity, $.6745(\theta_2/N)^{1/2}$ is said to be the probable error of the m_x. That is, the probability is $\frac{1}{2}$ that m_x is within the interval $.6745(\theta_2/N)^{1/2}$ measured from θ_1. Or if any other constant k be used in place of .6745, other similar statements may be made; if $k=1$, the probability is 68/100 that m_x is within $(\theta_2/N)^{1/2}$ of θ_1; if $k=3$, the probability is 997/1000 that m_x is within $3(\theta_2/N)^{1/2}$ of θ_1, etc.

The situation is fairly satisfactory for moderately large values of N if the parent even remotely resembles a normal distribution. But one point invites further discussion, namely, that in $(\theta_2/N)^{1/2}$ the $\theta_2^{1/2}$ is itself known only by an empirical determination. It is apparent that the probable error itself has a probable error, which in turn has its probable error, and so on ad infinitum. It is true that we do not know and cannot ever know the true probable error of m_x. Then what can we say?

2. Discussion in case the parent is normal. In case the parent distribution is normal the results needed for a discussion of this question are known. They are,

$$\lambda_1(m_x) = \theta_1\,; \qquad \lambda_2(m_x) = \theta_2/N\,;$$

$$\lambda_3(m_x) = \lambda_4(m_x) = \cdots = \lambda_r(m_x) = 0, \qquad r > 2$$

in which $\lambda_r(m_x)$ is the rth semi-invariant of the distribution of sample m_x's. Thus the distribution of sample m_x's is normal. Now also in this case[1]

$$\lambda_1(\sigma_x) = \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\left(\frac{2\theta_2}{N}\right)^{1/2}, \qquad \lambda_2(\sigma_x) = \frac{N-1-2a^2}{N}\theta_2$$

with

$$a = \Gamma\left(\frac{N}{2}\right) \div F\left(\frac{N-1}{2}\right).$$

Now then for the $\theta_2^{1/2}$ in $\theta_1 = m_x \pm k(\theta_2/N)^{1/2}$, what is the best value we can use in terms of our empirically observed quantities? From the equation

$$\lambda_1(\sigma_x) = a(2\theta_2/N)^{1/2}$$

the value of $\theta_2^{1/2}$ is exactly

$$\lambda_1(\sigma_x)\cdot(1/a)(N/2)^{1/2}\,.$$

---

[1] Romanowsky, V., *On the moments of the standard deviation*, etc.; Metron, Vol. 5, No. 4 (1925), p. 12.

But what do we have to put in for $\lambda_1(\sigma_x)$? Only our observed σ_x! But this is also an empirically observed quantity; if it is set in for $\lambda_1(\sigma_x)$ it will have to go in with $\pm$ its error expression which is

$$k[(N-1-2a^2)\theta_2/N]^{1/2}$$

and we shall write

$$\begin{aligned}\theta_1 &= m_x \pm \frac{k}{N^{1/2}}\left[\frac{1}{a}\left(\frac{N}{2}\right)^{1/2}\left\{\sigma_x \pm k\frac{(N-1-2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right\}\right]\\ &= m_x \pm \frac{k}{a\sqrt{2}}\left[\sigma_x \pm k\frac{(N-1-2a^2)^{1/2}}{N}\theta_2^{1/2}\right].\end{aligned}$$

But the quantity $\theta_2^{1/2}$ still occurs in the last terms; if we replace it again by

$$\frac{1}{a}\left(\frac{N}{2}\right)^{1/2}\left[\sigma_x \pm k\frac{(N-1-2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right],$$

we have

$$\begin{aligned}\theta_1 &= m_x\\ &\pm \frac{k}{a\sqrt{2}}\left\{\sigma_x \pm k\frac{(N-1-2a^2)^{1/2}}{N^{1/2}}\left[\frac{1}{a}\left(\frac{N}{2}\right)^{1/2}\left\{\sigma_x \pm k\frac{(N-1-2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right\}\right]\right\}\\ &= m_x \pm \frac{k}{a\sqrt{2}}\left\{\sigma_x \pm \frac{k}{a\sqrt{2}}(N-1-2a^2)^{1/2}\sigma_x\left[1 \pm k\frac{(N-1-2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right]\right\}.\end{aligned}$$

And repeating this indefinitely, we get,

$$\begin{aligned}\theta_1 = m_x \pm \frac{k}{a\sqrt{2}}\sigma_x\Bigg[1 &\pm \frac{k}{a\sqrt{2}}(N-1-2a^2)^{1/2}\\ &\pm \left(\frac{k}{a\sqrt{2}}\right)^2(N-1-2a^2) \pm \cdots\Bigg].\end{aligned}$$

Suppose all the signs were taken positive. Would the resulting series converge? Obviously this depends on

$$\left|\frac{k}{a\sqrt{2}}(N-1-2a^2)^{1/2}\right| \text{ being less than } 1,$$

with

$$a = \Gamma\left(\frac{N}{2}\right) \div \Gamma\left(\frac{N-1}{2}\right).$$

The easiest way to answer this question seems to be to remember that

$$a = \frac{\left(\frac{N}{2}\right)^{1/2} \lambda_1(\sigma_x)}{\theta_2^{1/2}},$$

and that

$$(N - 1 - 2a^2)^{1/2} = \frac{N^{1/2}\sigma_\mu}{\theta_2^{1/2}}, \text{ where } \mu = \sigma_x,$$

so that

$$\frac{k}{\sqrt{2}} \frac{(N - 1 - 2a^2)^{1/2}}{a} = k\frac{\sigma_\mu}{\lambda_1(\sigma_x)}, \text{ where } \mu = \sigma_x.$$

Thus for k less than the ratio of *the mean of sample standard deviations* to the *standard deviation of sample standard deviations*, the series does converge and to the value,

$$\theta_1 = m_x + \frac{k\sigma_x}{a\sqrt{2}} \frac{1}{1 - k\frac{\sigma_\mu}{\lambda_1(\sigma_x)}}, \text{ where } \mu = \sigma_x.$$

This can be approximately evaluated for particular values of k by use of the approximate expressions

$$\lambda_1(\sigma_x) = \theta_2^{1/2}\left(1 - \frac{3}{4N} - \frac{7}{32N^2} - \frac{9}{128N^3} - \cdots\right)$$

$$\sigma_u = \left(\frac{\theta_2}{2N}\right)^{1/2}\left(1 - \frac{1}{4N} - \frac{3}{8N^2} - \cdots\right)^{1/2}, \text{ where } \mu = \sigma_x,$$

which I have also obtained independently of Romanowsky (loc. cit.).

3. Interpretation of the results. But now what is the meaning of such an expression as the one given just above? The answer is, first of all, that it does not mean so much as one is tempted to think at first glance. It is as true as ever that no formula can give us a better guess at θ_1 than m_x, or a better estimate of $\theta_2^{1/2}$ than $\sigma_x/(1-3/(4N)-7/(32N^2)- \cdots)$. Then what do we have? The answer to this seems to be reached thru considerations like this: Our expression $\theta_1 = m_x \pm .6745(\theta_2/N)^{1/2}$ is a good one for θ_1 if the probable error of m_x is small compared with m_x itself. And our estimate of the value of $\theta_2^{1/2}$ to be used in the formula is satisfactory if its probable error is small compared with itself. And if, further, the like can be said of the probable error of this new probable error and so on indefinitely, the whole situation is what is desired. But if this is not true, if in particular the series set up above had not converged, our probable error of m_x and the value of θ_1 become meaningless. Of course not only con-

vergence is desired but rapid convergence. However for even moderate values of N this is effected for the values of k used in practice in the case we have considered.

Of course it was to be expected that the result of the investigation in this one of the most favorable of cases would be satisfactory. But it is important to emphasize that by so turning out it has added nothing to the expression for θ_1 or to the expression for the probable error of m_x beyond the usual meaning given to them. If this investigation or any like it should turn out unfavorably it would be too bad. And the difficulties to be encountered in most other cases do not lessen the necessity of knowing that such an investigation in each case would turn out favorably.

AN ELEMENTARY EXAMPLE OF A CONTINUOUS NON-DIFFERENTIABLE FUNCTION

By FRED W. PERKINS, Harvard University

The first example showing that the continuity of a function of the real variable x does not imply the existence of the derivative of this function at a single point is due to Weierstrass.[1] Other authors have also written upon this subject, but, in view of the importance of the phenomenon, the following discussion, which the writer has endeavored to make as elementary as possible, may not be without interest.

The method by which we propose to construct the function $f(x)$ constituting our example depends upon an interpolation process by means of which we extend the definition of $f(x)$ after we have defined this function for certain pairs of values of x. Let $x=x'$ and $x=x''$ be such a pair of values $(x' \neq x'')$ and denote by y' and y'' the numbers $f(x')$ and $f(x'')$ respectively. We set

$$f(x' + \tfrac{1}{3}[x'' - x']) = y' + \tfrac{5}{6}[y'' - y']$$

and

$$f(x' + \tfrac{2}{3}[x'' - x']) = y' + \tfrac{1}{6}[y'' - y'].$$

---

[1] Weierstrass gave an example in lectures at Berlin as early as 1861, but it was not published until 1875. Weierstrass' discussion may be found in his *Werke* (Berlin 1895) Vol. 2, page 71ff. Wiener (Journal für Mathematik vol. 90, 1881, p. 221ff.) made a study of the function constituting this example; in connection with this paper, see a note by Weierstrass, (loc. cit. vol. 2, page 228ff.).

Riemann gave a function which enables us to infer that a continuous function may fail to have a derivative on an everywhere dense set of points. (*Gesammelte Mathematische Werke*, p. 225ff.). See also Schwartz, *Gesammelte Mathematische Abhandlungen* (Berlin, 1890), vol. 2, p. 269ff. Among the more recent contributions on this subject may be mentioned two examples of nowhere differentiable functions published in the Jahresbericht der Deutschen Mathematiker Vereinigung, by Faber, vol. 16 (1907), p. 538ff. and by Landsberg, vol. 17 (1908), and a memoir on the Weierstrass' function by Hardy in the Transac-

It will be convenient to represent by P' and P'' the points (x', y') and (x'', y'') respectively, and also to denote by P_1 the point (x_1, y_1) and by P_2 the point (x_2, y_2). Excluding the case that $y' = y''$, we see that y_1 and y_2 lie between y' and y'', and that each of the quantities $|y_1 - y'|$, $|y_2 - y_1|$, $|y'' - y_2|$ is positive, but not greater than $5|y'' - y'|/6$. Furthermore, the lines $P'P_1$, P_1P_2 and P_2P'' have slopes which in each case are at least twice as great, numerically, as the slope of $P'P''$.

We are now ready to define the function $f(x)$ on the interval $0 \leqq x \leqq 1$. We set $f(0) = 0$ and $f(1) = 1$, and use the process described above to determine $f(1/3)$ and $f(2/3)$. We shall call the two points thus obtained the "interpolated points of the first order." By the same method we interpolate two more points between each pair of adjacent points now known on the graph of $f(x)$. The six points thus determined we shall call the "interpolated points of the second order," and so on, indefinitely. In this way, the value of $f(x)$ is specified for all values of x of the form $p/3^n$, where n is a positive integer, and p is a non-negative integer, not greater than 3^n.

Let X be any number of the interval $0 \leqq x \leqq 1$ not of the form $p/3^n$. Since $f(x)$ is defined for values of x differing from X by less than any preassigned positive quantity, and since the function $f(x)$, insofar as it has already been defined, has the lower bound 0 and the upper bound 1, the set of all known points on the graph of $f(x)$ has at least one limit point on the line-segment determined by the relations $x = X$, $0 \leqq y \leqq 1$. We shall now show that there is in fact just one such limit point. From properties of the interpolation process already stated, it follows that for any positive integer n, and any non-negative integer p such that $p < 3^n$ we have

$$\left| f\left(\frac{p+1}{3^n}\right) - f\left(\frac{p}{3^n}\right) \right| \leqq \left(\frac{5}{6}\right)^n.$$

Any interpolated point with an abscissa lying between $p/3^n$ and $(p+1)/3^n$ has an ordinate lying between $f[p/3^n]$ and $f[(p+1)/3^n]$. Given any $\epsilon > 0$, we can choose n so large that $(5/6)^n < \epsilon$, and then choose p so that X is an interior point of the interval bounded by $p/3^n$ and $(p+1)/3^n$. Then, in so far as $f(x)$ is defined on this interval, its least upper and greatest lower bounds here differ by less than ϵ, which shows that the set of all known points on the graph of $f(x)$ cannot have more than one limit point with the abscissa X. Let $f(X)$ be defined as the ordinates of this unique limit point. We extend in this manner the definition of $f(x)$ throughout the entire unit interval. The resulting function is clearly continuous at each of the non-interpolated points.

---

tions of the American Mathematical Society, vol. 17 (1916), p. 301. See also Hobson, The Theory of Functions of a Real Variable, (Cambridge, University Press, 1907), p. 620ff. In the paper by Hardy will be found further references to the literature on this subject.

Further, if $x^* = p/3^n$ is the abscissa of an interpolated point each of the two adjacent interpolated points of the $(n+k)$-th order has an ordinate differing from $f(x^*)$ by not more than $(5/6)^{n+k}$. By choosing the positive integer k sufficiently large, this difference can be made less, in absolute value, than any preassigned positive ϵ. It follows that for such a choice of k, the relation $|f(x^*) - f(x)| \leqq \epsilon$ holds for all interpolated points of the interval

$$\frac{3^k p - 1}{3^{n+k}} \leqq x \leqq \frac{3^k p + 1}{3^{n+k}},$$

and so at their limit points as well. Hence, $f(x)$ is continuous at the interpolated points also. The same sort of reasoning shows that $f(x)$ is continuous at $x=0$ and $x=1$.

We are now ready to show that at no point of the interval $0 \leqq x \leqq 1$ does the continuous function $f(x)$ possess a derivative. We understand this to mean that for no value of x in the interval in question does the difference quotient $[f(x+h) - f(x)]/h$ approach a limit as h approaches zero.[1]

Let x_0 be any number of the interval $0 \leqq x \leqq 1$. We choose any positive integer n, and then determine the positive integer p so that $(p-1)/3^n \leqq x_0 \leqq p/3^n$. One of the points

$$R: \left[\frac{p-1}{3^n}, f\left(\frac{p-1}{3^n}\right)\right], \qquad S: \left[\frac{p}{3^n}, f\left(\frac{p}{3^n}\right)\right]$$

determines with the point P_0: $[x_0, f(x_0)]$ a line with slope at least as great numerically as that of the line RS. But, from a property of the interpolation process, we know that the absolute value of the slope of RS is at least 2^n. Hence, we see that by choosing n sufficiently large we can find a point on the curve as near as we like to P determining with P a line with slope greater numerically than any preassigned quantity. This proves that $f(x)$ has no derivative for $x = x_0$. It does not imply, of course, that the difference quotient $[f(x_0+h) - f(x_0)]/h$ becomes infinite independently of the manner in which h approaches zero. It may be noted, however, that for $x_0 = 0$ or $x_0 = 1$, the difference quotient does become positively infinite independently of the manner in which the point $x_0 + h$ of the unit interval approaches x_0.

---

[1] Some writers, on the contrary, use a definition of differentiability allowing the difference quotient either to approach a limit or to become positively or negatively infinite.

PI MU EPSILON MATHEMATICAL FRATERNITY

By TOMLINSON FORT, Lehigh University

As a college teacher who has found the Pi Mu Epsilon fraternity of great benefit to his work and in the belief that the organization has potentialities for a widespread benefit to American mathematics, I presume to tell the readers of the MONTHLY something about it.

The fraternity was founded at Syracuse in 1914 by Professor E. D. Roe, Jr. and incorporated under the laws of the State of New York on May 25 of that year. Its headquarters have remained at Syracuse and Professor Roe has remained its Director General or chief officer. His deep interest in the fraternity and his wise handling of its affairs constitute no small contribution to the cause of mathematics and scholarship.

Pi Mu Epsilon is not a secret fraternity nor a fraternity based on anything but interest and achievement in mathematics. It is not an undergraduate mathematics club but numbers among its members undergraduates, graduates, and faculty members on a plane of equality within the fraternity. It aims to combine with the ideals of the more widely known Phi Beta Kappa a working organization within the colleges and universities for study and for the promotion of all mathematical interests within the community. It undertakes to rival no organization. It has concentrated on work at the institution and its great good is unquestionably to inspire enthusiasm in the student and to give him stimulating associations and an opportunity for independent study. There could be no greater mistake than to feel that Pi Mu Epsilon is or ever should be, in thought or deed or appearance, a rival of the American Mathematical Society or of the Mathematical Association of America. It works at the universities bringing the faculty, graduates, and leading undergraduates together for the common cause.

Why Pi Mu Epsilon rather than local clubs only? It is a matter of results. We must meet American college life as it is. Those of us who have tried both, find that Pi Mu Epsilon succeeds the better. We are able to set up strict scholarship qualifications and the student works for the honor of membership and of wearing the key. A campus reputation and recognition is quickly obtained for Pi Mu Epsilon and it becomes second to but few college honors. The student understands the national organization. The very fact that it is national makes him rate it higher. The fact that it is called "fraternity" tends to break down the barrier between him and his teachers quicker than anything else that I know. My own association with my students in Pi Mu Epsilon has been one of the keenest pleasures of my experience as a teacher.

What do the chapters of Pi Mu Epsilon do? There is considerable variation. Each chapter has its own by-laws but is subject to the national constitution.

Usually no one is admitted who has not completed at least two semesters of calculus or who has a grade less than "B" in any mathematics course. He must in addition be reported on by a scholarship committee as to scholarship in all his work and enthusiasm and probable success in mathematics. Some chapters conduct their work much as is usual in mathematics clubs with added social features. I, of course, can speak particularly only of my own experience. At Hunter College Pi Mu Epsilon, in addition to being an award for merit, has been primarily a study organization conducted at times somewhat as a seminar and at times in a way that might better be described as a socialized recitation. We have universally given all of one semester to a single topic. These topics have been: Non-Euclidean geometry, relativity, statistics, history of calculus, and vector analysis. Everybody prepares. We have had delightful social gatherings in addition to the study hours. At Hunter College there is a general mathematics club open to all, having many members who would not be eligible for Pi Mu Epsilon. The Pi Mu Epsilon members are all members of the "Mathematics Club." We regard its support as a duty incumbent on every member of the fraternity. We have also a mathematics club with membership limited to freshmen and programs of corresponding advancement. Pi Mu Epsilon members undertake to organize and assist the freshman club. Our chapter also awards each year a prize of ten dollars to the student graduating with the highest grades in mathematics from Hunter High School which is under different management from the College and is in a different part of the city. A condition of the awarding of the prize is that an undergraduate member of Pi Mu Epsilon be permitted to award it in the name of the fraternity from the High School commencement stage. We hope that this prize has done something for mathematics in the high school. Pi Mu Epsilon undergraduate members recently were instrumental in organizing a mathematics club at Washington Irving High School, New York City. At the University of Alabama, Pi Mu Epsilon members, in addition to making arrangements for their own meetings, were instrumental each year in organizing the Newtonian (freshman mathematics) Club, sponsored some outside lectures, and tried in various other ways to stimulate interest in mathematics on the campus.

What is necessary for a successful chapter of Pi Mu Epsilon? Every organization derives its spirit and owes its success to a few enthusiastic persons. Certainly one and preferably several members of the faculty who in age, position, and personality command the respect of every one and who will enter into Pi Mu Epsilon whole-heartedly and genuinely are necessary for its success. A department where research is the whole thing, teaching a necessary drudgery and undergraduate students regarded without true respect will never do. This is not to imply, however, that the fraternity does not recognize the importance of research and hope to lead the best of its young members to become research

mathematicians. To turn Pi Mu Epsilon over to a young instructor as director and to have it neglected by the older members of the staff will cause it to be a languishing mathematics club. One might just as well have it such in name. Pi Mu Epsilon will go but, of course, the enthusiastic teacher must lead. He will be repaid, I am sure, in those things which a true teacher prizes. A weak department without scholarship will not be granted a charter. The chapters of the fraternity decide this matter, each chapter casting one vote, but the spirit of the fraternity now is to vote favorably only on chapters where there is assurance of a scholarly faculty and of enthusiasm for the fraternity. No charter has ever been voted to an organization of students only.

The chapters of Pi Mu Epsilon are: Syracuse University, Ohio State University, University of Pennsylvania, University of Missouri, University of Alabama, Iowa State College, University of Illinois, Bucknell University, University of Montana, Hunter College of the City of New York, Washington University, University of California at Los Angeles, University of Kentucky, Ohio Wesleyan University.

The general officers are: E. D. ROE, Jr., Syracuse University, Director General; TOMLINSON FORT, Hunter College, Vice-Director General; H. S. EVERETT, Bucknell University, Secretary General; I. S. CARROLL, Syracuse University, Deputy Secretary General; LOUISA M. LOTZ, University of Pennsylvania, Treasurer General; MABEL KESSLER, University of Pennsylvania Librarian General; R. D. CARMICHAEL, University of Illinois; E. R. HEDRICK, University of California at Los Angeles; S. E. RASOR, Ohio State University; W. H. ROEVER, Washington University, Members of Executive Committee General.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. LINE VALUES OF POWERS OF TRIGONOMETRIC FUNCTIONS AND THEIR USE IN CONSTRUCTING CURVES[1]

By Dr. H. A. BENDER, University of Illinois

The line values of the different powers of the trigonometric functions readily lend themselves to the graphical construction of certain curves, and from the

[1] Read before the Illinois Section of the Association, May 9, 1925, under a different title.

simple geometric construction used to locate one point of the curve one can, in most cases, determine the exact graph by perceiving the locus of the point constructed as the angle be allowed to vary. For example, if at a point a units from the pole on the initial line a perpendicular be drawn to the radius vector, then the foot of the perpendicular is a point of the curve $\rho = a \cos \theta$. The graph, which is the locus of the point constructed as the angle be allowed to vary, is at once apparent from the construction.

The most familiar illustration of this type of graphing of Cartesian equations is afforded by the eccentric angle construction of a point of an ellipse from its parametric equations. Similar construction of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ from its parametric equations $x = a \sec \theta$, $y = b \tan \theta$ is not so familiar but is given in some of the text-books.

In the more complex curves, if one can not determine the exact shape of the curve from a single construction, a second or third point may be located with ease. In case accuracy is desired, one can locate several points with considerable ease and accuracy and hence avoid the computing and plotting a table of points.

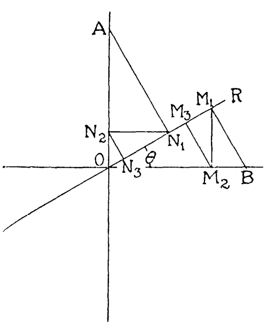


Fig. 1

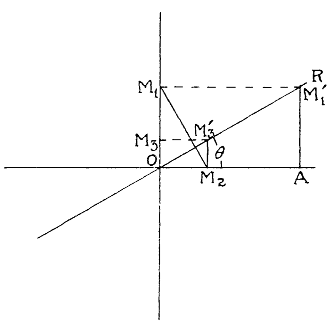


Fig. 2

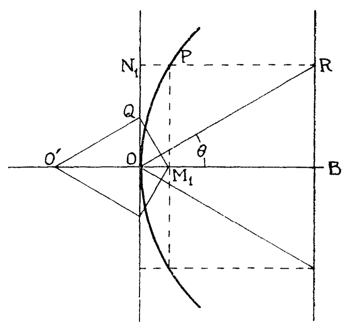


Fig. 3

The line values of $a \sin^n \theta$, $a \cos^n \theta$, and $a \tan^n \theta$ can readily be constructed. In Fig. 1 let OA be equal to a, and from A draw a perpendicular to OR; then $ON_1 = a \sin \theta$, $ON_2 = a \sin^2 \theta$, $ON_3 = a \sin^3 \theta$, etc. Similarly if $OB = a$, then $OM_1 = a \cos \theta$, $OM_2 = a \cos^2 \theta$, $OM_3 = a \cos^3 \theta$, etc. In Fig. 2 at A $(a,0)$ erect a perpendicular and let M_1 be the projection of M_1' on the y-axis, and from M_1 draw a perpendicular to OR; then we have $OA = a$, $OM_1 = a \tan \theta$, $OM_2 = a \tan^2 \theta$, $OM_3 = a \tan^3 \theta$, etc. The reciprocals of these functions may be constructed in the reverse order, or we may in Fig. 1 suppose $ON_3 = a$, $ON_2 = a \csc \theta$, $ON_1 = a \csc^2 \theta$, $OA = a \csc^3 \theta$, etc. Similarly for $a \sec^n \theta$ and $a \cot^n \theta$. The parabola $y^2 = 4ax$ has the parametric equations $x = a(\sec^2 \theta - 1)$, $y = 2a \tan \theta$. In Fig. 3 draw the line $x = 2a$. At O' $(-a,0)$construct an angle equal to θ, then $O'Q = a \sec \theta$ and $O'M_1 = a \sec^2 \theta$, hence $x = OM_1 = a(\sec^2 \theta - 1)$ and $y = ON_1 = 2a \tan \theta$.

The astroid $x^{2/3}+y^{2/3}=a^{2/3}$ has the parametric equations $x=a\cos^3\theta$, $y=a\sin^3\theta$; and from the construction in Fig. 4, $x=OM_3=a\cos^3\theta$, and $y=ON_3=a\sin^3\theta$. The curve $(x/a)^2+(y/b)^{2/3}=1$ has the parametric equations $x=a\cos\theta$,

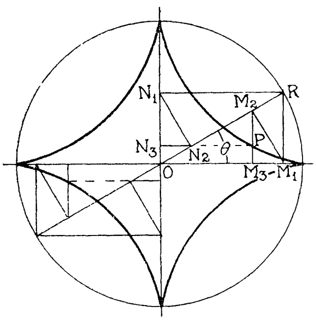


FIG. 4

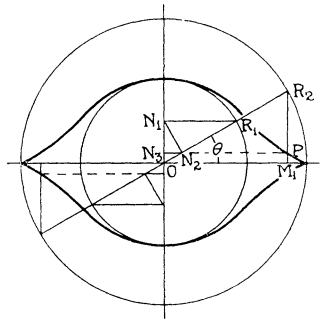


FIG. 5

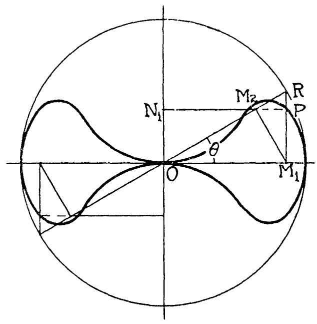


FIG. 6

$y=b\sin^3\theta$; and in Fig. 5, $x=OM_1=a\cos\theta$, $y=ON_3=b\sin^3\theta$. The curve $a^4y^2=a^2x^4-x^6$ has the parametric equations $x=a\cos\theta$, $y=a\cos^2\theta\sin\theta$. In Fig. 6, $x=OM_1=a\cos\theta$, and $OM_2=a\cdot\cos^2\theta$, $y=ON_1=a\cos^2\theta\sin\theta$.

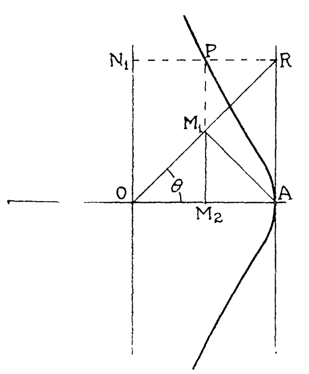


FIG. 7

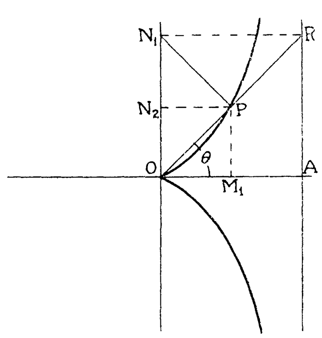


FIG. 8

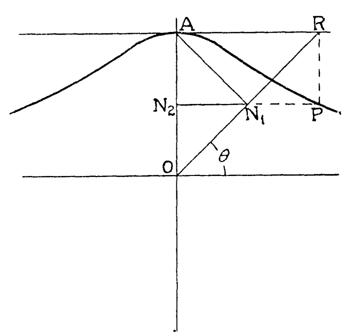


FIG. 9

We might extend the application to the curve $xy^2=4a^2(2a-x)$ as given in Fig. 7. It has the parametric equations $x=2a\cos^2\theta$, $y=2a\tan\theta$. If $OA=2a$, then $x=OM_2=2a\cos^2\theta$ and $y=ON_1=2a\tan\theta$. The cissoid $y^2(2a-x)=x^3$ in Fig. 8 has the parametric equations $x=2a\sin^2\theta$, $y=2a\tan\theta\sin^2\theta$. If $OA=2a$, then $ON_1=2a\tan\theta$, $OP=2a\tan\theta\sin\theta$, $x=OM_1=2a\tan\theta\sin\theta\cos\theta=2a\sin^2\theta$, and $y=ON_2=2a\tan\theta\sin^2\theta$. Since the witch $x^2y=4a^2(2a-y)$ has the parametric equations $x=2a\cot\theta$, $y=2a\sin^2\theta$, its construction in Fig. 9 is at once apparent.

Similar construction is applicable to curves in polar coordinates. The cardioid $\rho=a(1-\cos\theta)$ may be constructed by drawing, in Fig. 10, the circle

O' with radius a. At O' construct an angle equal to θ. Then $O'M = a \cos \theta$, and $OM = OP = \rho = a(1 - \cos \theta)$. To construct the curve $\rho = a \sin 2\theta$, let $OA = 2a$ in Fig. 11; then $OM_1 = 2a \sin \theta$, and $OM_2 = OP = \rho = 2a \sin \theta \cos \theta$ (note that ρ is negative for those values of θ for which M_2 is to the left of the origin). The curve $\rho = a \tan (\theta/2)$ has a very simple geometric construction. In Fig. 12, if OR' bisects the angle θ, then if from $A(0,a)$ a perpendicular be drawn to OR', the point P at the intersection of this perpendicular with OR is a point on the curve, for $OM = a \sin (\theta/2)$ and $OP = \rho = a \sin (\theta/2) \sec (\theta/2) = a \tan (\theta/2)$.

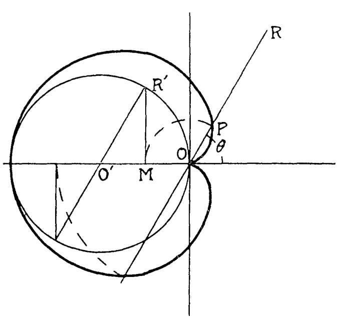


FIG. 10

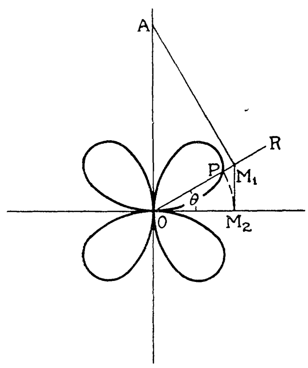


FIG. 11

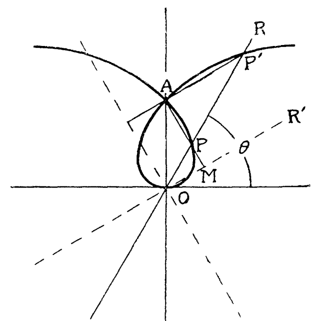


FIG. 12

It follows from the illustrations above that this method is applicable to practically all the standard curves, as well as many of the problems, which are given in our average analytic geometry and calculus text-books. However, from the periodic nature of the trigonometric functions it is evident that the method is best applicable to those curves which are symmetrical with respect to one or both of the coordinate axes.

II. AN EXAMPLE IN PROBABILITY

By R. E. MORRIS, University of Montana

It has been shown that the probability of winning in the case of two people A and B, A having 3 coins, B having 2 coins, A winning when he throws more heads than B and B winning when he throws the same number of heads as A or more than A, is $1/2$. This is true for 4 coins and 3 coins, or 5 and 4 coins, or in fact for n and $n-1$ coins. The purpose of this is to show that the probability is $1/2$ and in the course of this procedure to bring to light some interesting relations among binomial coefficients. The expression for the probability of A's winning is

$$\frac{1}{2^n}+\frac{{}_nC_1}{2^n}\,\frac{2^{n-1}-1}{2^{n-1}}+\frac{{}_nC_2}{2^n}\,\frac{[(2^{n-1}-1)-{}_{n-1}C_1]}{2^{n-1}}$$
$$+\frac{{}_nC_3}{2^n}\,\frac{[(2^n-1)-({}_{n-1}C_1+{}_{n-1}C_2)]}{2^{n-1}}+\cdots$$
$$+\frac{{}_nC_{n-1}}{2^n}\,\frac{[(2^{n-1}-1)-({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})]}{2^{n-1}}.$$

Factoring $2^{n-1}-1$, we obtain

$$\frac{1}{2^n}+\frac{1}{2^n2^{n-1}}\{(2^{n-1}-1)({}_nC_1+{}_nC_2+\cdots+{}_nC_{n-1})-[{}_nC_2\,{}_{n-1}C_1$$
$$+{}_nC_3({}_{n-1}C_1+{}_{n-1}C_2)+\cdots+{}_nC_{n-1}({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})]\}.$$

Examination shows us that the theorem will be proved if we show that this expression is equal to

$$\frac{1}{2^n}+\frac{1}{2^n2^{n-1}}[(2^{n-1}-1)(2^n-2)-(2^{n-1}-1)(2^n-2-2^{n-1}])$$

plus an expression which clearly reduces to 1/2.
The problem resolves itself into showing that:

$$(1)\qquad (2^{n-1}-1)(2^n-2-2^{n-1})\equiv {}_nC_2\,{}_{n-1}C_1+{}_nC_3({}_{n-1}C_1+{}_{n-1}C_2)+\cdots$$
$$+{}_nC_{n-1}({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})].$$

Clearly $2^{n-1}-1\equiv{}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1}$ and $2^n-2\equiv{}_nC_1+{}_nC_2+\cdots+{}_nC_{n-1}$. Hence

$$(2)\qquad (2^{n-1}-1)(2^n-2-2^{n-1})\equiv({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})[{}_nC_1+{}_nC_2+$$
$$\cdots+{}_nC_{n-1}-(1+{}_{n-1}C_1+\cdots+{}_{n-1}C_{n-1})].$$

If we perform the multiplication indicated in (2) and omit the members contained in (1) we obtain

$${}_{n-1}C_1({}_nC_1-1-{}_{n-1}C_1\,{}_{n-1}C_2-\cdots-{}_{n-1}C_{n-1})+{}_{n-1}C_2({}_nC_1+{}_nC_2-1$$
$$-{}_{n-1}C_1-{}_{n-1}C_2-\cdots-{}_{n-1}C_{n-1})+\cdots+{}_{n-1}C_{n-1}({}_nC_1+{}_nC_2+\cdots$$
$$+{}_nC_{n-1}-1-{}_{n-1}C_1-{}_{n-1}C_2-\cdots-{}_{n-1}C_{n-1}).$$

If we show this last expression to be identical with zero, (1) will follow immediately.
To show that it is equal to zero, let us write it in the form

$${}_nC_1({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})+{}_nC_2({}_{n-1}C_2+{}_{n-1}C_3+\cdots$$
$$(3)\qquad +{}_{n-1}C_{n-1})+\cdots+{}_nC_{n-1}\,{}_{n-1}C_{n-1}-(1+{}_{n-1}C_1+{}_{n-1}C_2+\cdots$$
$$+{}_{n-1}C_{n-1})({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1}).$$

Rewriting the negative expression, changing order of multipliers, (3) becomes

$$(4)\quad \begin{aligned}(1 + {}_{n-1}C_1)({}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1}) + ({}_{n-1}C_1 + {}_{n-1}C_2)({}_{n-1}C_2 \\ + {}_{n-1}C_3 + \cdots + {}_{n-1}C_{n-1}) + \cdots + ({}_{n-1}C_{n-2} + {}_{n-1}C_{n-1})({}_{n-1}C_{n-1}).\end{aligned}$$

If we make the substitution ${}_nC_r \equiv {}_{n-1}C_r + {}_{n-1}C_{r-1}$ for the left-hand factors in (4) we obtain an expression from which (3) is shown immediately to be zero.

Hence (1) is true and we obtain 1/2 as the probability that either A or B will win. There are some interesting possibilities to this problem with n and $n-2$ coins, n and $n-3$ coins, etc.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

NEW BOOKS RECEIVED

BENNEY, L. B. *Mathematics for Students of Technology.* Oxford, University Press, 1927. vii+451 pages.

BIGGS, H. F. *Wave Mechanics.* Oxford, University Press, 1927. 77 pages. $1.50.

CAREY, F. S. and GRACE, S. F. *Four-Place Tables with Forced Decimals.* London, Longmans, Green, and Company, 1927. 39 pages. $0.40.

DOWSETT, J. F. *Advanced Constructive Geometry.* Oxford, University Press, 1927. viii+340 pages. $9.00.

GIBBS, R. W. M. *Algebra to the Quadratic.* Oxford, University Press, 1927. 160 pages. $1.00.

GODFREY, C. and SIDDONS, A. W. *Four-Figure Tables.* Cambridge, University Press, 1927. 40 pages.

HOBSON, E. W. *Theory of Functions of a Real Variable and the Theory of Fourier's Series.* Volume I, third edition. Cambridge, University Press, 1927. xv+736 pages.

LITTLE, A. S. *A Table of Interpolation Multipliers.* Boston, Financial Publishing Company, 1927. 29 pages.

LOVE, C. E. *Analytic Geometry.* Revised edition. New York, The Macmillan Company, 1927. xiv+257 pages.

MASON, T. E. and HAZARD, C. T. *Analytic Geometry.* Boston, Ginn and Company, 1927. xi+224 pages. $2.40.

OTIS, A. S. and WOOD, B. D. *Columbia Research Bureau Algebra Test.* Yonkers-on-Hudson, World Book Company, 1927.

PERRON, O. *Algebra.* I. *Die Grundlagen*; II. *Theorie der algebraischen Gleichungen.* Berlin, Walter de Gruyter and Company, 1927. viii+307 pages and viii+243 pages.

SMITH, D. E. and REEVE, W. D. *The Teaching of Junior High School Mathematics.* Boston, Ginn and Company, 1927. viii+411 pages. $2.00.

WEIDA, F. M. *The Logarithmic Slide-Rule.* New York, The Macmillan Company, 1927. 7 pages.

REVIEWS

Cremona Transformations in Plane and Space. By HILDA P. HUDSON. Cambridge, England, University Press, 1927. xx+454 pages. Price £2 2s.

The Cambridge University Press is famous for its comprehensive list of standard works on fundamental subjects of mathematics. Geometers will welcome this one to a worthy place in that distinguished company. The general plan is an elevated one; in the opening chapter the reader is introduced to homaloidal nets, postulation and equivalence of plane systems, the direct and inverse fundamental systems, followed in the second by Clebsch's theorem and its arithmetic consequences. Quadratic transformations are discussed under three headings; planes distinct, planes superposed, involutions, in 24 pages, yet nearly every known result is included. Then follows the discussion of series of composition, involution, and the application to the resolution of singularities of plane curves. This last chapter could have been made more useful and attractive by the use of figures and of more illustrative examples. The theory of plane transformations is well rounded out in the first third of the book.

In space, one immediately feels a different atmosphere—here so much remains still undone, and the road is by no means clear how to proceed further. Postulation and equivalence are treated at length, but the various genera of a surface are hardly mentioned. In the discussion of particular elements the concept of contact has a prominent position; it is pointed out that not only do the ordinary formulas fail in this case, but that higher singularities involving contact cannot be expressed in terms of simpler ones linearly. Then follows a detailed discussion of a number of special transformations of cubics, most of which is the work of the author. The minute study of a (3, 4) transformation defined by cubics with a basic conic and osculating a line is a gem. A chapter is devoted to the resolution of singularities of surfaces, following the methods of Levi and Chisini. It represents a great amount of careful work, yet the presentation is sketchy and frequently not convincing. There is still plenty of room for the efforts of the most skillful investigators in this field.

At the end of the book the characteristics of the plane Cremona transformations of orders 2 to 16 are given, and the 75 forms of cubic transformations in space. The book contains no foot-notes. At the end appear the titles, names of authors, and complete reference to every article made use of in the book. In

the text, usually in a paragraph heading, appears a number in type used for no other purpose, referring to a specific entry in the list at the end, while after each paper appear the numbers of the pages in which that article is cited. The list is not complete, but is nearly enough so to make the work a veritable hand-book. Indeed the author states explicitly that the list is incomplete, and generously leaves a few blank pages for further entries. Typographically the work has attained a degree of excellence hardly found elsewhere; errors in the numerous formulas do not exist. It is particularly appropriate that an author who has enriched the field by so many important contributions has now put them in a proper setting by presenting this well-proportioned and carefully elaborated treatise on the whole subject.

VIRGIL SNYDER

Mathematical Statistics. By H. L. RIETZ. The Carus Mathematical Monographs. Number Three. Chicago, The Open Court Publishing Company, 1927. xi+181 pages.

This little book serves as an admirable introduction to Mathematical Statistics. Although the mathematics involved does not go beyond the calculus, and in most chapters not often beyond college algebra, nevertheless, very fundamental concepts in the theory of statistics and probability are set forth in unusually clear form. This is no easy task to perform and the ability to carry it through successfully comes only from long experience in dealing with practical problems in statistics and familiarity with the literature on the subject. It is difficult enough to get these concepts straightened out in one's own mind not to mention getting them across to others, particularly if they are not familiar with the elements of statistics. This monograph is a striking illustration of the fairly successful carrying out of the idea and purpose of the Carus Foundation. There are seven short chapters dealing with the following topics: the underlying concepts of mathematical statistics, relative frequencies in simple sampling, frequency functions of one variable, correlation, random sampling fluctuations, the Lexis theory, and a development of the Gram-Charlier series. The author has not attempted a detailed exposition, but he has emphasized and brought into relief the high spots under these captions.

In the first chapter stress is laid on two general types of problems which occur in mathematical statistics. In connection with problems falling under these two types the author gives detailed consideration to certain underlying concepts, taken in pairs, as follows: relative frequency and probability, observed and theoretical frequency distributions, arithmetic mean and mathematical expectation, mode and most probable value, moments and mathematical expectations of a power of a variable. The second chapter brings out the important points in

connection with the binomial description of frequency together with the mathematical expectation and standard deviation of the number of successes. The theorem of Bernoulli is explained and the proof by the use of the Bienaymé-Tchebycheff criterion pointed out. Some attention is also given to the DeMoivre-Laplace theorem and the Poisson exponential function.

A great deal of matter is compressed in the chapter of thirty pages on frequency functions of one variable. The Pearson frequency curves and the Gram-Charlier series of Type A and Type B are considered together with some remarks on determining the coefficients and on skewness and excess. Simple, multiple, and partial correlation are treated in the next chapter both with regard to the regression method and the correlation surface method of description. The treatment of multiple and partial correlation undoubtedly suffers on account of the condensation. A very good explanation is given of the correlation ratio in non-linear regression. The important subject of random sampling fluctuations is unusually well set forth in view of the limited space at the author's disposal. The reviewer would regard this as about the best chapter in the book.

Urn schemata are employed in the next chapter to illustrate the Bernoulli, Poisson, and Lexis types of statistical series arising in practice. The usual formulas for the mathematical expectation and standard deviation are obtained for these types with some illustrations of subnormal, normal and supernormal dispersion as obtained by the Lexis ratio. This last chapter contains a development of Type A and Type B Gram-Charlier series. The values of the coefficients in the Type A series are obtained by the use of the biorthogonal property of the normal function and its derivatives and the Hermite polynomials. The rigid and particular form of the analysis, due to Wicksell, is very satisfactory because in most texts dealing with this subject the proofs are sloppy. It is to be regretted that the author could not devote a chapter to semi-invariants and explain their elegant and useful properties in connection with the series development of frequency distributions. The value of these developments is their obvious extension to distributions of two or more variates.

The author concludes with several pages of notes giving references to important sources of material; this will prove very useful to readers who wish to go beyond the book itself. Although this monograph was not written as a textbook for class room use, nevertheless, it appears to the reviewer to serve the purpose very well in a first course if supplemented by lectures, examples and laboratory work. The clarity of the text on fundamentals makes it most desirable for the use of beginning students whose mathematical equipment is equivalent to the first year, or better, the first and second years of college mathematics.

Few errors or misprints were observed. On page six and seven the letter d represents the number of deaths in one year among an initial group of l persons exposed—those who die cannot be exposed for a year. On page eighteen in formula (7) the exponent κ should be replaced by k. The general appearance of the monograph, both as to printing and binding, is excellent.

JAMES W. GLOVER

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Annals of Mathematics, volume 28, no. 3. July 1927: "Sextic surfaces with a double septimic curve" by B. C. Wong, 251–262: "Determination of a basis foi the integral elements of certain generalized quaternion algebras" by M. D. Darkow, 263–270; "Expansions in Bessel functions" by M. H. Stone, 271–290; "On the Green's function for systems of differential equations" by W. M. Whyburn, 291–300; "On Lamé families of surfaces" by C. E. Weatherburn, 301–308; "Derivation of the Fredholm theory from a differential equation of infinite order" by H. T. Davis, 309–317; "Asymptotic expression for the probability of trials connected in a chain" by E. D. Pepper, 318–326; "On certain indefinite quaternary forms representing all integers" by C. G. Latimer, 327–329; "Applications of algebraic number theory to congruences involving binomial coefficients" by H. S. Vandiver, 330–332; "Ternary quadratic forms and congruences" by L. E. Dickson, 333–341; "Correspondences between algebraic curves" by S. Lefschetz, 342–354; "On the existence and properties of the solution of a certain differential equation of the second order" by T. H. Gronwell, 355–364; "A new definition of almost periodic functions" by N. Wiener, 365–367; "A theory of integers, in relation to the iteration of algebraic functions," by O. E. Glenn, 368–378; "On systems of total differential equations" by J. M. Thomas, 379–385; "On positive solutions of a system of linear equations" by L. L. Dines, 386–392; "On completely signed sets of functions" by L. L. Dines, 393–395; "Concerning continuous curves and correspondences" by W. L. Ayres, 396–418.

Messenger of Mathematics, volume LVI, no. 12, April 1927: "A modification of Glaisher's proof of Stirling's Theorem" by R. E. Moritz, 181–184.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 8, August 1927: "Contact transformations of three-space which convert a system of paths into a system of paths" by Jesse Douglas, 605–606; "Motion and collineations in general space" by M. S. Knebelman, 607–610; "Felix Klein and the history of modern mathematics" by G. A. Miller, 611–613; "The residual set of a complex on a manifold and related questions" by S. Lefschetz, 614–622. No. 9, September 1927: "Remark on a theorem of R. L. Moore" by B. Knaster and C. Kuratowski, 647–649; "Concerning the open subsets of a plane continuous curve" by Gordon T. Whyburn, 650–656; "On the functional independence of ratios of theta functions" by S. Lefschetz, 657–658.

Transactions of the American Mathematical Society, volume 29, no. 3, July 1927: "The contact of a cubic surface with an analytic surface" by E. P. Lane, 471–480; "Dynamical space-times which contain a conformal euclidean 3-space" by H. P. Robertson, 481–496; "Reduction of the ordinary linear differential equation of the nth order whose coefficients are certain polymonials in a parameter to a system of n first-order equations which are linear in the parameter" by C. E. Wilder, 497–506; "On the 'third axiom of metric space'" by C. W. Niemptzki, 507–513; "Implicit functions and differential equations in general analysis" by L. M. Graves, 514–552; "Concerning acyclic continuous curves" by H. M. Gehman, 553–568; "On a general formula in the theory of Tchebycheff polynomials and its applications" by J. A. Shohat, 569–583; "A factorization theory for functions $\sum^{n}_{=1} a_i e^{\alpha_i x}$" by J. F. Ritt, 584–596;

3294. Proposed by A. A. Bennett, Brown University.

Suppose a gambling house establishes an artificial "horse race" with six counters numbered from 1 to 6 representing six competing horses. The banker throws a single die repeatedly, and at each throw one horse is advanced one step from the starting line toward the finishing line. If the die turns up a 1, the counter marked "1" is advanced one step, the other counters being left alone. If 2 turns up, the counter marked "2" is advanced one step, and so on. The race is not an even one, however, since the several horses are given different handicaps. Thus the counter marked "1" has but five steps to go to win the race, while the counter marked "2" has six steps to go, the next, seven, and so on, the counter marked "6" having ten steps to go. A single race terminates as soon as any one horse completes his course. Limited bets may be placed by any player on any horse, and successive races occur unendingly. At the close of each race the banker pays all winning bets. Assume that the banker's profits are derived wholly from general admission fees, determine what odds the banker should offer on each horse, that is, how many dollars should he return to a successful better for each dollar placed on the winning horse, supposing that in the long run the banker may come out approximately even on the bets?

SOLUTIONS

3200 [3188; 1926, 338]. Proposed by B. F. Finkel, Drury College.

Find the equation of the curve whose radius of curvature at any point of the curve is n times the radius vector to the same point.

SOLUTION BY P. J. FEDERICO, Washington, D. C.

If the equation of a curve is in polar coordinates, its radius of curvature at any point can be written $R=\rho(d\rho/du)$ where u is the perpendicular distance from the pole to the tangent at the point. (See. Williamson, *Differential Calculus* (1912), page 296.) From the given condition, $R=n\rho$, it follows that $(d\rho/du)=n$, and hence $\rho=nu+c$.

The expression for u is

$$u=\rho^2(d\theta/ds),\quad ds^2=d\rho^2+\rho^2d\theta^2. \tag{1}$$

(Ibid., pages 223, 224). These, when substituted in the equation $\rho=n\,u+c$, give

$$(\rho-c)\rho^{-1}[n^2\rho^2-(\rho-c)^2]^{-1/2}d\rho=d\theta. \tag{2}$$

The integral of (2) is the equation of the required curve in polar coordinates; (2) can always be integrated (using formulas 160 or 161 and 182 or 183 in B. O. Peirce's *Short Table of Integrals*), but the result is, in general, too cumbersome to determine the nature of the curve readily. It will be noticed that the curve is real for all real values of n except $n=0$.

The special case when $c=0$ gives

$$d\rho/\rho=(n^2-1)^{1/2}d\theta;\ \rho=k\epsilon^{\mu},\quad \text{where } \mu=(n^2-1)^{1/2}\cdot\theta,$$

which is the partial solution discussed on page 274 of this Monthly for May, 1927.

3220 [1926, 480]. Proposed by W. L. Ayres, University of Pennsylvania.

Find the value of

$$2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{2!}2^{n-4}-\frac{(n-3)(n-4)(n-5)}{3!}2^{n-6}+\cdots$$
$$+(-1)^{r+1}\frac{(n-r+1)(n-r)\cdots(n-2r+3)}{(r-1)!}2^{n-2r+2}+\cdots,$$

where the number of terms is the greatest integer in $(n+2)/2$ and n is a non-negative integer.

SOLUTION BY HARRY LANGMAN, Brooklyn, N. Y.

The sum may be written

$$A=\sum_{r=0}^{s}(-1)^r\frac{(n-r)(n-r-1)\cdots(n-2r+1)}{r!}2^{n-2r}=2^n\sum_{r=0}^{s}(-1)^rC_r^{n-r}\cdot\frac{1}{4^r}=2^n\sum_{r=0}^{s}a_r,$$

where s is the greatest integer in $n/2$.

Now a_r = coefficient of x^r in $[1-(x/4)]^{n-r}$ = coefficient of x^n in $[x-(x^2/4)]^{n-r}$. Hence $A=2^n$ coefficient of x^n in

$$\sum_{r=0}^{s}[x-(x^2/4)]^{n-r}.$$

We require then the coefficient of x^n in

$$\Big[[x-(x^2/4)]^{n-s}-[x-(x^2/4)]^{n+1}\Big](1-x/2)^{-2}.$$

The term $[x-(x^2/4)]^{n+1}$ will not affect the result. Hence we require but the coefficient of x^s in $(1-x/2)^{-2}(1-x/4)^{n-s}$ or

$$\left(\sum_{a=0}^{\infty}(a+1)x^a/2^a\right)\left(\sum_{b=0}^{n-s}(-1)^b C_b^{n-s}x^b/4^b\right).$$

This is evidently

$$\sum_{b=0}^{s}(-1)^b C_b^{n-s}\cdot\frac{1}{4^b}\cdot\frac{s-b+1}{2^{s-b}}$$

$$=\frac{s+1}{2^s}\sum_{b=0}^{s}(-1)^b\frac{(n-s)!}{b!(n-s-b)!}\cdot\frac{1}{2^b}+\frac{n-s}{2^{s+1}}\sum_{b=1}^{s}(-1)^{b-1}\frac{(n-s-1)!}{(b-1)!(n-s-b)!}\cdot\frac{1}{2^{b-1}}.$$

If $n=2s$, this sum becomes

$$\frac{s+1}{2^s}\left(1-\frac{1}{2}\right)^s+\frac{s}{2^{s+1}}\left(1-\frac{1}{2}\right)^{s-1}=\frac{n+1}{2^n}.$$

If $n=2s+1$, we have

$$\frac{s+1}{2^s}\left[\left(1-\frac{1}{2}\right)^{s+1}+(-1)^{s+2}\cdot\frac{1}{2^{s+1}}\right]+\frac{s+1}{2^{s+1}}\left[\left(1-\frac{1}{2}\right)^s+(-1)^{s+1}\cdot\frac{1}{2^s}\right]$$

$$=\frac{2s+2}{2^{2s+1}}=\frac{n+1}{2^n}.$$

Hence in either case $A=n+1$.

Also solved by LEONARD CORLITZ and the PROPOSER.

NOTE BY OTTO DUNKEL, Washington University

If we denote the series of the problem by $P_2(n+1)$ it will be found by setting $n=0, 1, 2, 3$ that $P_2(1)=1, P_2(2)=2, P_2(3)=3, P_2(4)=4$. This suggests an examination of the second difference. Forming the difference $P_2(n+2)-2P_2(n+1)$, we easily verify that this difference is equal to $-P_2(n)$. Therefore the second difference is zero for all values of n and hence $P_2(n+1)=An+B$. The particular values above give $A=B=1$. The value of the series of the problem is then $n+1$.

This series is a special case of a previous result which may be stated as follows:

In n tosses of a single coin the number of ways of tossing at first r or more consecutive heads and in the subsequent tosses no set of a tail followed by as many as r consecutive heads is

$$P_2(n)=\sum_{k=1}^{k=i}(\mp 1)^{k+1}{}_{n-kr}C_{k-1}2^{n-k(r+1)+1},\qquad i=\left[\frac{n+1}{r+1}\right],$$

where ${}_mC_j=m!/j!(m-j)!$ *and* $[\alpha]$ *denotes the greatest integer in* α. Here $P_2(n)$ is a function of n and r. See *Solutions of a Probability Difference Equation* by Otto Dunkel in this Monthly, vol. 32 (1925), pp. 354–370. A different derivation is referred to in that paper.

3224[1926, 481]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find the locus of the center of gravity of the variable triangle determined by three skew lines in a plane turning about a fixed axis.

SOLUTION BY MICHAEL GOLDBERG, Washington, D.C.

Let the axis of x be taken along the fixed axis. As the plane turns about this axis it will reach a position, π_n defined by its angle t_n with a fixed reference plane, such that the line l_n projects upon π_n

in a line parallel to the axis. Let l_n make an angle θ_n with π_n and cut it in a point with the coordinates (a_n, p_n) in that plane. When the plane has the position defined by t, it cuts l_n in a point (x_n, y_n) such that

$$y_n = p_n \sec (t - t_n), \quad (x_n - a_n) \tan \theta_n = p_n \tan (t - t_n).$$

If we now shift the origin to the point $\{\frac{1}{3}(a_1+a_2+a_3), 0\}$, the locus of the center of gravity is given by the parametric equations

$$3x = k_1 \tan (t - t_1) + k_2 \tan(t - t_2) + k_3 \tan (t - t_3),$$
$$3y = p_1 \sec (t - t_1) + p_2 \sec (t - t_2) + p_3 \sec (t - t_3)$$

where $k_n = p_n/\tan \theta_n$.

Also solved by THEODORE BENNETT.

3225[1926, 481]. Proposed by C. N. Schmall, New York, N. Y.

If $ABCD$ is a cyclic quadrilateral, and P any point in its plane prove that

$$PA^2 \cdot \Delta BCD + PC^2 \cdot \Delta ABD = PB^2 \cdot \Delta ACD + PD^2 \cdot \Delta ABC.$$

SOLUTION BY ROSCOE WOODS, University of Iowa.

Let (x_i, y_i), $i=1, 2, 3, 4$, be the coordinates of the points A,B,C,D respectively. No three of these points are collinear. The necessary and sufficient condition that the quadrilateral be cyclic is expressed thus by the vanishing of a fourth order determinant:

$$\begin{vmatrix} x_1^2 + y_1^2, & x_1, y_1, 1 \\ x_2^2 + v_2^2, & x_2, y_2, 1 \\ x_3^2 + y_3^2, & x_3, y_3, 1 \\ x_4^2 + y_4^2, & x_4, y_4, 1 \end{vmatrix} = 0.$$

There is no loss of generality if P be taken at the origin. Note that the elements of the first column are OA^2, OB^2, OC^2, OD^2 respectively. Further note that the minor of OA^2 is twice the area of the triangle BCD and so on for the others. Hence the expansion of the above determinant according to the elements of the first column gives $OA^2 \cdot \Delta BCD + OC^2 \cdot \Delta ABD = OB^2 \cdot \Delta ACD + OD^2 \cdot \Delta ABC$ which is the relation sought. One should consult the following references if interested in this problem from a slightly different viewpoint: Charles Smith, *Conic Sections by the Methods of Coordinate Geometry* (Macmillan and Co., London, 1910), pp. 99–100 and J. L. Coolidge, *A Treatise on the Circle and Sphere* (Clarendon Press, Oxford, 1916), p. 138.

Consider now four points P, Q, R, S in the plane of $ABCD$. From the above we have

$$PA^2 \cdot \Delta BCD + PC^2 \cdot \Delta ABD = PB^2 \cdot \Delta ACD + PD^2 \cdot \Delta ABC$$

and three similar relations for Q, R and S. Now the quantities ΔBCD, ΔABD, etc., may be eliminated and we have a relation connecting the squares of the distances from four points and the vertices of a cyclic quadrilateral. This relation is

$$\begin{vmatrix} PA^2 & PB^2 & PC^2 & PD^2 \\ QA^2 & QB^2 & QC^2 & QD^2 \\ RA^2 & RB^2 & RC^2 & RD^2 \\ SA^2 & SB^2 & SC^2 & SD^2 \end{vmatrix} = 0.$$

If P coincides with A, Q with B, etc., this relation becomes

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 \\ BA^2 & 0 & BC^2 & BD^2 \\ CA^2 & CB^2 & 0 & CD^2 \\ DA^2 & DB^2 & DC^2 & 0 \end{vmatrix} = 0$$

which gives rise to Ptolemy's Theorem, namely $AB \cdot CD \pm AC \cdot BD \pm AD \cdot BC = 0$.

Evidently the same procedure may be extended to five co-spherical points in a space of three dimensions. If the points be A, B, C, D, E, no four of which are co-planar, we have immediately

$$\begin{vmatrix} 0 & AB^2 & AC & AD^2 & AE^2 \\ BA^2 & 0 & BC^2 & BD^2 & BE^2 \\ CA^2 & CB^2 & 0 & CD^2 & CE^2 \\ DA^2 & DB^2 & DC^2 & 0 & DE^2 \\ EA^2 & EB^2 & EC^2 & ED^2 & 0 \end{vmatrix} = 0.$$

Consult an article by Cayley on the relation between the distances of five points in space in the Messenger of Mathematics, vol. 18 (1889), pp. 100–102.

If the distance between two points $P(x_1, x_2, \cdots, x_n)$, $P(y_1, y_2, \cdots, y_n)$ in a space of n dimensions is given by

$$\overline{P_1P_2}^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2,$$

then this problem may be extended to a space of n dimensions where the $n+2$ points lie on a hypersphere and no $n+1$ of these points are on a hyperplane.

Also solved by Michael Goldberg and Harry Langman.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Miss Gwenthalyn Jones of Chicago has made a gift of $200,000 for the endowment of a professorship in mathematical physics at Princeton University.

Professor J. S. Ames of Johns Hopkins University has been elected chairman of the National Advisory Committee on Aeronautics.

Professor G. H. Cresse of the University of Arizona is spending his year of sabbatical leave at the University of Göttingen.

Assistant Professor W. L. Crum of Harvard University has been appointed professor of statistics in the Graduate School of Business, Stanford University.

Dr. B. F. Dostal of the University of Michigan has been appointed assistant professor of mathematics at the University of Florida.

Assistant Professor W. V. N. Garretson of Rutgers University has been appointed head of department of mathematics at Ouachita College, Arkadelphia, Arkansas.

Assistant Professor Jewel C. Hughes has been promoted to an associate professorship of mathematics at the University of Arkansas.

Professor J. J. Knox, head of the department of mathematics of the Senior High School of South Bend, Indiana, has been appointed professor of mathematics and head of the department of mathematics at Dakota Wesleyan University.

Dr. F. W. Kokomoor of the University of Michigan has been appointed assistant professor of mathematics at the University of Florida.

Dr. C. H. LANGFORD of Howard University has been appointed assistant professor of philosophy at the University of Washington.

Mr. C. A. MESSICK has been promoted to an assistant professorship of mathematics at the University of Florida.

Dr. E. B. MILLER of the University of Wisconsin has been appointed professor of mathematics at Illinois College, Jacksonville, Ill.

Associate Professor FRANK R. MORRIS has been promoted to a full professorship of mathematics at Fresno State College.

Professor F. R. MOULTON has been elected a member of the executive board of the National Research Council.

Dr. J. M. THOMAS has been appointed assistant professor of mathematics at the University of Pennsylvania.

Dr. W. J. TRJITZINSKY has been appointed assistant professor of mathematics at the University of Valparaiso, Valparaiso, Indiana.

Assistant Professor L. A. H. WARREN has been promoted to a full professorship of mathematics at the University of Manitoba.

Dr. LOUIS WEISNER has been appointed assistant professor of mathematics at Hunter College.

Dr. GORDON WHYBURN has been promoted to a full professorship of mathematics at the University of Texas.

The following appointments to instructorships in mathematics are announced:

Johns Hopkins University, Dr. LEONARD M. BLUMENTHAL;

University of Florida, Mr. A. M. CRAIG;

University of Texas, Mr. John H. SIMESTER.

Dr. GEORGE ANDREW HILL, senior astronomer at the United States Naval Observatory, died August 29, 1927, aged 69 years.

Professor H. E. RUSSELL of the University of Denver, since 1918 a member of the Association, died on May 31, 1927, as the result of an automobile accident.

Dr. H. D. THOMPSON, for more than thirty years professor of mathematics at Princeton University, has died, aged 63 years.

Dr. ANNA LAVINIA VAN BENSCHOTEN, professor of mathematics and head of the department of mathematics at Wells College from 1901 to 1920, died at Whittier, California, September 18, 1927.

TWO NOTABLE GIFTS TO THE ASSOCIATION

I. THE CARUS MATHEMATICAL MONOGRAPHS.

The entire expense for publishing and distributing these MONOGRAPHS is provided by MRS. MARY HEGELER CARUS as a gift to the ASSOCIATION. The sale of these books at cost to its members by the ASSOCIATION is thus made possible, and the receipts from such sales are used to build up an endowment fund of the ASSOCIATION to be known as the CARUS PUBLICATION FUND. Hence, when a member buys a CARUS MONOGRAPH he not only gets full value at minimum cost but he also contributes to a fund which will ultimately be of the utmost value to the ASSOCIATION. Can any member show good reason for not rendering this service to the ASSOCIATION? The first and second and third Monographs are still available to members at the cost price. The fourth is in preparation.

II. THE RHIND MATHEMATICAL PAPYRUS.

CHANCELLOR ARNOLD BUFFUM CHACE, of Brown University, who has repeatedly shown his vital interest in the Association by cash contributions to its depleted budget, has now made a notable gift which was fully explained in the September, 1926 issue of the MONTHLY. He has done the ASSOCIATION signal honor by publishing at great expense his RHIND MATHEMATICAL PAPYRUS under its auspices. The entire receipts from the sale of this work will be devoted to an endowment fund of the ASSOCIATION to be known as the ARNOLD BUFFUM CHACE FUND. Individuals and institutions not now members of the ASSOCIATION may secure the special rate to members by making application for membership before the sale begins, early in December.

A more detailed description of this great work is to be found in this November issue of the MONTHLY.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Annual Meeting, Nashville, Tenn., December 29, 30, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 14.

LOUISIANA, MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7, and Georgetown University, December 3.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 6.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November 26.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Austin or Houston, February, 1928.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

INDEX TO VOLUME XXXIV, 1927

THE AMERICAN MATHEMATICAL MONTHLY

By H. S. Everett, The University of Chicago, and Harriet Smull, Bucknell University

PAPERS, REPORTS OF MEETINGS

QUESTIONS AND DISCUSSIONS—DISCUSSIONS

QUESTIONS AND DISCUSSIONS—QUESTIONS

RECENT PUBLICATIONS—NEW BOOKS RECEIVED

RECENT PUBLICATIONS—REVIEWS

RECENT PUBLICATIONS—PERIODICALS

MATHEMATICAL CLUBS—ACTIVITIES

MATHEMATICAL CLUBS—TOPICS

PROBLEMS—AUTHORS

Numbers refer to pages, black-face type indicating a problem solved and solution published, italics a problem solved but solution not published, ordinary type a problem proposed.

PROBLEMS—SOLUTIONS

Numbers in black-face type refer to problems, those in light-face to pages.

NOTES AND NEWS

PERSONAL MENTION

PERSONAL MENTION—NECROLOGY

ADDENDA AND CORRIGENDA

P. 29, 3d line, replace the first C^2 by V^2.
P. 330, 24th line, delete the first "e" in "judgement."
P. 340, 22nd line delete the first "r" in "Tarmarkin."
P. 321, 28th line replace the comma after "typical" by ")".
P. 339, 11th line from the bottom, replace "u" by "i" in "Mullikan".

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXIV, 1927

NUMBER 10, DECEMBER

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND MINNEAPOLIS, MINN

Entered as second class matter at the Postoffice at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

THE FOURTH ANNUAL MEETING OF THE SOUTHEASTERN SECTION

The fourth annual meeting of the Southeastern Section of the Mathematical Association of America was held at Emory University, March 19–20, 1926. There were seventy present, including the following twenty members: E. A. Bailey, D. F. Barrow, L. M. Blumenthal, F. Cumming, L. E. Dickson, M. D. Earle, Floyd Field, R. W. Frary, H. K. Fulmer, Ruby Hightower, J. B. Jackson, A. B. Morton, Frank Ordway, W. P. Ott, W. W. Rankin, Caroline Reaves, D. Rumble, D. M. Smith, R. P. Stephens, Fredrick Wood.

Professor W. P. Ott, vice-chairman, presided on account of the death of the chairman, Professor M. T. Peed. A special dinner was given at the Henry Grady Hotel in honor of Professor L. E. Dickson of the University of Chicago. Professor A. B. Morton was Toastmaster. The following officers were elected: Chairman, W. P. Ott, University of Alabama; Vice Chairman, J. B. Coleman, University of South Carolina; Secretary-Treasurer, W. W. Rankin, Agnes Scott College. The Chairman appointed the following members to act on the Program Committee: W. W. Rankin, chairman, D. F. Barrow, J. W. Lasley, Frank Ordway.

Resolutions were prepared and read by Professors R. P. Stephens and J. F. Messick in memory of Professors S. M. Barton, University of the South and M. T. Peed, Emory University, who had died since the last meeting.

The following program was presented, after which a delightful luncheon was served to the delegates by Emory University.

1. "A chapter in the theory of numbers from the standpoint of quaternions." by Professor L. E. Dickson, University of Chicago. (By invitation.)
2. "Generalizations of certain types of boundary value problems of differential equations," by Professor W. W. Elliott, Duke University.
3. "Transfinite cardinal numbers," by Ruth Lineberry, Agnes Scott College. (By invitation.)
4. "New results on algebras and their arithmetics," by Professor L. E. Dickson, University of Chicago.
5. "Applications of mathematics to engineering," by Professor J. B. Peebles, Emory University. (By invitation.)
6. Committee Report: "Closer correlation of college and high school mathematics," by Professor W. W. Rankin, Chairman.

At the special dinner, informal talks were made by Professors Dickson, Field, and Stephens.

W. W. Rankin, *Secretary*

THE FIFTH ANNUAL MEETING OF THE SOUTHEASTERN SECTION

The fifth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of South Carolina, Columbia, S. C., April 15–16, 1927. There were forty present, including the following nineteen members: D. F. Barrow, E. C. Coker, J. B. Coleman, Julia Dale, M. D. Earle, W. Benjamin Fite, J. B. Jackson, A. V. Martin, J. F. Messick, A. B. Morton, W. P. Ott, D. D. Peele, W. W. Rankin, Caroline Reaves, H. A. Robinson, G. M. Robison, R. P. Stephens, W. L. Williams.

A special dinner was given in honor of Professor W. B. Fite of Columbia University. Professor J. B. Coleman was Toastmaster. The following officers were elected: Chairman, J. B. COLEMAN; Vice Chairman, A. W. HOBBS; Secretary-Treasurer, W. W. RANKIN. The invitation to hold the spring meeting of 1928 at Duke University was accepted. The following program was presented, after which the delegates were guests of the University of South Carolina at luncheon.

1. "Mathematical research in the United States," by Professor W. B. FITE, Columbia University.

2. "Graduate study in mathematics in southern universities," by Professor R. P. STEPHENS, University of Georgia.

4. "Summability of infinite products," by Professor G. M. ROBISON, Duke University.

5. "Uniform convergence of infinite series," by Professor W. B. FITE, Columbia University.

6. "The application of the exponential mean to Fourier series," by Professor JULIA DALE, Delta State College.

7. "Poles and polars and their representation," by Professor J. F. MESSICK, Emory University.

W. W. RANKIN, *Secretary*

THE FOURTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fourth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Centenary College, Shreveport, La., March 4–5, 1927, Professor S. T. Sanders of Louisiana State University presiding.

There were eighty registered delegates including the following twenty-eight members of the Association: Zelda Allen, Leora Blair, Julia Dale, Mrs. A. P. Daspit, Hal Fox, John A. Hardin, J. T. Harwell, Sydney M. Kil-

patrick, Mrs. B. E. Longmire, A. C. Maddox, I. Maizlish, B. E. Mitchell, I. C. Nichols, Ralph L. O'Quinn, Amy F. Preston, W. D. Reeve, S. T. Sanders, H. E. Slaught, C. D. Smith, H. L. Smith, Ida K. Smith, P. K. Smith, N. Smylie, W. B. Stokes, Norma E. Touchstone, B. A. Tucker, Yetta Velinsky, J. F. Welch.

The guests of the section were H. E. Slaught, of Chicago; W. D. Reeve, of New York City; Marie Gugle and Amy F. Preston of Columbus, Ohio.

At the dinner tendered by the Chamber of Commerce of Shreveport and Centenary College, the principal address was given by Professor Slaught on the subject "Mathematics and sunshine." Professor Reeve gave an address at each of the regular sessions. The first dealt with the subject matter of "Junior high school mathematics"; the second was on the subject "What mastery should we expect in mathematics."

Miss Marie Gugle gave an illustrated lecture Saturday morning on the subject "Dynamic symmetry." Professor Roaten made a report on segregation of high school students on the basis of ability. One of the most interesting features of the meeting was a mathematical exhibit prepared by Professor Maizlish of Centenary College and Miss Preston of Columbus, Ohio. At the Saturday morning session, a branch of the National Council of Teachers of Mathematics was formed for the states of Louisiana and Mississippi, also a branch of the American Academy of Science.

Professor S. T. Sanders was re-elected chairman of the section, Dean John A. Hardin vice-chairman for Louisiana and Professor Hal Fox for Mississippi. Professor P. K. Smith of Mississippi Woman's College, Hattiesburg, Miss., was elected secretary-treasurer.

The following papers were read:

1. "Maximum efficiency in mathematical training," by Professor A. C. Maddox, Louisiana State Normal College.

2. "Application of the exponential mean to Fourier's series," by Professor Julia Dale, Delta State Teachers' College.

3. "Some applications of certain functional equations," by Professor I. Maizlish, Centenary College.

4. "A substitute for Duhamel's theorem," by Professor H. L. Smith, of Louisiana State University.

Abstracts of the third and fourth papers are as follows:

3. The following four functional equations were discussed by Professor Maizlish:

$$g(x+y) = g(x) + g(y), \qquad g(xy) = g(x) + g(y),$$
$$g(x+y) = g(x)g(y), \qquad g(xy) = g(x)g(y).$$

Applications of these equations were made to certain problems in thermodynamics and the kinetic theory of gases. Among other things, a proof was given of the following interesting theorem: "A necessary and sufficient condition in order that a function $F(x, y, w, z)$ be invariant under any orthogonal transformation is that it be a function of the radius vector only; that is, that $F(x, y, w, z) = G(x^2+y^2+w^2+z^2)$."

4. Professor Smith gave various sets of conditions which are sufficient for the validity of the equality

$$\operatorname{Lim} \sum_{r=1}^{n} \varphi(x_r\, x_{r-1}) = \int_a^b f(x)\, dx,$$

where $a(=x_0) < x_1 < x_2 < \cdots < x_{n-1} < b(=x_n)$, and where the limit is taken with respect to the norm of the partition just indicated. The proof depends on a lemma which also yields the usual existence theorem for the Riemann integral. Applications were made to the fundamental theorem of the integral calculus and to the length of arc.

B. E. MITCHELL, *Secretary*

THE THIRD ANNUAL MEETING OF THE NEBRASKA SECTION

The third annual meeting of the Nebraska section of the Mathematical Association of America was held on May 6, 1927, at the University of Nebraska. Twenty-two persons were present including twelve members of the Association. The following program was presented:

1. "Functions which represent primes for almost all integral values of the variable" by O. C. COLLINS, University of Nebraska (By invitation).
2. "Solutions of quadratic congruences modulo a power of 2" by LAURENCE HAMPTON, University of Nebraska.
3. "On substitution groups" by F. L. WILMER, Omaha, Nebraska. (By title).
4. "The principle of cross-classification" by EVELYN HESSELTINE, University of Nebraska.
5. "Polynomials associated with those of Legendre" by Mrs. FLORENCE B. Young, University of Nebraska. (By invitation).
6. "Failures in freshman mathematics" by A. L. CANDY, University of Nebraska.

1. Mr. Collins showed that any algebraically prime function with integer coefficients will represent primes for all integer values of the variable except certain ones which can be expressed in terms of the coefficients in such a way that substitution in the function will yield an algebraically factorable quantity. Of these integer values of the variable, some are independent of the arithmetic

form of the coefficients, and functions exist which represent composite numbers for no other integer values of the variables.

2. Mr. Hampton discussed the solution of the quadratic congruence modulo a power of 2, and obtained a few novel results.

4. Miss Hesseltine gave a discussion of the principle of cross-classification and showed how many results of the theory of numbers may be brought under this principle.

5. Mrs. Young discussed the properties of polynomials arising from those of Legendre by repeated differentiation or integration.

6. Mr. Candy presented statistics covering several years, showing that from thirty to forty percent of students who register in the freshman courses in mathematics in the University of Nebraska failed to complete their course satisfactorily. Some withdrew in good standing, others withdrew because of delinquency, but the larger number remained in class to the end of the semester and fail to make a passing grade.

Officers elected for the year 1927–28 are: W. C. Brenke, University of Nebraska, Chairman; Ellen H. Frankish, North High School, Omaha, Secretary-Treasurer; H. C. Feemster, York College, Member of Executive Committee.

Ellen H. Frankish, *Secretary*

THE ELEVENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The eleventh annual meeting of the Kentucky Section of the Mathematical Association was held May 14, 1927, at the University of Kentucky, Lexington, Kentucky. Dean P. P. Boyd presided at both sessions. Dinner was served at 12 o'clock in the University Cafeteria.

After a short business meeting Dean Boyd called attention to the December, 1927 meeting of the National Association at Nashville, Tennessee, and urged each member to be present at the meeting. Professor Guy Stevenson of the University of Louisville was made chairman and Professor A. R. Fehn was made secretary-treasurer for the year 1927–1928. The section expressed its appreciation to Dean Boyd and the University of Kentucky for the hospitable reception and delightful dinner given to the attending members. There were twenty-five present including the following members of the Association: P. P. Boyd, J. L. Clayton, J. M. Davis, D. S. Dearman, A. R. Fehn, W. R. Hutcherson, Florence E. LeStourgeon, C. A. Maney, E. L. Rees, C. H. Richardson, Guy Stevenson, B. B. Vance.

Papers were presented on the program as follows, abstracts of most of these being given, numbered to correspond:

1. "Equivalence and classification of pairs of quadratic forms" by Miss MARY COOPER, University of Kentucky (by invitation).

2. "On the multiplication of sigma functions" by Professor C. H. RICHARDSON, Georgetown College.

3. "Congruences and complexes of line geometry" by Professor W. R. HUTCHERSON, Berea College.

4. "Expansions of the Neumann type in terms of products of Bessel functions" by Professor GUY STEVENSON, University of Louisville.

5. "On the trisection of an angle" by Professor A. R. FEHN, Centre College.

6. "On the use of formulas" by Professor E. L. REES, University of Kentucky.

7. "A new emphasis in teaching of freshman mathematics" by Professor C. A. MANEY, Transylvania College.

8. "Survey courses in mathematics" by Professor D. S. DEARMAN, Kentucky Wesleyan College.

9. "Sectioning in mathematics" by Professor J. M. DAVIS, University of Kentucky.

10. Business.

1. It is shown that if two pencils of quadratic forms have the same elementary divisors, then one pencil can be transformed into the other by means of linear substitutions. After the proof that there exists a pair of forms whose matrix has any preassigned invariant factors, Weierstrass's canonical pair of quadratic forms are introduced. The paper is concluded with a classification of pairs of ternary and quaternary quadratic forms.

2. Professor Richardson showed how the partition notation simplified the multiplication of sigma functions, particularly when applied to the expression $\sum x^a \cdot \sum x_i^b x_j^c x_l^d \cdots x_m^k$ when a is different from, as well as when a is equal to, $b, c, d, \cdots, k$.

3. This paper gave some general statements defining complexes and congruences. Then it was shown how two points $P(y_1y_2y_3y_4)$ and $P(z_1z_2z_3z_4)$, which determine a straight line, give rise to six line coordinates. Some properties of these coordinates were then shown and an analytic development of complexes and congruences was thus derived.

4. A generalization of the expansions of arbitrary analytic functions obtained by Neumann (*Theorie der Bessel'schen Funktionen*) and Gegenbauer (Wiener Sitzungsberichte (2), vol. 74. (1877), pp. 124–130) in terms of Bessel functions and products of Bessel functions. This paper extends the expansion theories of Neumann and Gegenbauer in four general directions: (1) the number of Bessel functions occurring as factors in the terms of the expansions is extended from 2 to any number n; (2) the order; and (3) the argument of each of the Bessel functions in the expansions is given a more general form;

and, finally, (4) the extension is made to the expansion of functions of several variables. Also, a relation between these expansions and those of Taylor and Laurent is obtained, and certain orthogonality properties are derived.

5. This paper (called forth by questions from some of the writer's students concerning a recent solution of the trisection of an angle by means of the straight edge and compasses) is based on theorems in L. E. Dickson's *Elementary Theory of Equations.*

6. In this paper it was shown how many of the so-called fundamental formulas of analytics and calculus could be dispensed with, with little or no inconvenience to the student.

7. The mathematics survey course as taught at Transylvania College is one of a group of survey courses in a nearly uniform freshman program of study. Selected applications of mathematics to the fields of the various sciences, engineering, and business, are developed as illustrations of the quantitative nature of the structure of modern civilization. Especial emphasis is given to the point of view that the combination of the experimental method with the quantitative thinking of mathematics constitutes the basis of scientific progress. Among the topics covered in a year's course of four semester-hours with sections meeting twice weekly are the following: (1) logarithms and applications; (2) portions of trigonometry with applications to surveying and statics; (3) simple notations of the calculus with applications to algebraic functions; the law of growth; velocity and acceleration; (4) graphical methods with applications to various functions, with use of logarithms and the calculus; (5) energy, its measure and applications.

8. This paper dealt with a description of a survey course in mathematics. This course was designed to meet the demands of two classes of students: (1). those who wish to obtain a knowledge of mathematical thought and the contributions of mathematics to the world but who do not have the time or inclination to give to the study of mathematics the time required of a student to obtain this knowledge; (2). those who wish a survey of the field before going into the study of the details, this to be followed by the traditional courses.

A. R. FEHN, *Secretary*

CERTAIN MATHEMATICAL QUESTIONS SUGGESTED BY THE TRUE-FALSE TEST

By HELEN M. WALKER, Teachers College, Columbia University

Although most of the difficulties encountered by the maker of the true-false examination[1] are psychological in their nature, yet there arise in this con-

[1] In the usual "true-false examination" the student is told to mark a set of statements in some specified manner, indicating whether he believes them to be true or false. He is generally instructed

nection a few questions which can be satisfactorily answered only after a mathematical analysis. The object of this paper is to propose several such questions, to answer them from the point of view of the theory of probability, and to offer the results of a set of two thousand spins of a coin as an empirical check on the theoretical results.[1]

1. *In a true-false test of n statements, if the order is determined wholly by chance, what is the probability that there will be at least one run of k or more consecutive statements all of which are "true?"*

As a matter of actual practice, these tests are not often written with long runs of similar answers and students are sometimes troubled when they find themselves recording what they think to be a suspiciously long string of answers all alike. Long runs facilitate the passing of information from one class to another if the test is to be used more than once, and there may be many other reasons for avoiding them, but it does not appear that they should be entirely excluded on the ground that they are theoretically improbable.

For the sake of clarity and brevity we shall restate the problem as follows: *An unbiased coin is spun on its edge n times, a plus sign is recorded for each showing of heads, and a minus sign is recorded for each showing of tails. What is the probability that there will be at least one run of k or more consecutive plus signs in the record?*

Let $p_{n,k}$ = the probability that there will be at least one run of k or more heads in a set of n throws. We shall then find a difference equation by means of which $p_{n,k}$ may be obtained from $p_{n-1,k}$.

A sequence of k heads may occur in the first $n-1$ spins or it may be completed on the nth spin, the two cases being mutually exclusive. The probability that a k-sequence shall occur in the first $n-1$ spins is $p_{n-1,k}$. In order that a sequence of k heads be completed on the nth spin, it is necessary that tails shall appear on the $(n-k)$th spin, that the following k spins shall exhibit heads, and that there shall be no k-sequence of heads in the first $(n-k-1)$ throws.

---

to place a plus sign to the left of each statement which he believes to be true without qualification, and a different specified symbol, either zero or a minus sign, to the left of each statement which he believes to be false, or to be true only under conditions other than those stated. The use of zero makes for ease and rapidity of scoring; the minus sign has certain other advantages. To phrase a satisfactory set of statements for such a test is a task requiring considerable skill. For a further discussion of the use of the true-false test and the technique of its construction, the following references are suggested: D. G. Paterson, *Preparation and Use of New-Type Examinations* (World Book Company); G. M. Ruch, *Improvement of the Written Examination* (Scott, Foresman and Company); Ben D. Wood, *Measurement in Higher Education* (World Book Company).

[1] One thousand of the spins were made by Miss Ada M. Beckey, a student in Teachers College, who also assisted in tabulating the empirical data and in computing the averages obtained from them. No trial was counted in which the coin did not actually spin on its edge. Miss Beckey used one coin and the writer used another. In spite of preliminary trials to discover bias in the coins, those selected were probably imperfect. Both showed a preference for heads, the mean numbers of heads per hundred spins being 50.2 and 52.2, with a final average of 51.2.

The probability for this situation is the product of the various probabilities, or $(1-p_{n-k-1,\,k})\,(1/2)^{k+1}$. Therefore

$$p_{n,k} = a^{k+1} + p_{n-1,k} - a^{k+1}p_{n-k-1,k}, \quad \text{where } a = \tfrac{1}{2}. \tag{1}$$

Now for any k, we have
$p_{n,k}=0$ when n is less than k; $p_{k,k}=a^k=2a^{k+1}$; $p_{k+1,k}=3\,a^{k+1}$; $p_{k+2,k}=4\,a^{k+1}$; $\cdots$; $p_{2k,k}=(k+2)\,a^{k+1}$; $p_{2k+1,k}=(k+3)a^{k+1}-2a^{2(k+1)}$; etc. This may be continued indefinitely. The number of terms in the expression for $p_{n,k}$ is increased by one whenever the number of throws is increased by $k+1$. Thus, when $n=3k+2$, a third term will appear in the expression; and when $n=rk+r-1$, the expression will consist of r terms, the sign of the last one being $(-1)^{r-1}$.

The general expression for $p_{n,k}$ may be discovered by writing down a succession of the probabilities for different n's for the same value of k, and then this

TABLE I.

$$
\begin{aligned}
p_{4,4} &= 2a^5\\
p_{5,4} &= 3a^5\\
p_{6,4} &= 4a^5\\
p_{7,4} &= 5a^5\\
p_{8,4} &= 6a^5\\
p_{9,4} &= 7a^5 - 2a^{10}\\
p_{10,4} &= 8a^5 - 5a^{10}\\
p_{11,4} &= 9a^5 - 9a^{10}\\
p_{12,4} &= 10a^5 - 14a^{10}\\
p_{13,4} &= 11a^5 - 20a^{10}\\
p_{14,4} &= 12a^5 - 27a^{10} + 2a^{15}\\
p_{15,4} &= 13a^5 - 35a^{10} + 7a^{15}\\
p_{16,4} &= 14a^5 - 44a^{10} + 16a^{15}\\
p_{17,4} &= 15a^5 - 54a^{10} + 30a^{15}\\
p_{18,4} &= 16a^5 - 65a^{10} + 50a^{15}\\
p_{19,4} &= 17a^5 - 77a^{10} + 77a^{15} - 2a^{20}\\
p_{20,4} &= 18a^5 - 90a^{10} + 112a^{15} - 9a^{20}\\
p_{21,4} &= 19a^5 - 104a^{10} + 156a^{15} - 25a^{20}\\
p_{22,4} &= 20a^5 - 119a^{10} + 210a^{15} - 55a^{20}\\
p_{23,4} &= 21a^5 - 135a^{10} + 275a^{15} - 105a^{20}\\
p_{24,4} &= 22a^5 - 152a^{10} + 352a^{15} - 182a^{20} + 2a^{25}\\
p_{25,4} &= 23a^5 - 170a^{10} + 442a^{15} - 294a^{20} + 11a^{25}
\end{aligned}
$$

general formula may be established by mathematical induction first for n and then for k. Let $k=4$, and let $p_{n,4}$ be the probability of a sequence of four heads in n spins. Table I shows the successive probabilities for at least one sequence of four or more heads in n spins, as n increases from 4 to 25.

It is readily apparent that the numbers which form the coefficients of a^{k+1} are the first differences of the numbers which form the coefficients of $a^{2(k+1)}$, the second differences of the numbers which form the coefficients of $a^{3(k+1)}$, and so on. In the *Chart to Facilitate the Computation of* $p_{n,k}$, these coefficients have been rearranged to form a difference table. The first differences for any column are to be found in the column directly to the left of it. The general terms of the columns are:

Column I $\quad (s+1)$

Column II $\quad \dfrac{(s)(s+3)}{2!}$

Column III $\quad \dfrac{(s)(s+1)(s+5)}{3!}$

Column IV $\quad \dfrac{(s)(s+1)(s+2)(s+7)}{4!}$

Column V $\quad \dfrac{s(s+1)(s+2)(s+3)(s+9)}{5!}$

Column VI $\quad \dfrac{s(s+1)(s+2)(s+3)(s+4)(s+11)}{6!}$

. . . .

. . . .

Column r $\quad \dfrac{s(s+1)(s+2)\cdots(s+r-2)(s+2r-1)}{r!}$

These general terms were found by taking differences and applying the formula[1]

$$(2) \qquad u_n = u_0 + n\Delta u_0 + \frac{n(n-1)}{2!}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 u_0 + \cdots$$

The *Chart to Facilitate the Computation of* $p_{n,k}$ was constructed from the numerical coefficients in Table I. The coefficients of a^5 are found in Column I; of a^{10} in Column II; of a^{5r} in Column r. The pth entry in the rth column is the sum of the first p numbers in the $(r-1)$st column, and thus the chart is a difference table with the customary order reversed. Once the chart is constructed,

[1] See Whittaker and Robinson, *The Calculus of Observations* (Blackie and Son, Ltd.), page 11, or Forsyth, *Introduction to the Mathematical Analysis of Statistics* (John Wiley and Sons), page 19.

CHART TO FACILITATE THE COMPUTATION OF $P_{n,K}$.

$n=25, K=8$; $n=25, K=7$; $n=25, K=6$; $n=25, K=5$; $n=25, K=4$; $n=25, K=3$; $n=25, K=2$

S	I	II	III	IV	V	VI	VII	VIII	IX
1	2	2	2	2	2	2	2	2	2
2	3	5	7	9	11	13	15	17	19
3	4	9	16	25	36	49	64	81	100
4	5	14	30	55	91	140	204	285	385
5	6	20	50	105	196	336	540	825	1210
6	7	27	77	182	378	714	1254	2079	3289
7	8	35	112	294	672	1386	2640	4719	8008
8	9	44	156	450	1122	2508	5148	9867	17875
9	10	54	210	660	1782	4290	9438	19305	37180
10	11	65	275	935	2717	7007	16445	35750	72930
11	12	77	352	1287	4004	11011	27456	63206	
12	13	90	442	1729	5733	16744	44200		
13	14	104	546	2275	8008	24752	68952		
14	15	119	665	2940	10948	35700			
15	16	135	800	3740	14688	50388			
16	17	152	952	4692	19380	69768			
17	18	170	1122	5814	25194	94962			
18	19	189	1311	7125	32319				
19	20	209	1520	8645	40964				
20	21	230	1750	10395	51359				
21	22	252	2002	12397	63756				
22	23	275	2277	14674	78430				
23	24	299	2576	17250	95680				
24	25	324	2900	20150					
25	26	350	3250	23400					
26	27	377	3627	27027					
27	28	405	4032	31059					

it may be used to read off the equation for any $p_{n,k}$ without further computation. Since we have here reproduced but a small section of the chart, which is theoretically unlimited in extent, it can be used for small values of n only.

The general method of finding the equation for $p_{n,k}$ from the chart is as follows:

(1) Look in Column I for the number $(n-k+2)$. This will be the coefficient of the first term in the expression for $p_{n,k}$, that is, the coefficient of $(1/2)^{k+1}$.

(2) Locate the exact center of the rectangle in which this number stands, and through that point draw a line with a slope of $(k+1)$, "slope" being interpreted as on a graph in which the two variables have been plotted with different units of measure. As this line crosses Column II it selects the coefficient of the second term, as it crosses Column III it selects the coefficient of the third term, etc., and the expression for the probability may be written down at once.

The family of lines belonging to a single k is the set of parallels having a slope of $(k+1)$.

The family of lines belonging to a single n is the pencil of lines passing through the center of that rectangle in Column s for which $s=n+2$. The pencil of lines drawn in the chart is a part of the family for which $n=25$.

The formula for $p_{n,k}$ is found by ordinary induction to be as follows:

$$
\begin{aligned}
p_{n,k} = {} & [s+1]_{n-k+1}a^{k+1} - \left[\frac{s(s+3)}{2!}\right]_{n-2k} a^{2(k+1)} \\
& + \left[\frac{s(s+1)(s+5)}{3!}\right]_{n-3k-1} a^{3(k+1)} \\
& - \left[\frac{s(s+1)(s+2)(s+7)}{4!}\right]_{n-4k-2} a^{4(k+1)} \\
& + \left[\frac{s(s+1)(s+2)(s+3)(s+9)}{5!}\right]_{n-5k-3} a^{5(k+1)} + \cdots \\
& + (-1)^{r-1}\left[\frac{s(s+1)(s+2)(s+3)\cdots(s+r-2)(s+2r-1)}{r!}\right]_{n-rk-r+2} a^{r(k+1)}.
\end{aligned} \tag{3}
$$

In the expressions such as $[s+1]_{n-k+1}$, the quantity within the square brackets is the general term of the column; the subscript gives the value to be assigned to s for particular values of n and k. When these particular values are substituted for the quantities within the brackets, the resulting formula for $p_{n,k}$ is:

$$
\begin{aligned}
p_{n,k} = {} & (n-k+2)a^{k+1} - \frac{(n-2k)(n-2k+3)a^{2(k+1)}}{2!} \\
& + \frac{(n-3k-1)(n-3k)(n-3k+4)a^{3(k+1)}}{3!}
\end{aligned}
$$

(4)
$$-\frac{(n-4k-2)(n-4k-1)(n-4k)(n-4k+5)a^{4(k+1)}}{4!}+\cdots$$
$$+(-1)^{r-1}\frac{(n-rk-r+2)(n-rk-r+3)\cdots(n-rk)(n-rk+r+1)a^{r(k+1)}}{r!}.$$

This formula may be established, with no unusual difficulties, by the method of mathematical induction, by making use of the difference equation (1).[1]

2. *What is the most probable distribution of any set of n questions as regards the number of k-sequences* ($k=1, 2, 3, 4, \cdots, n$) *occurring in it?* If the order is due to chance alone, how many sequences of exactly k records of the same sign are to be expected, or, in other words, what is the most probable number of k-sequences in a set of n spins of a coin? Here we are considering runs of exactly k terms, not of k or more.

The probability that a k-sequence will begin on the first throw is $(1/2)^k$, since it is necessary that the $(k-1)$ next succeeding terms shall have the same sign as the first term, and the kth term shall have the opposite sign. The sign of the first term itself is immaterial. The probability that a k-sequence will begin on the $(n-k+1)$st term is also $(1/2)^k$, since here it is necessary that the preceding term shall have the opposite sign and the $(k-1)$ succeeding terms shall have the same sign. The probability of a k-sequence beginning on any of the $(n-k-1)$ remaining terms is $(1/2)^{k+1}$, since the $(k-1)$ terms immediately following the initial term of the sequence must agree in sign, while both the term immediately preceding it and the kth term following it must be of the opposite sign. Therefore the mathematical expectation of a k-sequence is approximately

$$(5)\qquad N_k=(1/2)^k+(1/2)^k+(n-k-1)(1/2)^{k+1}=(n+3-k)(1/2)^{k+1}.$$

This formula evidently contains some error, since the probabilities of runs beginning on different terms are not all mutually exclusive. If p_i is the probability of a run beginning on the ith row, then the complete formula[2] for N_k should be

[1] After this paper was written, my attention was called by Dr. M. H. Stone to a note by Cramér, *Sur quelques points du calcul des probabilités*, in the London Mathematical Society Proceedings, 2nd series, vol. 23 (1924–25) pages lviii–lx. Cramér says that the problem of having a sequence of white balls in the course of n drawings of balls from an urn containing black and white balls has been treated by De Moivre, Condorcet, and Laplace, and that they have given the recurrence formula
$$a_{n,\mu}=a_{n-1,\mu}+p^{\mu}q(1-a_{n-\mu-1,\mu}),$$
where $q=1-p$; and the generating function
$$F_\mu(x)=\sum_{n=0}^{\infty}\beta_{n,\mu}x^n=\frac{1-p^\mu x^\mu}{1-x+p^\mu qx^\mu},\quad \text{where } \beta_{n,\mu}=1-a_{n,\mu}.$$
Dr. Stone points out that formula (3) could have been reached by the method of generating functions.

[2] See Coolidge, *Introduction to Mathematical Probability* (Clarendon Press), page 23.

$$N_k = \sum_{i=1}^{n} p_i - \frac{1}{2!} \sum_{i,j=1}^{n} p_{ij} + \frac{1}{3!} \sum_{i,j,l=1}^{n} p_{ijl} - \cdots$$

of which only the first term has been used in the approximation just given. To find the magnitude of the error committed we will again make use of a difference equation.

If the formula by which N_k is computed is a correct one, then $\sum kN_k$, from $k=1$ to $k=n$, should be equal to n. The difference between n and this sum furnishes an expression for the error made in using this approximation. This error, it must be remembered, is distributed over the n terms, all of which are positive.

We shall now prove that $\sum kN_k$, from $k=1$ to $k=n$, is $n-n(1/2)^{n-1}$, and that therefore the error in the computation of any single N_k cannot be greater than $n(1/2)^{n-1}$.

$$\begin{aligned}
\sum_{k=1}^{n} kN_k &= \sum_{k=1}^{n} k(n+3-k)(1/2)^{k+1} \\
&= (n+3)\sum_{1}^{n} k(1/2)^{k+1} - \sum_{1}^{n} k^2(1/2)^{k+1} \\
&= \left[(n+3)\left\{-(1/2)^k(k+1)\right\} + (1/2)^k(k^2+2k+3)\right]_1^n \\
&= \left[(1/2)^k(k^2-nk-k-n)\right]_1^n \\
&= (1/2)^n(-2n) - (1/2)(-2n) \\
&= n - n(1/2)^{n-1}.^*
\end{aligned}$$

The sum of the probabilities as given by formula (5) is slightly too small, but since all of the n probabilities making up the sum are positive, the error in any one cannot be greater than $n(1/2)^{n-1}$. This error is very small, even for relatively low values of n.

The most probable distribution of a set of 100 questions, or of 100 spins of a coin, as to the number of sequences occurring in it, is shown in Table II. Table III presents similar information for a set of 50 questions. The third line of Table II may be interpreted thus: Two thousand spins of a coin were divided into 20 sets of 100 spins each. In these 20 sets the average number of times that exactly three heads or exactly three tails appeared in sequence was 6.05. The theoretically most probable number of such sequences in 100 spins is 6.25. If there were 6.25 runs of three similar records that would account for 18.75 spins out of the total 100 spins.

---

* The summation is effected by parts, by use of the formula $\sum v_x \Delta u_x = u_x v_x - \sum u_{x+1}\Delta v_x$. To evaluate $\sum k(1/2)^{k+1}$, we let $k=v_x$ and $(1/2)^{k+1}=\Delta u_x$. Then $\Delta v_x=1$, and $u_x=-(1/2)^k$, and $u_{x+1}=-(1/2)^{k+1}$. $\sum k(1/2)^{k+1}=-k(1/2)^k+\sum(1/2)^{k+1}$. As a check on the correctness of the solution given above, we see that $f(k)=(1/2)^k \cdot (k^2-nk-k-n)$, $f(k+1)=(1/2)^{k+1}\cdot(k^2+k-nk-2n)$; and also that $f(k+1)-f(k)=k(1/2)^{k+1}(n-k+3)$.

The column headed *Theoretical value of* N_k gives the most probable distribution of 100 spins in regard to sequences, and indicates the amount of bunching which is most probable in a set of 100 true-false questions where the order is determined by chance alone. No wise maker of tests would attempt to conform slavishly to this theoretical distribution, yet he may be interested to know that the laws of chance would provide for the occurrence of a run of seven consecutive similar answers in about one out of every three tests of this length, and that about one in every ten such tests might have a sequence of nine similar answers. There is considerable reason to think that it may be psychologically desirable to arrange test questions in such a way that a reasonable number of runs occur. The chief objection to the use of long runs is probably the one previously stated, that they increase the ease with which one student may illicitly purvey information to another.

TABLE II

Number of k-sequences in a set of 100 spins.

k	Empirical results of 20 sets of spins.	Theoretical value of N_k.	$k \cdot N_k$
1	25.40	25.500	25.50
2	12.00	12.625	25.35
3	6.05	6.250	18.75
4	3.05	3.094	12.37
5	1.69	1.531	7.66
6	.65	.758	4.55
7	.30	.375	2.63
8	.55	.186	1.49
9	.15	.091	.81
10	.05	.045	.45
11	.00	.022	.24
12	.00	.011	.13
			$\sum_1^{12} kN_k = 99.93$

TABLE III

Number of k-sequences in a set of 50 spins.

k	Empirical results of 40 sets of spins.	Theoretical value of N_k.	$k \cdot N_k$
1	12.950	13.000	13.000
2	6.000	6.375	12.750
3	3.125	3.125	9.375
4	1.475	1.531	6.125.
5	.850	.750	3.750
6	.325	.367	2.203
7	.125	.180	1.258
8	.225	.089	.703
9	.075	.043	.387
10	.025	.021	.210
			$\sum_1^{10} kN_k = 49.761$

The following conclusions do not contribute much that is new to the technique of constructing the true-false examination, but they serve to illustrate the way in which mathematical reasoning may be used to check or to verify conclusions based on experience.

3. *Should exactly half of the statements be true and exactly half false? If this is not necessary, by how much may the number of true statements deviate from* $(1/2)n$ *?* What is the probability that the number of heads appearing in n spins will deviate from $(1/2)n$ by a number whose absolute value is greater than t ?

There is an impression current among students who have become expert in the technique of passing true-false tests that half of the statements are to be of one sort and half of the other. Occasionally a student reports that he changed some of his answers because on counting he discovered that he had too many plus signs!

The probability that exactly r heads will appear in a set of n spins of an unbiased coin is

$$p = \frac{n!}{r!(n-r)!}(1/2)^n, \tag{6}$$

which, by reasoning that is familiar to all students of probability, is a maximum when r is the integer nearest to $(1/2)n$. Therefore the most probable number of true statements, if that number is to be determined by chance alone, is $(1/2)n$. Formula (6) is the general term of the expansion $[(1/2)+(1/2)]^n$, and the standard deviation of this series is $(1/2)\sqrt{n}$. Thus the chances are roughly two to one that the number of true statements should be between the limits $(1/2)n \pm (1/2)\sqrt{n}$. The numerical values of these limits for a few selected values of n are shown in Table IV.

TABLE IV

n=number of questions	25	49	64	100
$\frac{1}{2}n$=mean number of questions	12.5	24.5	32	50
$\sigma=(1/2)\sqrt{n}$	2.5	3.5	4	5
Mean$\pm\sigma$	10–15	21–28	28–36	45–55
Mean$\pm$p. e.	10.8–14.2	22.1–26.9	29.3–34.7	46.6–53.4

4. *How should the test be scored? Is it better to use the total number of right answers as the score, or the difference between the number right and the number wrong? Should the students be encouraged to guess at answers they do not know?*

For reasons of a non-mathematical nature it is now generally conceded that it is better to use the difference between the number right and the number wrong rather than the number right and that guessing should be discouraged. These are not primarily mathematical questions, yet several mathematical considerations may be noted.

(a) *If all the questions in a test are answered, there is perfect correlation between the scores obtained by the two methods.*

Let A_i=number of correct responses made by a given pupil; B_i=number of incorrect responses made by the same pupil; t=number of questions in the test; N=number of pupils; $\bar{A}=\sum A/N$, or average number of correct responses; $a_i=A_i-\bar{A}$; $\sigma_a=\sum a^2/N$; let $\bar{B}$, b_i, and σ_b have meanings similar to those for $\bar{A}$, a_i, and σ_a. Let $A_i+B_i=t$; then $a_i+b_i=0$, $a_i-b_i=2a_i$. Let r_{ab}=coefficient

of correlation between A and B and $r_{a(a-b)}$ = coefficient of correlation between A and $(A-B)$. Then by the Pearson product-moment method[1] of computing a coefficient of correlation, we have

$$r_{a(a-b)} = \frac{\Sigma a(a-b)}{N\sigma_a\sigma_{a-b}} = \frac{\Sigma a(2a)}{N\sigma_a\sigma_{2a}} = \frac{2\sigma_a^2}{2\sigma_a^2} = 1.$$

Thus there is perfect correlation between the number right and the result of subtracting the number wrong from the number right if all the questions are answered, that is if $A_i+B_i=t$. If, however, some of the statements are left without a mark, then the correlation between the two forms of scores is probably not perfect, and the method of marking becomes an important matter.

(b) *If all the questions are answered, the standard deviation of the scores is twice as great when the number of wrongs is subtracted from the number of rights as when the number of rights alone is used.* If the results of several tests are to be combined or compared, the same method of scoring ought to be used for all.[2]

When $B=t-A$, it follows by the simplest of algebra that $\sigma_b=\sigma_a$, and that $r_{at}=-1$.

Then

$$\sigma^2{}_{(a-b)} = \sigma_a^2 - 2r_{ab}\sigma_a\sigma_b + \sigma_b^2 = 4\sigma_a^2, \quad \text{or} \quad \sigma_{(a-b)} = 2\sigma_a.$$

(c) *The practice of guessing reduces the amount of dependence which can be placed on the results of the tests.* This statement merely reiterates the outcome of thoughtful observation, yet the instructions accompanying true-false tests frequently tell the student to "be sure to mark every question, whether you are certain of the answer or not."

Let k = the number of questions whose answer a given student knows completely; m = the number of questions about which he is misinformed, so that he will, without much hesitation, set down the wrong answer, o = the number he omits; g = the number at which he guesses; pg = the number at which he guesses

k	pg	qg	m	o

$\longleftarrow \cdots\cdots\cdots\cdots R \cdots\cdots\cdots\cdots \longrightarrow \longleftarrow W \longrightarrow$

successfully; and qg = the number at which he guesses unsuccessfully. The total number of right answers is $R=k+pg$. The total number of wrong

[1] The meaning and method of computing a coefficient of correlation, as well as the meaning of the standard deviation (σ), are discussed in almost every text on statistics. The reader is also referred to two papers which have appeared in this Monthly: E. V. Huntington *Mathematics and statistics, with an elementary account of the correlation coefficient and the correlation ratio*, vol. 26 (Dec. 1919), pp. 421–435; Dunham Jackson *The algebra of correlation*, vol. 31 (1924), pp. 110–121.

[2] This last statement obviously does not apply when the test scores are given weights in inverse proportion to their standard deviations.

answers is $W=m+qg$. If the student guesses, his score is $k+pg$ or $k-m+(p-q)g$. If he does not guess at all, his score is k or $k-m$. It is highly probable that p is greater than q, for even in unfamiliar material there are usually enough clues coming from the form of the questions and from one's general stock of information to make the chances for a correct answer somewhat better than for an incorrect one. If guessing is encouraged, therefore, the average score of the class is likely to be raised but the individual scores become less reliable, as will be shown.

Let us suppose that for a particular student in a particular test, k and m are fixed quantities. The uncertain element in his score arises from his chance marking of g. The most probable number of these questions to be marked correctly is pg, where p is some function of the student's general familiarity with the material, his degree of intelligence, the clarity with which the questions are phrased, and numerous other factors. It is almost certain that p varies from question to question, and the assumption made here of a p constant throughout the test is not altogether satisfactory. Moreover, while k and m may be constant and definite for a particular test, they are anything but that when we think of this test as a sample from the general subject matter with which the test deals.

$$\sigma_{k+pg} = \sigma_{pg} = (pqg)^{1/2} \leqq \tfrac{1}{2}(g)^{1/2}.$$

$$\sigma^2_{k-m+pg-qg} = \sigma^2{}_{(p-q)g} = \sigma^2{}_{pg} + \sigma^2{}_{qg} - 2r_{pg.qg}\,\sigma_{pg}\sigma_{qg} = 4\sigma^2{}_{pg}.$$

Therefore

$$\sigma_{k-m+pg-qg} = 2\sigma_{k+pg} = 2\sigma_{pg} = 2(pq)^{1/2}g^{1/2} \leqq g^{1/2}.$$

Thus for any particular set of questions the uncertainty as to a student's correct score decreases with a decrease in g, the number of questions at whose answer he guesses. If this area can be reduced by instructions which do not encourage guessing, the scores become increasingly more reliable.[1]

Without more data than is offered by the test itself, it is impossible to measure the amount of this fluctuation, because it is impossible to tell how many of the wrong answers belong to category g, and because it is impossible to get an *a priori* estimate of the size of p. No attempt is made here even to suggest the probable value of p, which may change from test to test, from pupil to pupil, and even from question to question. This would of course be an important consideration in deciding upon the method of scoring, since the optimum value of the constant t in the formula, *Score* $=R+tW$, is dependent upon the value

[1] This discussion of the uncertainty of a student's score does not take into account at all the unreliability that is due to sampling in the choice of test material. That is an entirely different matter. For a discussion of this, see Kelley, *The reliability of test scores*, Journal of Educational Research, vol. 3 (May, 1921).

of p. The value of t is probably between 0 and -1, and its "best" value for any given test could be determined statistically without great difficulty.

We can, however, set a maximum limit to the amount of fluctuation in pg, because we know that pq is a maximum when $p=q=1/2$.

$$\sigma_{pg} = (pqg)^{1/2} = p^{1/2}(qg)^{1/2} \leqq p^{1/2}(qg+m)^{1/2} = p^{1/2}W^{1/2} < W^{1/2};$$

$$\sigma_{pg} < W^{1/2} \text{ and } \sigma_{R-W} < 2(W)^{1/2}.$$

As an illustration, suppose that in a test of 100 statements, 5 are left unmarked and 25 are marked incorrectly. Then $R=70$. $R-W=70-25=45$. The standard deviation of $R-W$ is not greater than $2\sqrt{25}$, or 10. Thus the chances are better than 2 to 1 that the score should lie between 35 and 55; but how much better than 2 to 1 we cannot determine from this data.

For corroborating evidence, the results of the coin-spinning experiment have been again invoked. The two thousand records were divided into twenty sets of one hundred each, each set being considered as the answers to a true-false test. An additional hundred spins were made to furnish a "key" against which the other "tests" were scored. The results are shown in Table V. Theoretically, if the coin is unbiased, we should expect a mean score of 50 rights, with $\sigma=5$, σ of the mean $=0.5$, and σ of $\sigma=0.35$. When scoring rights minus wrongs, we should expect a mean value of 0, $\sigma=10$, σ of the mean $=1$, and σ of $\sigma=0.71$. The empirical results recorded in Table V are possible, but not highly probable from the theoretical point of view. Even for these figures the inequality $\sigma_{pg}=\sigma_R<\sqrt{W}$ seems to be a generous limit. For a score of 59, which is the highest in the list, this would make $\sigma<\sqrt{41}$, which is a limit considerably larger than 5.92.

TABLE V.

Rights		Rights minus wrongs.	
Score	Frequency	Score	Frequency
39	1	−22	1
40	0	−20	0
41	2	−18	2
42	0	−16	0
43	3	−14	3
44	0	−12	0
45	0	−10	0
46	0	− 8	0
47	3	− 6	3
48	0	− 4	0
49	1	− 2	1
50	1	0	1
51	1	2	1
52	1	4	1
53	1	6	1
54	3	8	3
55	0	10	0
56	1	12	1
57	0	14	0
58	0	16	0
59	2	18	2
Mean = 49.1 σ = 5.92		Mean = −1.8 σ = 11.85	

ON A PROBLEM ARISING OUT OF THE THEORY OF A CERTAIN GAME[1]

By J. V. USPENSKY, Carleton College

The mathematical theory of a certain game[2] requires the solution of the following problem:

Is it possible to find n positive numbers a, b, c, $\cdots$, l in such a way that n series

$$\begin{array}{llllll}
(A) & [a], & [2a], & [3a], & [4a], & \cdots \\
(B) & [b], & [2b], & [3b], & [4b], & \cdots \\
(C) & [c], & [2c], & [3c], & [4c], & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \\
(L) & [l], & [2l], & [3l], & [4l], & \cdots
\end{array}$$

reproduce, without repetition, the series of natural numbers 1, 2, 3, 4, $\cdots$? Note: The symbol $[k]$ denotes the largest integer which is less than or equal to k.

The final answer to this question is very simple: namely, excluding the trivial case $n=1$, $a=1$, the problem is possible only when $n=2$, and then a and b must be irrational numbers satisfying the condition: $1/a+1/b=1$.

But the proof of this simple result requires somewhat elaborate analysis, based to a large extent on the following well known theorem due to Kronecker: *Denoting by $a_1, a_2, \cdots, a_k$ and $b_1, b_2, \cdots, b_k$ real numbers, it is possible to find integers $x_0, x_1, x_2, \cdots, x_k$ satisfying the system of inequalities*

$$|x_1 - a_1x_0 - b_1| < \epsilon; \quad |x_2 - a_2x_0 - b_2| < \epsilon, \cdots; \quad |x_k - a_kx_0 - b_k| < \epsilon,$$

where the positive number ϵ may be taken arbitrarily small, provided integers $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_k$, not all zero do not exist, so that

$$\lambda_0 + \lambda_1a_1 + \lambda_2a_2 + \cdots + \lambda_ka_k = 0.^3$$

In the first place it is easy to establish certain necessary conditions which should be fulfilled if the series (A), (B), $\cdots$, (L) satisfy the requirements of our problem.

---

[1] Read before the Minnesota Section of the Association, May 21, 1927.

[2] This game is described in Ahrens' book *Mathematische Unterhaltungen und Spiele* as Wythof's game.

[3] Kronecker, "*Näherungsweise ganzzahlige Auflösung linearer Gleichungen*, Kronecker's Werke, Bd. III, pp. 47–109.

Consider the totality of numbers of all these series which do not exceed an arbitrary positive integer N. As all these numbers are supposed to be different and as all other numbers of the series (A), (B), $\cdots$, (L) are greater than N, it follows that they represent in their totality exactly the numbers 1, 2, 3, $\cdots$, N. Now, the number of all the terms in the series (A) which do not exceed N is obviously given by the *greatest* integer m, satisfying the condition $[ma] \leqq N$, whence the following expression for m can be easily derived:

$$m = (N/a) + \epsilon \text{ where } -1 < \epsilon < 1/a.$$

In the same way we find the following expressions for the number of terms, not exceeding N, in the series (B), (C), $\cdots$, (L) respectively:

$$n = \frac{N}{b} + \eta \text{ where } -1 < \eta < \frac{1}{b};$$

$$p = \frac{N}{c} + \zeta, \text{ where } -1 < \zeta < \frac{1}{c}; \cdots; \ w = \frac{N}{l} + \omega, \text{ where } -1 < \omega < \frac{1}{l}\cdot$$

From the preceding we conclude that the sum $m+n+p+\cdots+w$ is exactly equal to N, that is

$$N(1/a + 1/b + \cdots + 1/l) + \Omega = N \tag{1}$$

where

$$\Omega = \epsilon + \eta + \zeta + \cdots + \omega.$$

Now this number Ω may vary only within the finite limits

$$-n \text{ and } 1/a + 1/b + \cdots + 1/l;$$

dividing therefore both sides of the equation (1) by N and making N increase indefinitely we find that the numbers $a, b, c, \cdots, l$ must satisfy the exact relation:

$$1/a + 1/b + 1/c + \cdots + 1/l = 1. \tag{2}$$

This is the first necessary condition the numbers $a, b, c, \cdots, l$ must satisfy.

The second necessary condition is that not all these numbers are rational. For, supposing them all rational, we can represent them as fractions with the same denominator

$$a = p/s, \ b = q/s, \cdots l = u/s.$$

Let M be the least common multiple of $p, q, \cdots, u$; it is obvious that the following terms in the series (A), (B), $\cdots$, (L) represent one and the same number:

$$\left[\frac{sM}{p}a\right] = M, \quad \left[\frac{sM}{q}b\right] = M, \cdots, \quad \left[\frac{sM}{u}l\right] = M,$$

which is contrary to our hypothesis. One, at least, of the numbers $a, b, \cdots, l$ must be irrational; let it be a. Let us introduce the abbreviated notation $1/a=\alpha$ and correspondingly $1/b=\beta$, $1/c=\gamma, \cdots, 1/l=\lambda$. By Kronecker's theorem we conclude the existence of integers m, n, k satisfying the inequalities

$$|m - k\alpha - \alpha/2| < \epsilon \tag{3}$$

$$|n - k\beta - \beta/2| < \epsilon \tag{4}$$

for an arbitrary small ϵ, *unless β and α are connected by the equation*

$$\beta = s - r\alpha \tag{5}$$

with certain *rational* s and r. By taking $\epsilon<\alpha/2$ and $\epsilon<\beta/2$ the inequalities (3) and (4) imply the following two,

$$k\alpha < m < k\alpha + \alpha, \quad k\beta < n < k\beta + \beta,$$

which are equivalent to

$$k < ma < k + 1, \quad k < nb < k + 1.$$

That is

$$[ma] = [nb] \tag{6}$$

while neither of the numbers ma, nb is an integer.

For positive k the numbers m and n are also positive, and from (6) we conclude the existence of two equal terms in series (A) and (B) respectively, contrary to our hypothesis. For negative k both m and n are negative, and from (6) it follows that

$$[-ma] = [-nb], \tag{7}$$

owing to the fact that neither of the numbers ma, nb is an integer. The equation (7) leads to the same contradiction as before. It is therefore necessary that α and β be connected by the equation (5).

Now we shall prove that $z>0$. Denoting by v the common denominator of the fractions z and s we can find positive integers k' and m' satisfying the inequality $0<m'-k'\alpha<\epsilon/v$ with arbitrary small ϵ (because α is an irrational number) whence, putting $m'v=m$, $k'v=k$, it follows that

$$k\alpha = m - \theta, \quad 0 < \theta < \epsilon. \tag{8}$$

From the equation (5) we get

$$k\beta = ks - mr + r\theta, \tag{9}$$

and $ks-mr=n$ is an *integer*. Supposing $r\leqq 0$ and taking $\epsilon<\alpha$, $|r|\epsilon<\beta$, from (8) and (9) we conclude

$$k\alpha < m < k\alpha + \alpha, \quad k\beta \leqq n < k\beta + \beta$$

or

$$k < ma < k + 1, \quad k \leqq nb < k + 1.$$

That is $[ma] = [nb]$, contrary to hypothesis. Therefore, $r > 0$.

As to the number s, we can prove that $s \leqq 1$. Suppose, on the contrary, $s > 1$ and denote by ϵ a positive number satisfying the inequalities

$$3\epsilon < (s - 1)/r, \quad 3\epsilon < \beta/r. \tag{10}$$

By Kronecker's theorem it is possible to find integers k', m' satisfying the inequality

$$|m' - k'\alpha - gv^{-1}| < \epsilon v^{-1}, \quad g = \alpha - r^{-1}(s - 1) + 2\epsilon$$

whence, for $k = k'v$, $m = m'v$, it follows that $|m - k\alpha - g| < \epsilon$; that is

$$\theta = m - k\alpha = g + \nu\epsilon\,; \; -1 < \nu < 1. \tag{11}$$

At the same time we have

$$k\beta = n - (1 - r\theta), \quad \text{where} \quad n = ks - mr + 1 \quad \text{is an integer.}$$

As $\theta < g + \epsilon = \alpha - r^{-1}(s-1) + 3\epsilon$ and ϵ satisfies the first of the inequalities (10), it follows that $\theta < \alpha$. On the other hand

$$\theta > \alpha - \tfrac{1}{2}(s - 1) + \epsilon = r^{-1}(1 - \beta) + \epsilon > 0$$

because $\beta < 1$, as can be easily seen from (2). As $0 < \theta < \alpha$, we derive from (11)

$$[ma] = k. \tag{12}$$

Considering the number $1 - r\theta$ we find first

$$1 - r\theta > 1 - r[\alpha - r^{-1}(s - 1) + 3\epsilon] = \beta - 3\epsilon r > 0$$

because of the second of the inequalities (10), and second

$$1 - r\theta < 1 - r[\alpha - r^{-1}(s - 1) + \epsilon] = \beta - \epsilon r < \beta,$$

whence we can conclude $[nb] = k$ and further, comparing this with (12)

$$[ma] = [nb] \quad \text{and at the same time} \quad [-ma] = [-nb].$$

That is, we reach again the same contradiction as before, and therefore we must assume $s \leqq 1$.

Now we can express α in terms of β in the following way: $\alpha = sr^{-1} - \beta r^{-1}$, and as β proves to be an irrational number we can reason as before and obtain the important inequality

$$s/r \leqq 1 \text{ or } s \leqq r.$$

Applying the same reasoning to couples of numbers $\gamma, \alpha;\ \delta, \alpha; \cdots$, we finally reach the conclusion: All the numbers $\alpha, \beta, \gamma, \cdots$, are irrational and connected by the equations

$$\begin{array}{lll} \beta = s - r\alpha\,; & s \leqq r, & r > 0 \\ \gamma = s' - r'\alpha\,; & s' \leqq r', & r' > 0 \\ \delta = s'' - r''\alpha\,; & s'' \leqq r'', & r'' > 0 \\ \cdots\cdots & & \\ \cdots\cdots & & \end{array} \tag{13}$$

Now we have the equation (2) $\alpha+\beta+\gamma+\cdots+\lambda=1$, and if we substitute here for $\beta, \gamma, \delta, \cdots$, their expressions (13) we get

$$s + s' + s'' + \cdots - 1 = (r + r' + r'' + \cdots - 1)\alpha,$$

whence, α being an irrational number, it follows necessarily that

$$s + s' + s'' + \cdots = 1 \text{ and } r + r' + r'' + \cdots = 1$$

and this necessitates

$$s = r, \quad s' = r', \quad s'' = r'', \cdots$$

because

$$s \leqq r, \quad s' \leqq r', \quad s'' \leqq r'', \cdots$$

On the other hand, if $n \geqq 3$ we have $r < 1$. The fact that $s = r$ is of fundamental importance; it shows that β is connected with α by the equation $\beta = s(1-\alpha)$ and, interchanging β and α, we have an equation of the same kind,

$$\alpha = \sigma(1 - \beta) \text{ or } \beta = 1 - (\alpha/\sigma).$$

Comparing this expression with the preceding we get

$$1 - (\alpha/\sigma) = s - s\alpha, \text{ whence } s = \sigma = 1.$$

That is $r = 1$. This shows that in case $n \geqq 3$ we reach the contradiction with the inequality $r < 1$ and therefore we have to admit that $n \leqq 2$. Now if $n = 1$ we have a trivial case $a = 1$, but in case $n = 2$ we find that the numbers a and b, both *irrational*, must satisfy the equation $1/a + 1/b = 1$. This necessary condition is at the same time sufficient.

To prove this we only need to establish the following two statements:

(i). For a given m it is impossible to satisfy simultaneously the two inequalities

$$m\alpha < k < m\alpha + \alpha, \qquad m\beta < l < m\beta + \beta$$

by integers k and l. For, adding these inequalities and taking into account that $\alpha+\beta = 1$ we get

$$m < k + l < m + 1$$

which is impossible.

(ii). If it is impossible to satisfy the inequality $m\alpha < k < m\alpha + \alpha$ by an integer k, then it is possible to find an integer l satisfying the inequality $m\beta < l < m\beta + \beta$. In this case it is possible to find an integer k by the conditions $m\alpha + \alpha < k < m\alpha + 1$.

Now introduce for α an equal number $1-\beta$, which gives $m\beta < m+1-k < m\beta + \beta$ and it is obvious that the inequality $m\beta < l < m\beta + \beta$ is satisfied by taking $l = m+1-k$, which completes the proof of our theorem.

TSCHIRNHAUS TRANSFORMATIONS ON CERTAIN RATIONAL CUBICS

By RAYMOND GARVER, University of Rochester

1. **Introduction.** Any rational, algebraic transformation on the roots of an equation of the nth degree can be reduced to an integral transformation of degree[1] less than or equal to $(n-1)$. Such transformations were first studied by Tschirnhaus in the seventeenth century and are named for him.[2] Simple examples are found in the transformations of the elementary theory of equations. Thus a preliminary linear transformation on the roots of any cubic equation can be employed to remove the second term; hence we shall consider the rational cubic only in the form

(1) $$f(x) = x^3 + a_2 x + a_3 = 0, \quad (a_2 \text{ and } a_3 \text{ rational})$$

which has its discriminant $D = -4a_2^3 - 27a_3^2$.

The most general Tschirnhaus transformation on the roots of (1) can be expressed as

(2) $$y = m(x^2 + k_2 x + k_3) = P(x),$$

where m, k_2, k_3 are real or complex numbers and $m \neq 0$. In most of our work we take $m = 1$. Transformation (2) leads to a new equation in y, whose second coefficient will be zero provided

(3) $$\Sigma y = m(\Sigma x^2 + k_2 \Sigma x + 3k_3) = m(s_2 + k_2 s_1 + 3k_3)$$

is equal to zero, where $\sum x^k$ means the sum of the kth powers of the roots of (1) and is denoted by s_k.

However, for equation (1) we have

(4) $$s_1 = 0,\ s_2 = -2a_2,\ s_3 = -3a_3,\ s_4 = -2a_2^2,\ s_5 = 5a_2 a_3,$$
$$s_6 = -2a_2^3 + 3a_3^2.$$

The condition $\sum y = 0$ therefore gives

(5) $$k_3 = 2a_2/3.$$

---

[1] Dickson, *Modern Algebraic Theories*, p. 211.

[2] Tschirnhaus, Acta Eruditorum, vol. 2 (1683), pp. 204–207.

If we square both sides of (2), sum over the roots of (1), and substitute from (4) and (5), we obtain

$$\Sigma y^2 = m^2[(2/3)a_2^2 - 2k_2^2 a_2 - 6k_2a_3], \tag{6}$$

and similarly

$$\Sigma y^3 = m^3[3a_3^2 + (2/9)a_2^3 + 3k_2a_2a_3 + 2k_2^2a_2^2 - 3k_2^3 a_3]. \tag{7}$$

The coefficient of y in the transformed equation will be $-(\sum y^2)/2$, and the constant term will be $-(\sum y^3)/3$.

2. **Reduction to Binomial Form.** Tschirnhaus first reduced the general cubic, which can be taken in form (1), to the binomial form $y^3+n=0$, by a method similar to, though more laborious than, the one employed here.[1] This binomial form can be secured by setting $\sum y^2=0$, solving the resulting quadratic in k_2, and computing $\sum y^3$ for one of these values of k_2. We obtain, after reducing (7) to a linear expression in k_2 by using $\sum y^2=0$, and then substituting

$$k_2 = [-9a_3 \pm (-3D)^{1/2}]/6a_2,$$

the following convenient form:

$$\Sigma y^3 = (1/54)Dm^3a_2^{-3}[3D \pm 9a_3(-3D)^{1/2}]. \tag{7'}$$

The constant term of the transformed equation is then

$$(1/54)Dm^3a_2^{-3}[-D \mp 3a_3(-3D)^{1/2}],$$

which can be made rational for any cubic of type (1) if we allow m to have complex values. It will, on the other hand, be irrational, in general, if we consider only transformations with rational coefficients. In this connection we state

THEOREM I. *A necessary and sufficient condition that a rational cubic* $x^3+a_2x+a_3=0$ *be transformable by a rational Tschirnhaus transformation into* $y^3+R=0$, *where R is rational and different from zero, is that it be of the form*

$$x^3 + 3crm^2x + m^3(c^3r - r^2) = 0 \quad (m, c, r \text{ rational}). \tag{8}$$

PROOF: First, the condition is sufficient. For if to (8) we apply the transformation $y=x^2-mc^2x+2crm^2$, where k_2 and k_3 are selected as indicated above, (using the negative sign in the selection of k_2), we obtain the transformed equation

$$y^3 - r[m^2(c^3 + r)]^3 = 0, \tag{10}$$

which is of the desired form.

[1] Tschirnhaus eliminated x between (1) and (2), and then chose the parameters to obtain the form $y^3+n=0$. The *method* of Article 1–2 has been used by a number of writers but the actual results are believed to be new.

Conversely, suppose that a cubic in x is transformed into

$$y^3 + r = 0. \tag{11}$$

The most general Tschirnhaus transformation on (11) which gives a transformed equation of type (1) is easily seen to be $x=m(y^2+cy)$, and the transformed equation is exactly equation (8), except that here we cannot say definitely that m and c must be rational—they might be irrational, and yet the coefficients in (8) be rational.

We now state a lemma, which is given in a more general case as an exercise in L. E. Dickson's *Modern Algebraic Theories*, page 211:

LEMMA. *If the roots of the transformed equation in y (in the form (11) for example) are distinct, then the roots of the original equation in x can be expressed in the form $x_i=r(y_i)$, where $r(z)$ is a rational function of z whose coefficients are derived rationally from those of the $f(x)$ and $P(x)$.*[1]

For the greatest common divisor $d_i(x)$ of $f(x)$ and $P(x)-y_i$ $(i=1, 2, 3)$ is found by rational operations, so that its coefficients are rational functions of y_i whose coefficients are derived rationally from those of $f(x)$ and $P(x)$. Since both $f(x)$ and $P(x)-y_i$ vanish for $x=x_i$, we have $d_i(x_i)=0$, and $d_i(x)$ is of at least the first degree in x. And it cannot be of higher than the first degree. For x_1, x_2, x_3 are distinct, since $y_1=P(x_1)$, $y_2=P(x_2)$, $y_3=P(x_3)$ are distinct, and $d_i(x)$ would necessarily vanish for two distinct roots of $f(x)=0$, say x_i and x_j. Since $d_i(x)$ is a factor of $P(x)-y_i$, the latter vanishes both for $x=x_i$ and $x=x_j$. That is, $y_j\equiv P(x_j)=y_i$, contrary to hypothesis. Hence $d_i(x)$ is of the first degree.

The G. C. D. process, applied to $f(x)$ and $P(x)-y_i$, will give eventually a divisor of the first degree in x, which, by the lemma, is the G. C. D. This divisor is

$$x(a_2 - k_3 + y_i + k_2^2) + (a_3 + k_2k_3 - k_2y_i).$$

Since it vanishes for $x=x_i$, we have

$$x_i = (k_2y_i - a_3 - k_2k_3)/(y_i + a_2 + k_2^2 - k_3), \tag{12}$$

which can be considered as a transformation leading from the equation in y back to the equation in x. Also we know (see Section 1) that it is equivalent to an integral transformation, of which the coefficients must be rational. This follows since the coefficients in (12), considered as a rational function of y, are rational by hypothesis (though the lemma is obviously valid when this is not the case), and since the process of replacing a rational transformation by the equivalent integral transformation involves only rational operations.[2] Hence the given equation is one which can be obtained from (11) by a Tschirnhaus

[1] $f(x)$ and $P(x)$ are defined by (1) and (2).

[2] See L. E. Dickson, *Modern Algebraic Theories*, p. 211 and p. 151.

transformation, and it is, therefore, after its second term has been removed, of the form (8).

Example. The equation $x^3+6x-2=0$ is of the form (8) with $m=1$, $c=1$, $r=2$. From (10), the transformed equation is at once seen to be $y^3-54=0$. Equation (12) gives $x_i=(6-y_i)/(3+y_i)$, and corresponding to the root $y_1=3\cdot 2^{1/3}$ of $y^3-54=0$ we secure $x_1=(2-2^{1/3})/(1+2^{1/3})$ as a root of the given equation.

3. Reduction to Form $y^3=0$. The lemma of the preceding section, and the argument dependent on it, clearly do not apply if the r of (11) is zero, since the roots of the transformed equation are no longer distinct. We have, however:

THEOREM II. *A necessary and sufficient condition that a rational cubic $x^3+a_2x+a_3=0$ be transformable into $y^3=0$ is that it have a double root.*

PROOF: If we have an equation of the given type transformed by $y=m(x^2+k_2x+k_3)$ into $y^3=0$, we must have $y_i=m(x_i^2+k_2x_i+k_3)=0$ for $i=1, 2, 3$, so that x_1, x_2, x_3 cannot be distinct. Conversely, if the equation $x^3+a_2x+a_3=0$ has a double root, the transformation $y=x^2-(3/2)a_3a_2^{-1}x+(2/3)a_2$, which in general gives the equation $y^3+(1/12)a_2^{-1}Dy-(1/216)a_2^{-3}D^2=0$, clearly leads to $y^3=0$ in this case.

4. A Transformation on Cyclic Cubics. Every cubic with rational coefficients whose Galois group for the field of rational numbers is the cyclic group can be reduced to the form[1] $z^3-3(m^2+m+1)z+(m^2+m+1)(2m+1)=0$, or, setting $2m+1=p$ and $x=2z$, to the form

$$x^3 - 3(p^2+3)x + 2p(p^2+3) = 0. \tag{13}$$

That is, for the cyclic cubic (13) $a_2=-3(p^2+3)$, $a_3=2p(p^2+3)$, and the discriminant is easily calculated to be $18^2(p^2+3)^2$. If we now apply transformation (2), $\sum y^2=0$ has the two roots $k_2=p\pm(-3)^{1/2}$. Choosing the $+$ sign and using (7′), $\sum y^3$ is calculated to be equal to $72(-3)^{1/2}(p^2+3)[p+(-3)^{1/2}]$. If we choose the $-$ sign, we obtain $-72(-3)^{1/2}(p^2+3)[p-(-3)^{1/2}]$. The transformed equation is then

$$y^3 - 24(-3)^{1/2}(p^2+3)[p+(-3)^{1/2}] = 0. \tag{14}$$

Hence we state the following

THEOREM III. *Every rational cubic whose Galois group for the field of rational numbers is the cyclic group can be transformed into a binomial cubic of the form* (14), *where the coefficients of the transformation also involve* $(-3)^{1/2}$, *but no other irrationality.*

---

[1] See Florian Cajori, *Theory of Equations*, p. 197.

The general type of the reduced equation, though certainly not its very simple form, could have been foretold from the results at the beginning of section 2.

If we consider the transformed equation in the form (14), and use (12), where y_i is now a root of (14), we can express the roots of the original cyclic cubic (13) as follows:

$$x_i = [p + (-3)^{1/2}]\frac{y_i + 2(-3)^{1/2}p + 6}{y_i + 2(-3)^{1/2}p - 6}.$$

5. Rational Cubics with a Rational Root. It is easily seen that if a cubic of form (1) has a rational root, it must be of the type $x^3+(n-r^2)x-rn=0$, where r is the rational root and n is also rational. In this case we have

THEOREM IV. *Any rational cubic with a rational root can be transformed, by a rational Tschirnhaus transformation, into an equation of the form* $y^3+sy=0$, *where s is rational.*

The procedure in this case is somewhat different, for we are now seeking to make $\sum y^3$ equal to zero. If we substitute $a_2=n-r^2$, $a_3=-rn$, in (7), the equation $\sum y^3=0$ becomes

$$3rnk_2^3+2(n-r^2)^2k_2^2-3rn(n-r^2)k_2+3r^2n^2+(2/9)\ (n-r^2)^3=0.$$

It can be easily verified, though less easily derived, that this equation has the rational root $k_2=(r^2+2n)/(-3r)$. We note that if r is zero the given equation already has the required normal form, and no transformation is necessary. If we substitute this value of k_2 in (6) and take $m=1$, $\sum y^2$ reduces to $(2/9)r^{-2}(2r^2+n)^2(r^2-4n)$, and the transformed equation is $y^3-(1/9)r^{-2}(2r^2+n)^2\ (r^2-4n)y=0$, which is of the desired form.

BENJAMIN PEIRCE'S LINEAR ASSOCIATIVE ALGEBRA AND C. S. PEIRCE

By RAYMOND CLARE ARCHIBALD, Brown University

Since the greater part of an issue of this Monthly (January, 1925) was devoted to the life and work of Benjamin Peirce, it would seem appropriate to place on record in the same publication a vigorous document of his very able son, the late C. S. Peirce. This document, dated June 28, 1910, is a two page sheet which was in his copy of Jordan's *Traité des Substitutions et des Equations Algébriques*. It was written by Peirce in his seventy-first year. Except for the footnotes, which I have added, the transcription of the document is as follows:

"I will record a reminiscence about this book. It was published in 1870, the same year as the date of the original edition of my father's *Linear Associative*

from it. But he only cut the leaves of the first sheet, and remained to his dying day a superstitious worshipper of two hostile gods, Hamilton and the scalar $\sqrt{(-1)}$. As a professor of mathematics, one would have thought he might have fancied getting some insight to the mathematical advances of his day, most of which have involved the influence of this work; but that wasn't his nature. He was also largely a creature of feeling though his feelings were not of the violent kind. When he died, he left me, as his sole but sufficient legacy (I being the only poor member of the family) his mathematical books, having previously dispossessed himself of every one that he knew I particularly desired. He thought I had a copy of this."

July, 1927.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A GENERALIZATION OF THE STROPHOID

By F. H. HODGE, Purdue University

We start with a circle of radius $a+k$ tangent to the y-axis at the origin and a fixed point A with coordinates $(a, 0)$. A line through A meets the circle in the points P and P'. We lay off on this line $MP=PN=OP$ and $M'P'=P'N'=OP'$. (See Fig. 1) The locus traced by M, N, M', and N' as the line rotates about A (See Fig. 2) is the curve that we are to consider.

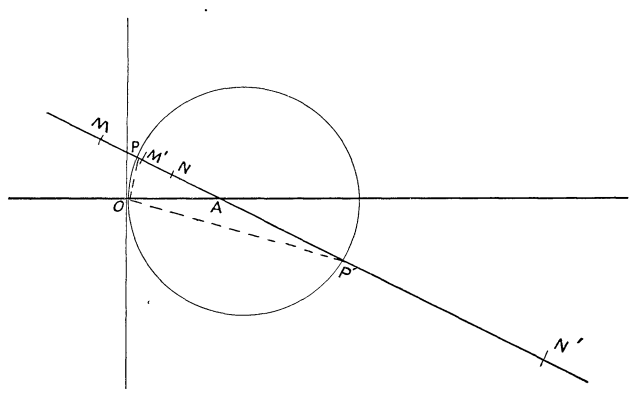


FIG. 1.

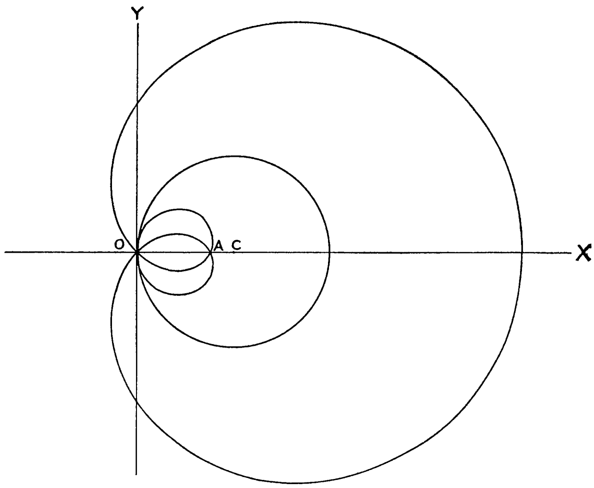


FIG. 2.

To derive its equation we note that the equation of the circle is $x^2-2(a+k)x+y^2=0$ and that the equation of the line is $y=m(x-a)$.

The coordinates of P and P', which we will designate by (ξ, η), must satisfy both of these equations which gives us

$$(1)\qquad \begin{aligned} \xi &= a + \frac{k \pm [(1+m^2)(a^2+2ak)+k^2]^{1/2}}{1+m^2}; \\ \eta &= m\frac{k \pm [(1+m^2)(a^2+2ak)+k^2]^{1/2}}{1+m^2}. \end{aligned}$$

If (x, y) represents a point on the curve in question we have, by the conditions of the problem, $(x-\xi)^2+(y-\eta)^2=\xi^2+\eta^2$ or

$$(2)\qquad x^2 + y^2 - 2x\xi - 2yn = 0.$$

Substituting the values of ξ and η given above we get

$$(3)\quad (1+m^2)(x^2+y^2-2ax) - 2(k \pm [(1+m^2)(a+2ak)+k^2]^{1/2})(x+my) = 0$$

as the required equation. This equation involves the parameters a and k which are characteristic of the curve and also the quantity m which is connected with x and y by the relation $y=m(x-a)$ and hence may be eliminated.

If now the value of k increases indefinitely the circle approaches the y-axis and this construction becomes the classic construction of the Strophoid as is given, for example, in Woods and Bailey's *Analytic Geometry and Calculus*, page 86. In this case we wish finite values of x and y to be determined by finite

values of m and accordingly we use the negative sign with the radical. For an infinite value of k

$$k - [(1 + m^2)(a^2 + 2ak) + k^2]^{1/2}$$

takes the form $\infty - \infty$.
Evaluating this by the methods of indeterminates, it reduces to $-a(1+m^2)$ which, when substituted in (3), reduces it to the form

$$x^2 + y^2 + 2amy = 0$$

or, since

$$m = \frac{y}{x - a},$$

to the form

$$y^2 = x^2\frac{a - x}{a + x},$$

which is a standard form of the equation of the Strophoid.

Returning to equation (3) we note that, since $m=y/(x-a)$, this equation contains only even powers of y, and the curve is symmetric with respect to the x-axis. If $y=0$, $m=0$ unless $x=a$. We wish to determine the points of the curve, other than $(a, 0)$, where it meets the x-axis. If $y=0$ and $m=0$, the equation becomes

$$x[x-2a-2k\pm 2(a+k)]=0.$$

On account of the double sign we have two equations $x(x-2a-2k-2a-2k)=0$ which gives $x=0$, $4(a+k)$ and $x(x-2a-2k+2a+2k)=0$ which gives $x=0, 0$. Thus the origin is a triple point and the curve also cuts the x-axis at $(0, 4a+4k)$ as is evident from the construction.

Consider now the point $(a, 0)$. For these values the equation becomes

$$a(1 + m^2) + k = \pm 2[(1 + m^2)(a^2 + 2ak) + k^2]^{1/2}$$

or, on rationalizing,

$$(1 + m^2)[a^2(1 + m^2) + 2ak - 4(a^2 + 2ak)] = 0,$$

which gives $m = \pm a^{-1}(3a+2k)^{1/2}$ or $\pm i$. This indicates that the point $(a, 0)$ is an ordinary double point and also an isolated double point.

If we put $m=y/(x-a)$ in the equation of the curve and rationalize, we get an equation of degree eight which, however, contains $(x-a)^2+y^2$ as a factor, indicating that $(a, 0)$ is an isolated double point of the curve.

II. On the Origin and Development of the Idea of "Per Cent"

By Bibhutibhusan Datta, University College of Science, Calcutta

"The origin and development of the idea of per cent, including the question of symbolism," has been put as a topic for discussion by Professor David Eugene Smith in his *History of Mathematics*. In fact the book abounds with many such interesting and valuable topics. As regards the question of symbolism, I know nothing more than what has been given by Professor Smith. I can, however, add to the stock of information relating to the origin and development of the idea.

It has been suggested in the above mentioned book that the idea of per cent probably originated with the early Romans (about the beginning of the christian era) who in fixing taxes in certain cases "made use of fractions which easily reduce to hundredths."[1] But in India instances of the actual use of per cents are found many years earlier. Thus in Kauṭilya's *Arthaśāstra*,[2] a work of the 4th century B. C., it is laid down: "An interest of a paṇa and a quarter per month per cent is just. Five paṇas per month per cent is commercial interest (vyāvahārikī). Ten paṇas per month per cent prevails among forests. Twenty paṇas per month per cent prevails among sea traders (sāmudrānām)." (III. 11, p. 214). In restoring valuable lost articles, such as precious stones, the king was used to demand from the owner a ransom amounting to five per cent of their value. (III. 16, p. 233). In case of gambling, betting, etc., five per cent of the stakes won by every winner was to go to the state. (III. 20, p. 241). Again, "The superintendent of commerce shall fix a profit of five per cent over and above the fixed price of local commodities, and ten per cent on foreign produce." (IV. 2, p. 252). Similar instances of the use of per cent are found in other early works of India, especially in the *Smriti Śāstras*.[3]

Thus it appears as an undoubted fact that the idea of per cent was current in India from the 4th century B. C. There are materials to prove that it must have originated a few centuries earlier. The original Sanskrit terminology for "five per cent," for instance, is *pañcakam śatam*, which literally means "five in the hundred." The expression *pañca śatam* means "five hundreds"; it may also mean "one hundred and five." The first expression differs from the second by the addition of the suffix *ka* to the number name. Similarly, *aṣṭakam śatam* means "eight in the hundred" whereas *aṣṭa śatam* means "eight hundreds" or "one hundred and eight"; and so on. Again, *nava-bhāgikam śatam* (or *nava-bhāgyam śatam*) means "one-ninth in the hundred," whereas *nava-*

---

[1] David Eugene Smith, *History of Mathematics*, vol. 2, p. 248.

[2] Kauṭilya's *Arthaśāstra*, edited and translated into English, R. Shamasastry, Mysore.

[3] See *Manu Smriti*, Bombay (1915), viii. 139–142, 152. This work in its present form was written in the first century B. C. The original work was composed much earlier. Compare also, *Yājñavalkya Smriti*, ii. 37, 38; *Kātyāyana*.

bhāgam śatam means "one-ninth of the hundred" or "hundred and one-ninth." The celebrated grammarian Pānini (*c.* 700 B.C.) has laid down rules validating the introduction of the suffix *ka* (also *ya* in case of fractions) to the number names in case of "an interest, a rent, a profit, a tax, or a bribe given" to signify "that there in."[1] Now Pānini must have known instances of the use of expressions of this kind in the literature of his age or anterior to his age. Otherwise he would not have framed rules for regulating the formations of the epithets of the aforesaid kind. Hence it follows that the idea of per cent was known in India in the 7th century B.C., or earlier.

It should also be noted that Pānini's sūtras can have probably wider applications; that is, in their applications the base in computation need not necessarily be hundred, it can be any number whatsoever. But in the absence of a specific instance, specially when we remember the highly compact nature of the Sanskrit Sūtra language, it is difficult to give a definite opinion one way or the other. There is, however, one instance in the *Gotama Sūtra*, where it is stated that "an interest of five *māṣās* per twenty per month is just."[2] Apparently the base is twenty. Though it can be easily made hundred, it is not easy to say whether per cent was being recognized as such in those ages.

In China, the idea of per cent is found for the first time in the second section of the *Chiu-chang Suan-shu* or the "Arithmetic in Nine Sections," "which treats of questions in simple percentage and proportion."[3] The use of per cent is found in the *Hsia-hou Yang Suan-ching* or the "Arithmetical Classic of Hsia-hou Yang" (*c.* 550); and in the *Chang Ch'iu-chien Suan-ching* or the "Arithmetical Classic of Chang Ch'iu-chien" (*c.* 575). In 1299, Chu Shih-chich also treated of examples involving percentage partition. Neither the authorship, nor the time of composition of the "Arithmetic in Nine Sections" can be settled definitely, for by an edict of the despotic emperor Shih Hoang-ti, all books were burned and all scholars were buried in 213 B.C. It has been stated in 263 A.D. by the commentator Liu Hui that the work was rewritten by Chang T'sang (died 152 B.C.), on the basis of some old writings. It was subsequently revised and enlarged by Ching Ch'ou-ch'ang (*c.* 60 B.C.). In such circumstances it cannot be safely asserted that the idea of per cent was known in China before the 2nd century B.C. There is nothing relating to it in the oldest mathematical compendium of the Chinese, *Chou-pei*, whose composition is believed to have commenced in the 12th century B.C. and which in course of time was augmented in volume owing to later interpolations by different writers in different ages.

[1] See *Pānini's Grammar*, v. i. 22, 47, 49.

[2] *Gotama Sūtra*, xii. 26.

[3] Yoshio Mikami, *The Development of Mathematics in China and Japan*, Leipzig (1913), p. 11. My information about the Chinese mathematics is derived from this source.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University to whom books and communications should be sent.

NEW BOOKS RECEIVED

BALL, W. R. *Récréations Mathématiques.* Second French edition: third part. Translated by J. Fitz-Patrick. Paris, J. Hermann, 1927. 363 pages.

BURGESS, R. W. *Introduction to the Mathematics of Statistics.* Boston, Houghton Mifflin Company, 1927. ix+304 pages. $2.50.

CARMICHAEL, R. D. and WEAVER, J. H. *The Calculus.* Boston, Ginn and Company, 1927. xiv+345 pages.

COURNOT, AUGUSTIN. *The Mathematical Principles of the Theory of Wealth.* Translated by N. T. Bacon, with notes and a bibliography by Irving Fisher. New York, The Macmillan Company, 1927. xxiv+209 pages.

DESCARTES, RENÉ. *La Géométrie.* Nouvelle edition. Paris, J. Hermann, 1927. 91 pages.

LAMÉ, G. *Exposé des Méthodes pour Résoudre les Problèmes de Géométrie.* Paris, J. Hermann, 1927. xii+124 pages.

POTRON, ABBÉ. *Exercises de Calcul Différentiel et Intégral.* Second volume, Solutions of the exercises. Paris, J. Hermann, 1927. iii+258 pages.

REVIEWS

Modern Algebraic Theories. By L. E. DICKSON. Chicago, Benjamin H. Sanborn, 1926. ix+276 pages. Price $3.50.

The modern algebra of 1800 was "a thing of shreds and patches." The fundamental theorem of algebra had but lately been proved and certainly was not generally understood. The theory of equations was to a large extent a formal or symbolic matter, for what could Newton's formulae or Cardan's solution mean to a man who had neither Argand's diagram nor Sir William Hamilton's fundamental method of creating complex numbers? The theory of groups was in its infancy, the word invariant was unknown and the phenomenon of invariancy had been observed in only a few special cases, even the possibilities of determinants were unexplored.

Then began the golden age of algebra; Lagrange was its precursor; pregnant words—transformation, group invariant, matrix, hypercomplex, rank—shining names—Gauss, Abel, Galois, Cayley, Jacobi, Hermite, Kronecker, Klein, Frobenius . . . . connote its developments. Whence came this new algebra, the modern algebra of to-day? It has its roots deep in the past, in the questions the Greeks and possibly the Egyptians asked about the regular solids, in the age-old speculations on the nature and meaning of number, in the Arabic solution of the quadratic, and in the terrific struggles of the Italians of the Renaissance with the cubic and the biquadratic. For centuries profound

Mathematics for Engineers. By R. W. DULL. New York, McGraw-Hill Book Company, 1926. xvii+780 pages.

This recapitulation of the mathematics employed in ordinary engineering practice is an expansion of a set of notes compiled by the author for his own convenience, and is intended to be a reference book for engineers. The subjects treated are: numerical computations; algebra, beginning with the rudiments and including cubic equations, logarithms and exponentials, series, determinants, and combinations and permutations; trigonometry, vectors, and hyperbolic functions; plane and solid analytic geometry; calculus, including nine pages on the differentiation of functions of more than one variable. Differential equations are not considered. Much of the material should be familiar to alumni of our engineering schools. Yet some of them may find it convenient to have a book in which they can look up methods which, through disuse, have become unfamiliar. The book is planned to facilitate the acquirement of mathematical technique. It is no text for the student of mathematics as such.

The subject matter is conveniently arranged for reference in fifty-seven chapters and over a thousand numbered articles. The explanations are in simple form, and are illustrated by practical applications. A cursory examination of the 767 pages of text revealed no serious flaws. The illustrative figures are good. But many of them are much too small. Fig. 574 of page 659 is an example.

J. E. TREVOR

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 50, no. 3, July 1927: "Linear ordinary self-adjoint differential equations of the second order," by Anna Pell Wheeler, 309–320; "The singularities of a function defined by a Dirichlet series," by D. V. Widder, 321–328; "On entire function interpolation," by I. M. Sheffer, 329–342; "Optics in space of constant non-vanishing curvature," by James Pierpont, 343–354; "On an imprimitive group of order 5184," by J. R. Musselman, 355–366; "Rational involutorial transformations in S_4 which leave invariant ∞^4 quadric surfaces," by H. C. Shaub, 367–382; "A three-dimensional quartic variety in four-space," by B. C. Wong, 383–388; "Compound singularities of the rational plane quartic curve," by J. H. Neelley, 389–400; "Pencils of conics in the Galois field of order 2^n," by A. D. Campbell, 401–406; "On generalized Lacunae," by J. J. Gergen, 407–418; "Properties of certain aggregate functions," by L. S. Hill, 419–432; "The theory of group-reduced distributions," by J. H. Redfield, 433–455; "Closure of the tangential process on the rational plane cubic," by F. E. Allen, 456–461.

Bulletin of the American Mathematical Society, volume 33, no. 5, September-October 1927: "On the geometry of linear displacement," by D. J. Strink, 523–564; "Concerning the boundaries of domains of a continuous curve," by W. L. Ayres, 565–570; "A theorem on connected point sets," by C. Kuratow-

ski and C. Zarankiewicz, 571–575; "On certain sufficient conditions for the convergence and Cesàro summability of the allied series of a double Fourier series," by G. M. Merriman, 576–582; "A general theorem on quantic determinants," by T. R. Rosebrugh, 583–590; "On the degree of approximation to a harmonic function," by J. L. Walsh, 591–598; "On inductive relations," by C. H. Langford, 599–607; "Three-parameter and four-parameter linear families of conics in the Galois field of order 2^n," by A. D. Campbell, 608–612.

Messenger of Mathematics, volume 57, no. 1, May 1927: "Some remarks on the wave-theory of matter," by Dr. H. Bateman, 7–11.

The Quarterly Journal of Mathematics, volume 50, no. 4, September 1927: "Periodic functions of n variables connected with an algebraic number field of degree n," by E. T. Bell; "The generalized Hessian," by T. R. Hollcroft, 362–372.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3295. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Given a complete quadrilateral it is possible to describe three circles having their centers at the vertices of the diagonal triangle so that the vertices of the quadrilateral shall be the center of similitude of the three circles taken in pairs. Moreover one of the three radii may be taken arbitrarily.

NOTE. This is the converse of the known proposition: The centers of similitude of three given circles taken in pairs are the vertices of a complete quadrilateral. See, for instance, Nathan Altshiller-Court, *College Geometry*, p. 160, Johnson Pub. Co., Richmond, Va., 1925.

3296. Proposed by J. Rosenbaum, Milford, Connecticut.

It is well known that the radius of the inscribed circle of a right triangle is equal to half the difference between the sum of the legs and the hypotenuse. Derive an analogous expression for the radius of the inscribed sphere of a right tetrahedron.

3297. Proposed by C. N. Mills, Normal, Illinois.

Suppose BOD to be a quadrant of a circle of radius R; find the radius of a circle inscribed therein. Also find the radius of a circle which will touch both circles and the line OB.

3298. Proposed by W. A. Thompson, Supt. of Schools, Webster, South Dakota.

Given two circles of unequal sizes, such circles intersecting each other, to draw a secant cutting both circles, such that the chords of the circles are equal to each other and equal to the segment of the line between the two chords.

3299. Proposed by H. W. Reddick, Cooper Institute of Technology, New York, N. Y.

Solve the differential equation

$$\frac{d^{n+1}y}{dx^{n+1}} - \frac{dy}{dx} - \frac{ny}{x} = 0,$$

a generalization of Problem 3227.

3300. Proposed by Emma M. Gibson, High School, Springfield, Missouri.

If u and v are particular integrals of the equation

$$\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0,$$

prove that $y=Au+Bv$ is the complete solution, where

$$A'/v = -B'/u = ce^{\mu}/(u'v-uv')^2 \quad (\mu = -\int Pdx),$$

c being a constant and the accents denoting differentiation with respect to x. (University of Edinburgh Examinations, 1903).

3301. Proposed by J. B. Reynolds, Lehigh University.

From the corners of a square sheet of tin are cut out quadrilaterals so that when the sides are turned up the pan formed will have a maximum volume. Find the top and bottom dimensions of the pan.

SOLUTIONS

Algebra 494 [1917, 466]. Proposed by N. P. Pandya, Sojitra, India.

Solve algebraically and also graphically, log sin x = sin log x.

Solution by Otto Dunkel, Washington University.

The logarithmic base will be taken as e. Setting $x=e^t$, $e^{\sin t}=f(t)$, $\sin e^t=\phi(t)$, the equation becomes $f(t)=\phi(t)$. In the interval from $2n\pi$ to $(2n+1)\pi$, $n=$ an integer, $f(t)$ increases from unity to e and then decreases to unity; in the interval from $(2n-1)\pi$ to $2n\pi$, $f(t)$ decreases from unity to $1/e$ and then increases to unity. In each interval the curve for $f(t)$ is symmetrical about the vertical at the mid-point of the interval. Since $\phi(t)$ is never more than unity, roots can occur only in the interval $(2n-1)\pi \leqq t \leqq 2n\pi$.

It will be shown that there are no negative roots. Set $t'=-t>0$. Then $\sin t'<t'$, $e^{\sin t'}<e^{t'}$, $e^{-t'}<e^{-\sin t'}$, and finally $\sin e^{-t'}<e^{-t'}<e^{\sin(-t')}$, or $\phi(t)<f(t)$. It remains to examine the intervals $(2n-1)\pi \leqq t \leqq 2n\pi$, for $n=1, 2, 3, \cdots$. As t passes from one end of the nth interval to the other, the number of periods of $\sin x=\phi(t)$ is the integral part of $e^{(2n-1)\pi}(e^{\pi}-1)/2\pi$; and the number of roots in the interval is at least twice this integer. The number of roots in the interval increases very rapidly with n. A pair of roots are separated by $k=\log_e[(4m+1)\pi/2]$, where m is a positive integer chosen so that k falls within the interval for t.

The first interval which contains roots, $\pi \leqq t \leqq 2\pi$, will be considered. Here $e^{\pi}=23.14061=7.36591\pi$, $e^{\pi}(e^{\pi}-1)=163.0851\pi$. There are then at least 162 roots in this interval. An examination of the ends of the interval shows that there are precisely 162 roots. To separate the first two roots, we take $m=4$, and then $k=\log_e(17\pi/2)=3.2848$. Since the two roots separated differ very little, this number suggests a first approximation to either root. The usual processes of approximation give the results: first root, $t=3.26633$, $x=26.2150$; second root, $t=3.30539$, $x=27.2593$. The third and fourth roots are separated by 3.4961. The second interval contains about 87,330 roots.

3212. [1926, 429]. Proposed by A. A. Bennett, Lehigh University.

The following scheme portrays an example in ordinary "exact" long division. The symbol x, merely indicates the presence of a digit. The divisor appears upon the left and the quotient on the right. A letter, such as a, b, c, etc., denotes a digit and when one of the letters, (not x) is used in two places, this signifies that the same digit is to be used in both places. However, distinct letters do not necessarily denote distinct digits.

Find the divisor and quotient and prove that your solution is the only one possible.

```
x x x x x a b x x x x ) x x x x x x x x x x x x x c c f x ( x c x x x x x
                        b x x x x x x x x x x
                        ---------------------
                          x x x x x x x x x x x x
                          x x x x b d a x e f x x
                          -----------------------
                            x x x x x x x x x x x c
                            x x x x x x x x x x x e
                            -----------------------
                              x x x x x x x x x x b c
                              x g g g g g g g g g g d
                              -----------------------
                                x x x x x x x x x x a f
                                x x x x x x x x x x x d
                                -----------------------
                                  x x x x x x x x x x b x
                                  x x x x x x x x x x b x
                                  -----------------------
```

Solution by Harry Langman, Brooklyn, N. Y.

The conditions given are somewhat redundant. In this solution we shall assume the letter f where it occurs in the statement of the problem to be replaced by an x, as well as the letter b in the last two lines of the division. In fact, the letter b of the second line may be replaced by an x, provided it is stated that not all the letters other than x are zero.

We shall denote the divisor, D, by $t_1t_2t_3t_4t_5a\ b\ t_8t_9t_{10}t_{11}$ and the quotient by $q_1cq_3q_4q_5q_6q_7$. We have $c=0$. Obviously, $b+e=10$ (since $b\neq 0$) and $a+d=0$ or 10. We shall consider first the possibility $a=b$. Here $ad\neq 0$, $a+d=10$, and $d=e$. The product q_3D is then $xxxxadaxdxxx$. Trying successively the values from 1 to 9 for a we find that the only possibilities are $a=b=3$, $d=e=7$, $q_3=4$ or 7; $a=b=6$, $d=e=4$, $q_3=4$ or 7; $a=b=9$, $d=e=1$. Let $a=3$. If $q_3=4$, $t_5=4$ or 9. For $t_5=4$, no value for q_5 will yield equal g's. For $t_5=9, t_4=0$ or 5, and the g's would be unequal. If $q_3=7, t_5=5, t_4=0$. Again no digit will suffice for q_5. Let $a=6$. If $q_3=4$, $t_5=3$ or 8. Multiplying by 4 will give an odd number to carry, requiring b odd in the product q_3D. But in this case $b=6$. If $q_3=7$, $t_5=0$, and no q_5 will do. Let $a=9$. Hence $g=9$. Here $d=1$; hence $q_5=3$, 7, or 9. For $q_5=7$, the adjacent letters a and b of D would be unequal. For $q_5=3$ or 9 every digit of D, except possibly the units' digit, would be 9. No multiple of such a number will give the indicated product, q_3D. Hence $a\neq b$ in any case, and $d\neq e$.

We obtain D by dividing $rgggggggggggd$ by q_5. Obviously, q_5 cannot be among the digits 2, 4, 5, 8, since this would require $a=b$ in D. Suppose for the moment that $r=g$, and consider the following table:

Cyclic Order of Digits in D.

g	$q_5=3$	$q_5=6$	$q_5=7$	$q_5=9$
1	3 7 0 3 7 0	1 8 5 1 8 5	1 5 8 7 3 0	0 1 2 3 4 5 6 7 9
2	7 4 0 7 4 0	3 7 0 3 7 0	3 1 7 4 6 0	0 2 4 6 9 1 3 5 8
3	1 1 1 1 1 1	5 5 5 5 5 5	4 7 6 1 9 0	0 3 7 0 3 7 0 3 7
4	4 8 1 4 8 1	7 4 0 7 4 0	6 3 4 9 2 0	0 4 9 3 8 2 7 1 6
5	8 5 1 8 5 1	9 2 5 9 2 5	7 9 3 6 5 0	0 6 1 7 2 8 3 9 5
6	2 2 2 2 2 2	1 1 1 1 1 1	9 5 2 3 8 0	0 7 4 0 7 4 0 7 4
7	5 9 2 5 9 2	2 9 6 2 9 6	1 1 1 1 1 1	0 8 6 4 1 9 7 5 3
8	9 6 2 9 6 2	4 8 1 4 8 1	2 6 9 8 4 1	0 9 8 7 6 5 4 3 2
9	3 3 3 3 3 3	6 6 6 6 6 6	4 2 8 5 7 1	1 1 1 1 1 1 1 1 1

This table gives various possibilities for the sequence of digits in D. Also, the quotients for each value of g are corresponding multiples of those for $g=1$. Excepting the cases of equal digits, all possible remainders occur during each division for $q_5=3$ or 9, so that different values for r will merely start the cycle with different digits; and for $q_5=6$, the same set of odd or even remainders occurs for all values

of r different from g, merely rotating the cycle. In the case of equal digits , a different r will also yield equal digits, and we have already excluded this case. For $q_5=7$, except for $2r+g$ divisible by 7, all remainders but one occur on obtaining each cycle. This one occurs when $2r+g$ is divisible by 7, and yields a succession of equal digits, requiring $a=b$, already excluded. Hence, excepting this value for r, the cycle will rotate also for the other values. For $g=0$, the successive digits of D are equal, except when $q_5=7$ and $r\neq 7$, yielding only the cycle 142857, already included in the table. The remaining case for $q_5=7$, where $g=7$ and $r\neq 7$, yields the additional sequence 253968.

With this one exception, the numbers in this table represent then all possible sequences of digits in D. Moreover, it is clear that any multiple of D, such as q_3D, will yield a sequence of digits found in the *same column* with that of D barring a possible succession of equal digits. The latter is barred out for q_3D, since $a\neq b$. Since $a+d=0$ or 10, and no two adjacent digits are 0, $a+d=10$ and $ad\neq 0$. We have then to locate those pairs of adjacent digits (representing da in q_3D) totalling 10. For $q_5=3$ or $q_5=6$, the only possibility is $d=3$, $a=7$. But the digit preceding d, namely b, would then be 0. Hence $q_5\neq 3$ or 6. The remaining possibilities are included in the table below.

$q_5=7$				$q_5=9$			
d	a	b	e	d	a	b	e
7	3	8	1	4	6	2	1
4	6	7	3	9	1	6	5
1	9	6	4	3	7	0	3
2	8	4	7	8	2	3	1
8	2	6	3	2	8	7	9
				6	4	8	9
				1	9	4	5

Here, for example, 73 occurs in the first sequence of the column headed $q_5=7$ in the previous table. The digit before d, 8, in the cycle must represent b, and the next but one after a, 1, must represent e, etc. The last set for $q_5=7$ is found in the exceptional sequence 253968. But we must have $b+e=10$. This excludes $q_5=9$, whence $q_5=7$. The only possibilities are then $d=4$ or 1. The former requires $a=6$, $b=7$. The pair 67 would then occur in D. But no such pair occurs among the permissible sequences under $q_5=7$. Hence $d\neq 4$. Hence the only possibility is $d=1$, $a=9$, $b=6$, $e=4$. Using the multiplier $q_5=7$ on the pair 96 in D, we obtain $g=7$. Since $d=1$, $t_{11}=3$. Proceeding from right to left, we find the remaining digits of D, giving $D=68253968253$, requiring $r=4$. Obviously, $q_1=1$, $q_4=8$, and $q_6=7$. Also q_3 must be 3. Further, the remainder previous to the last ends in $89x$. The product beneath ends in 771. Hence the 9th digit in the final remainder must be 1. For this the only possibility is $q_7=5$. The division is then automatically completed. The quotient is 1038775, and the dividend is readily found to be 70900515872010075.

Also solved by H. C. BRADLEY, J. M. BARBOUR, and F. L. WILMER.

3222. [1926, 481]. Proposed by Norman Anning, University of Michigan.

Under what conditions is it possible to choose five points in space such that the straight line joining any two shall be perpendicular to the plane which contains the remaining three?

SOLUTION BY OTTO DUNKEL, Washington University.

Suppose that the five distinct points A, B, C, D, E, satisfy the conditions of the problem; then no three lie in a straight line and no four lie in a plane. The conditions of the problem will be satisfied if $AE\perp BCD$, $BE\perp CDA$, $CE\perp DAB$, $DE\perp ABC$; for any other straight line, say AB, must then be such that $AB\perp CE$, $AB\perp DE$, and therefore $AB\perp CDE$. Hence if we consider any one of the five tetrahedrons, say $ABCD$, its altitudes must meet in the remaining point, E. Since $AB\perp CDE$, the plan CDE cuts the face ABC in an altitude of the triangle. Hence DE cuts ABC in its orthocenter. Similar results follow for the remaining altitudes of the tetrahedron.

Conversely, to select the five points, we take any three points A, B, C which form a triangle, and at its orthocenter D' we construct the perpendicular to its plane. For D we take any point, other than D', on this perpendicular. Then $DD'\perp ABC$ and thus $DD'\perp AB$. Also by construction $CD'\perp AB$. It now follows that $AB\perp CDD'$ and then that $AB\perp CD$. Similarly, we show that $AC\perp BD$. Since $AB\perp CDD'$, $ABD\perp CDD'$. Hence a perpendicular from C to ABD must lie in CDD' and it cuts DD' in a point which we take for E. It will be shown that A, B, C, D, E, selected in this way, satisfy the sufficient conditions stated above. Since $CE\perp ABD$, $CE\perp BD$; and it was shown above that $AC\perp BD$.

Hence $BD \perp ACE$. Let CE cut the face ABD in C'. Hence the plane ACC' cuts the face ABD in an altitude AC'. Since $AB \perp CDD'$, DC' is also an altitude of the face ABD, and therefore C' is the orthocenter of ABD. Reasoning as above, using first DD' and then CC', we see that the perpendicular from B to ADC must cut each of these lines; hence it passes through their intersection E. Similarly, the perpendicular from A to BCD passes through E, and this completes the proof. See 3187 [3175; 1926, 228].

Also solved by THEODORE BENNETT, W. L. AYRES, NATHAN ALTSHILLER-COURT, MICHAEL GOLDBERG, J. ROSENBAUM, and ROSCOE WOODS.

3226 [1926, 524]. Proposed by H. E. Trefethen, Colby College.

The cross section of a circular ring is crescent shaped. The outer arc is a semicircle. The inner arc has a radius of one inch and its center is at the center of the ring. Find the greatest possible volume for the ring.

SOLUTION BY OTTO DUNKEL, Washington University

The inner and outer arcs of the generating crescent have the equations $y_i^2 = 1 - x^2$ and $y_0 = (1-c^2)^{1/2} + (c^2 - x^2)^{1/2}$, where $2c$ is the length of the common chord. Hence $y_0^2 - y_i^2 = 2(1-c^2)^{1/2}(c^2-x^2)^{1/2}$. If we multiply this by $2\pi dx$ and integrate from 0 to c, we shall obtain the volume V. The result is obviously $\pi(1-c^2)^{1/2}$ times the area of a circle of radius c. Hence $V = \pi^2 c^2 (1-c^2)^{1/2}$. If $c = (2/3)^{1/2}$ we have the maximum value of V, namely, $2\pi^2 \cdot 3^{-3/2}$.

Also solved by H. S. UHLER, V. W. ADKISSON, R. P. AGNEW, T. ANDREW, W. L. AYRES, M. GOLDBERG, D. F. GUNDER, W. F. HUBERT, ELMER LATSHAW, W. K. NELSON, A. PELLETIER, G. E. RAYNOR, C. H. VEHSE, and F. L. WILMER.

3227 [1926, 524]. Proposed by H. W. Reddick, Cooper Union Institute of Technology.

Solve the differential equation

$$\frac{d^5y}{dx^5} - \frac{dy}{dx} - \frac{4y}{x} = 0.$$

SOLUTION BY FREDERIC H. MILLER, Cornell University

Employing the operator D to denote differentiation with respect to x, we may write the equation in the form

$$(D^5 - D - 4x^{-1})y = 0. \tag{1}$$

Assume a solution of the form $y = x^n e^{ax}$, the values of n and a to be determined by substitution in the differential equation. We then get, after dividing through by $x^{n-5} e^{ax}$,

$$a^5x^5 + 5a^4nx^4 + 10a^3n(n-1)x^3 + 10a^2n(n-1)(n-2)x^2 + 5an(n-1)(n-2)(n-3)x$$
$$+ n(n-1)(n-2)(n-3)(n-4) - ax^5 - nx^4 - 4x^4 = 0.$$

Equating the coefficients of the various powers of x to zero, we find that the only permissible values of n and a are $n=1$; $a = \pm 1, \pm i$. We therefore have the following known integrals belonging to the complementary function:

$$y = xe^x;\ y = xe^{-x};\ y = x \sin x;\ y = x \cos x.$$

Now introduce a new variable z by setting $y = xe^{-x} \cdot z$; upon substitution in (1) and reducing, we get

$$[xD^5 - 5(x-1)D^4 + 10(x-2)D^3 - 10(x-3)D^2 + 4(x-5)D]z = 0, \tag{2}$$

an equation of the fourth order. When $y = xe^x$,

$$z = y/xe^{-x} = e^{2x}; \quad Dz = 2e^{2x}.$$

Hence we may put $Dz=e^{2x}\cdot w$, where w is a second new variable. This, when substituted in (2), yields

$$[xD^4+(3x+5)D^3+2(2x+5)D^2+2(x+5)D]w=0. \tag{3}$$

For $y=x\cos x$, there is found in a similar manner $Dw=-2e^{-x}\cos x$; so that we may put $Dw=e^{-x}(\cos x)v$. Substituting in (3), we obtain

$$[x(\cos x)D^3-(3x\sin x-5\cos x)D^2-2(x\cos x+5\sin x)D]v=0. \tag{4}$$

Taking $y=x\sin x$, we find $Dv=-2\sec^2 x$. Thus we may set $Dv=(\sec^2 x)u$; equation (4) then becomes

$$[xD^2+(x\tan x+5)D]u=0.$$

the solution of which is

$$u=A\int x^{-5}\cdot\cos x\cdot dx+B.$$

Hence, retracing the above steps, we get finally

$$y=K_1xe^{-x}\int[e^{2x}\int\{e^{-x}\cos x\int[\sec^2 x\int x^{-5}\cdot\cos x\cdot dx)dx\}dx]dx$$

$$+K_2x\sin x+K_3x\cos x+K_4xe^x+K_1xe^{-x}.$$

Also solved by G. E. RAYNOR, J. B. REYNOLDS, J. D. TAMARKIN, and the PROPOSER.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

SPECIAL NOTE BY THE EDITOR-IN-CHIEF

During the year 1927, thirty six books were reviewed in this Monthly. From the authors of two of them, the editor received letters in reply to criticisms made by the reviewers. One of these letters was published in April (p. 204). The other was not published in this Monthly, but it was distributed recently by the publisher of the book as a circular letter addressed to members of the Association, with the unqualified statement that the editor refused to publish it. The fact of the matter is that the editor, in a note written to the author on May 16, offered to publish the letter if the writer of it would shorten it and eliminate a few objectionable phrases. No reply to that offer was ever received by the editor.

W. H. BUSSEY

Dr. ETHEL L. ANDERTON has been promoted to an assistant professorship of mathematics at Smith College.

Dr. TOBIAS DANTZIG, professor of mathematics at the University of Maryland, will conduct a course in advanced mathematics for physicists and chemists at the Bureau of Standards during the year 1927–28.

Professor H. C. GOSSARD of the University of Wyoming has been appointed professor of mathematics and dean of men at Nebraska Wesleyan University.

Professor CHARLES O. GUNTHER, head of the department of mathematics at Stevens Institute of Technology, has been appointed dean of sophomores to succeed Dr. Sevenoak.

Professor F. R. MOULTON, director of the department of astronomy at the University of Chicago, has resigned, to become associated in an executive capacity with the Utilities Power and Light Corporation of Chicago.

Dr. B. C. PATTERSON has been appointed associate professor of mathematics at Hamilton College.

Associate Professor J. B. REYNOLDS of Lehigh University has been promoted to a full professorship of mathematics and theoretical mechanics.

Miss GEORGIA E. ROBINSON has been appointed professor of physics at Ozark Wesleyan College.

The following appointments to instructorships in mathematics are announced:

Amherst College, Mr. B. LeF. BROWN;
Tulane University, Mr. J. F. THOMSON.

The following have been appointed National Research Fellows in mathematics for 1927–28: W. L. AYRES, R. W. BARNARD, ALONZO CHURCH, C. M. CRAMLET, JESSE DOUGLAS, D. A. FLANDERS, ORRIN FRINK, G. M. MERRIMAN, T. W. MOORE, HILLEL PORITSKY, H. P. ROBERTSON, C. F. ROOS, I. M. SHEFFER, M. M. SLOTNICK, W. M. WHYBURN. (This list includes reappointments).

CONTENTS

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

VOLUME XXXIV, 1927

NUMBER 10, DECEMBER SUPPLEMENT

PRICE 50 CENTS

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND MINNEAPOLIS, MINN.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members; $5.00 a Year, Single Copies 60 cents, to others.

The Chauvenet Prize

The ASSOCIATION has established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The first award, covering the five years preceding 1925, was made to GILBERT AMES BLISS for his paper on "Algebraic Functions and their Divisors" published in the *Annals of Mathematics*. The committee of award consisted of Professor W. C. Graustein, Harvard University, Anna J. Pell-Wheeler, Bryn Mawr College, and E. B. Van Vleck, Chairman, University of Wisconsin.

The next award will be made in 1930 covering the five years preceding.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION,—one more of the many good reasons for membership.

THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

FOR THE ACADEMIC YEAR

1927-1928

MENASHA, WIS. AND MINNEAPOLIS, MINN.
PUBLISHED BY THE ASSOCIATION
1927

For a list of all officers and Council Members or Trustees for preceding years see page 60.

* Not a member of the Board.

ASSOCIATE EDITORS OF THE OFFICIAL JOURNAL FOR 1927.

H. E. BUCHANAN, Tulane University.
ELIZABETH CARLSON, University of Minnesota.
W. B. CARVER, Cornell University.
OTTO DUNKEL, Washington University.
H. J. ETTLINGER, University of Texas.
H. S. EVERETT, University of Chicago.
B. F. FINKEL, Drury College.
H. W. KUHN, Ohio State University.
C. N. MILLS, Illinois State Normal University.
F. D. MURNAGHAN, Johns Hopkins University.
H. L. OLSON, Michigan State College.
D. E. SMITH, Columbia University.

ASSOCIATE EDITORS OF THE ANNALS OF MATHEMATICS.
(*representing the Association*)

W. B. FORD, University of Michigan.
Mrs. ANNA PELL WHEELER, Princeton, N. J.

SECTIONS OF THE ASSOCIATION.

ILLINOIS.

C. N. MILLS, Illinois State Normal University, *Chairman.*
A. E. GAULT, Bradley Polytechnic Institute, *Vice-Chairman.*
BESSIE I. MILLER, Rockford College, *Secretary-Treasurer.*
E. H. TAYLOR, Eastern Illinois State Teachers College, *Member Executive Committee.*

INDIANA.

J. E. DOTTERER, Manchester College, *Chairman.*
E. D. GRANT, Earlham College, *Vice- Chairman.*
H. T. DAVIS, Indiana University, *Secretary-Treasurer.*

IOWA.

ROSCOE WOODS, University of Iowa, *Chairman.*
E. E. MOOTS, Cornell College, *Vice-Chairman.*
J. F. REILLY, University of Iowa, *Secretary-Treasurer.*

KANSAS.

U. G. MITCHELL, University of Kansas, *Chairman.*
H. S. MYERS, Southwestern College, *Vice-Chairman.*
LUCY T. DOUGHERTY, Kansas City Junior College, *Secretary-Treasurer.*

KENTUCKY.

GUY STEVENSON, University of Louisville, *Chairman.*
A. R. FEHN, Centre College, *Secretary-Treasurer.*

LOUISIANA-MISSISSIPPI.

S. T. SANDERS, Louisiana State University, *Chairman.*
J. A. HARDIN, Centenary College, *Vice-Chairman.*
P. K. SMITH, Mississippi Woman's College, *Secretary-Treasurer.*

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.

J. R. MUSSELMAN, Johns Hopkins University, *Chairman.*
E. C. PHILLIPS, Georgetown University, *Secretary-Treasurer.*
T. H. TALIAFERRO, University of Maryland, *Member Executive Committee.*
C. C. BRAMBLE, U. S. Naval Academy, *Member Executive Committee.*

MICHIGAN.

E. R. SLEIGHT, Albion College, *Chairman.*
N. H. ANNING, University of Michigan, *Secretary-Treasurer.*
L. C. PLANT, Michigan State College, *Member Executive Committee.*

MINNESOTA.

Sister M. CLAUDETTE, College of St. Benedict, *Chairman.*
A. L. UNDERHILL, University of Minnesota, *Secretary-Treasurer.*
INEZ RUNDSTROM, Gustavus Adolphus College, *Member Executive Committee.*
F. J. TAYLOR, College of St. Thomas, *Member Executive Committee.*
A. BOGARD, College of St. Teresa, *Member Executive Committee.*

MISSOURI.

KATHRYN WYANT, University of Missouri, *Chairman.*
W. H. ROEVER, Washington University, *Vice-Chairman.*
P. R. RIDER, Washington University, *Secretary-Treasurer.*

NEBRASKA.

W. C. BRENKE, University of Nebraska, *Chairman.*
ELLEN H. FRANKISH, North High School, Omaha, *Secretary-Treasurer.*
H. C. FEEMSTER, York College, *Member Executive Committee.*

OHIO.

C. H. YEATON, Oberlin College, *Chairman.*
RUFUS CRANE, Ohio Wesleyan University, *Secretary-Treasurer.*
C. T. BUMER, Ohio State University, *Member Executive Committee.*

PHILADELPHIA SECTION.

H. H. MITCHELL, University of Pennsylvania, *Chairman.*
J. R. KLINE, University of Pennsylvania, *Secretary-Treasurer* pro tem.

ROCKY MOUNTAIN SECTION.

W. J. RISLEY, Colorado School of Mines, *Chairman.*
G. W. FINLEY, Colorado State Teachers College, *Vice-Chairman.*
PHILIP FITCH, North Side High School, Denver, *Secretary.*
G. H. LIGHT, University of Colorado, *Treasurer.*

SOUTHEASTERN SECTION.

J. B. COLEMAN, University of South Carolina, *Chairman.*
A. W. HOBBS, University of North Carolina, *Vice-Chairman.*
W. W. RANKIN, Duke University, *Secretary-Treasurer.*

SOUTHERN CALIFORNIA SECTION.

W. P. RUSSELL, Pomona College, *Chairman.*
E. T. BELL, California Institute of Technology, *Vice-Chairman.*
P. H. DAUS, University of California at Los Angeles, *Secretary-Treasurer.*

TEXAS.

W. H. WHYBURN, Texas Technological College, *Vice Chairman.*
H. J. ETTLINGER, University of Texas, *Secretary-Treasurer.*

INDIVIDUAL MEMBERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

Closed for printing November 15, 1927.

Note. This list gives data for the members of the Association as taken from the register issued during January, 1926, unless changes in position, mailing address, etc., have been reported to the Secretary. Records may be inexact or incomplete through the failure of members to return fresh information to the Secretary, but great efforts have been put forth to be full and correct. When members are known to be occupied in other lines than mathematics, this fact is indicated.

ADAMS, B. T., A.M. (Baylor). Head of Dept., Jr. Coll., Wichita Falls, Tex.
ADAMS, Asst. Prof. C. R., Ph.D. (Harvard). Brown Univ., Providence, R. I. *260 Doyle Ave.*
ADAMS, Prof. E. P., Ph.D. (Harvard). Princeton Univ., Princeton, N. J.
ADAMS, O. S., Sc.D. (Kenyon). Geodetic Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.
ADKINS, Prof. L. K., M.S. (Chicago). Head of Dept., State Teachers Coll., La Crosse, Wis.
ADKISSON, V. W., A.B. (Drury). Grad. Student and Instr., Univ. of Pennsylvania, Philadelphia, Pa. *3739 Walnut St.*
AGARD, Asst. Prof. H. L., Ph.D. (Yale). Williams Coll., Williamstown, Mass.
AGNEW R.P., M.S. (Iowa State). Instr., Cornell Univ., Ithaca, N. Y. *Dept. of Math.*
AITCHISON, BEATRICE, Senior, Goucher Coll., Baltimore, Md. *1929 S St., Washington, D. C.*
AKERS, Prof. O. P., Ph.D. (Cornell). Allegheny Coll., Meadville, Pa. *360 N. Main.*
ALBERT, J. P., A.B. (Miami). Instr., Miami Univ., Oxford, Ohio.
ALBERT, Prof. O. W., A.M. (Columbia). Univ. of Redlands, Redlands, Calif. *629 Buena Vista St.*
ALBRIGHT, Prof. G. H., A.M. (Harvard). Math. and Astr., Colorado Coll., Colorado Springs, Colo.
ALDERTON, Asso. Prof. NINA M., Ph.D. (California). Mills Coll., Calif.
ALEXANDER, Asso. Prof. J. W., Ph.D. (Princeton). Princeton Univ., Princeton, N. J. *29 Cleveland Lane.*
ALEY, Pres. R. J., Ph.D. (Penna.). Butler Univ., Indianapolis, Ind.
ALICE IRENE, Sister, A.M. (Columbia). Instr., Coll. of St. Catherine, St. Paul, Minn.
ALLEN, Prof. B. R., M.S. (Chicago). Galloway Coll., Searcy, Ark.
ALLEN, Prof. E. E., A.M. (Park Coll.). Occidental Coll., Los Angeles, Calif.
ALLEN, Asst. Prof. E. F., Ph.D. (Missouri). Univ. of Missouri, Columbia, Mo.
ALLEN, FLORENCE E., Ph.D. (Wisconsin). Instr., Univ. of Wisconsin, Madison, Wis.
ALLEN, GERTRUDE E., B.S. (California). Supervisor of Math., Univ. High School, Oakland, Calif.
ALLEN, Asso. Prof. JOSEPH, A.M. (Harvard). Coll. of the City of New York, New York, N. Y.
ALLEN, Prof. R. B., Ph.D. (Clark). Kenyon Coll., Gambier, Ohio.
ALLISON, Prof. ELI, M.S. (Wesleyan). Brenau Coll., Gainesville, Ga.
AMES, Prof. L. D., Ph.D. (Harvard). Univ. of Southern Calif., Los Angeles, Calif. *1328 W. 37th Drive.*
AMICK, Prof. T. C., Ph.D. (Central Univ.). Elon College, N. C.
AMIG, MARGARET, A.M. (Wellesley). Teacher, St. Catherine's School, Westhampton, Richmond, Va.
AMMERMAN, CHARLES, A.M. (Illinois). Teacher, Roosevelt High School, St. Louis, Mo.
ANDERSON, Prof. HARRIET, B.S. (Iowa State). Grand Island Coll., Grand Island, Nebr.
ANDERSON, H. E., A.M. (Augustana). Teacher, High School, Atascadero, Calif.
ANDERSON, Prof. MARY, A.M. (Illinois). Illinois Woman's Coll., Jacksonville, Ill.
ANDERSON, Prof. W. E., Ph.D. (Penna.). Miami Univ., Oxford, Ohio.
ANDERTON, Asst. Prof. ETHEL L., Ph.D. (Yale). Smith Coll., Northampton, Mass.
ANDREW, THURMAN, M.S. (Kentucky). Instr., Wentworth Inst., Boston, Mass.
ANDREWS, ANNA H., Ph.B. (Wesleyan). Teacher, Public High School, Hartford, Conn. *167 Beacon St.*
ANDREWS, C. W., A.M. (Harvard), LL.D. (Northwestern). Librarian, The John Crerar Libr., Chicago, Ill. *86 E. Randolph St.*
ANDREWS, VIOLET M., A.B. (Oberlin). Fellow, Northwestern Univ., Evanston, Ill. *718 Simpson St.*
ANDREWS, Prof. W. H., Ph.D. (Chicago). Educ., State Agric. Coll., Manhattan, Kans.
ANGELICA, Sister M., Ph.D. (Fordham). St. Joseph Coll. for Women, Brooklyn, N. Y. *245 Clinton Ave.*
ANKEBRANT, J. A., A.M. (Michigan). Head of Dept., Western State Coll., Gunnison, Colo.
ANNING, Asst. Prof. N. H., M.A. (Queen's). Univ. of Michigan, Ann Arbor, Mich. *1313 Henry St.*
ARCHER, Prof. PETER, S. J. Canisius High School, Buffalo N. Y. *651 Washington St.*
ARCHIBALD, Prof. R.C., Ph.D. (Strasbourg), Dr. Univ. Padua, LL.D. (Mt. Allison). Brown Univ., Providence, R. I.

ARMSTRONG, BEULAH M., Ph.D. (Illinois). Univ. of Illinois, Urbana, Ill. *364 Math. Bldg.*
ARMSTRONG, Prof. P. L., A.M. (Southwestern Presb. Univ.). W. Tennessee Teachers Coll., Memphis, Tenn.
ARNAUD, J. J., B.S. (Coll. City of N. Y.). Asst. Astronomer, U. S. Naval Observatory, Washington, D. C. *2520 Hall Pl. N. W.*
ARNDT, Prof. W. F. C., Ph.D. (Göttingen). Grey Univ. Coll., Univ. of S. Africa, Bloemfontein, S. Africa.
ARNOLD, Prof. C. L., M.S. (Ohio State). Ohio State Univ., Columbus, Ohio.
ARNOLD, Asst. Prof. H. E., A.M. (Wesleyan). Wesleyan Univ., Middletown, Conn.
ARNOLD, KATHERINE S., A.M. (Columbia). Registrar, Hood Coll., Frederick, Md.
ARNOLD, Asst. Prof. W. C., M.S., A.M. (Chicago). DePauw Univ., Greencastle, Ind. *Box 466.*
ASHCRAFT, Prof. T. B., Ph.D. (Johns Hopkins). Colby Coll., Waterville Me. *34 Pleasant St.*
ASHMUN, R. N., A.M. (Washington). Mathematician, Int. Boundary Commission, Washington, D. C. *205 New Jersey Ave.*
ASHTON, Prof. C. H., Ph.D. (Munich). Univ. of Kansas, Lawrence, Kans.
ATCHISON, Prof. C. S., Ph.D. (Johns Hopkins). Washington and Jefferson Coll., Washington, Pa. *442 E. Bean St.*
ATKIN, Asst. Prof. EDITH I., A.M. (Columbia). Illinois State Normal Univ., Normal, Ill. *611 Hester Ave.*
ATKINSON, Asst. Prof. JULIA F., A.M. (California). Univ. of Arizona, Tucson, Ariz. *Box 46, Univ. Sta.*
ATWATER, FRANCES A., A.M. (Cornell). Teacher, Columbus School for Girls, Columbus, Ohio.
AUDE, Asso. Prof. H. T. R., M.S. (Colgate). Colgate Univ., Hamilton, N. Y. *50 Broad St.*
AUERBACH, MATILDA, Grad. Teachers Coll. Ethical Culture High School, New York, N. Y. *33 Central Park W.*
AUSTIN, ALICE M., A.M. (North Dakota). Head of Dept., Jr. Coll., Independence, Kans. *Box 582.*
AUSTIN, W. A., A.M. (Indiana). Union High School, Venice, Calif.
AVERS, H. G., A.B. (George Washington). Geodetic Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.
AYRES, FRANK, M.S. (Chicago). Instr., A. and M. Coll., College Station, Tex. *Box 139.*
AYRES, W. L., Ph.D (Penn.). National Research Fellow, Univ. of Texas, Austin, Tex. *Univ. Sta.*

BABBITT, A. E., A.M. (Illinois). Actuary, Lamar Life Ins. Co., Jackson, Miss.
BABCOCK, Miss WEALTHY, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans, *1238 Miss.*
BACON, Prof. CLARA L., Ph.D. (Johns Hopkins). Goucher Coll., Baltimore, Md. *2316 N. Calvert St.*
BAGBY, Prof L. C., A.M. (Kansas). Linsly Inst. of Tech., Wheeling, W. Va.
BAIDAFF, B. I., Dr. (Buenos Aires). Analisis, Univ. Bs. As., Buenos Aires, Argentina. *Belgrano 887.*
BAILEY, Prof. E. A., M.S. (Emory). Dean and Registrar, Math., La Grange Coll., La Grange, Ga. *701 Vernon St.*
BAILEY, Prof. F. H., A.M. (Harvard). Mass. Inst. of Tech., Cambridge, Mass.
BAILEY, H. W., Ph.D. (Illinois). Instr., Univ. of Illinois, Champaign, Ill. *[illegible] S. 1st St.*
BAIRD, A. C., A.B. (Wooster). Vice-Prin., Fifth Ave. High School, Pittsburgh, Pa.
BAKER, FRANCES E., M.S. (Iowa). 929 Kirkwood Ave., Iowa City, Ia.
BAKER, Prof. R. P., Ph.D. (Chicago). Univ. of Iowa, Iowa City, Ia. *929 Kirkwood Ave.,*
BALDWIN, Asst. Prof. J. W., A.M. (Michigan). Registrar, Coll. of City of Detroit, Detroit, Mich. *6433 Roosevelt Ave.*
BALLANTINE, Asst. Prof. J. P., Ph.D. (Chicago). Univ. of Washington, Seattle, Wash. *4521 6th Ave. N. E.*
BALOF, C. A., M.S. (Iowa). Instr., Howard Coll., Birmingham, Ala. *7503 Underwood Ave.*
BANES, GLADYS L., A.B. (Butler). Instr., Butler Coll., Indianapolis, Ind.
BARBER, H. C., A.M. (Amherst). Teacher, Phillips Exeter Acad., Exeter, N. H. *75 Court St.*
BAREIS, Asst. Prof. GRACE M., Ph.D. (Ohio State). Ohio State Univ. *Canal Winchester, Ohio.*
BARNARD, Asst. Prof. R. W., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
BARNES, L. S., A.B. (Occidental). Asst. Instr., Lehigh Univ., Bethlehem, Pa. *715 4th Ave.*
BARNES, W. N., B.S. (Chicago). Prin., High School, Aransas Pass, Tex.
BARNETT, Asst. Prof. I. A., Ph.D. (Chicago). Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math.*
BARNEY, IDA, Ph.D. (Yale). Research Asst., Yale Univ. Observ., New Haven, Conn. *110 Linden.*
BARNHART, Prof. C. A., A.M, (Illinois). Univ. of New Mexico, Albuquerque, N. M. *115 S. Maple St.*
BARNHILL, Prof. J. F., A.M. (Columbia). Michigan State Normal Coll., Ypsilanti, Mich.
BARR, Asst. Prof. C. F., M.S. (Chicago). Univ. of Wyoming, Laramie, Wyo. *515 S. 8th St.*
BARRETT, A. J., A.B. (Arkansas). Instr., New York Univ. *8521 159th St., Jamaica, N. Y.*
BARRICK, Prof. D. L., A.B. (Okla. Baptist Univ.). Montezuma Coll., Montezuma, N. Mex.
BARROW, Prof. DAVID, Ph.D. (Harvard). Univ. of Georgia, Athens, Ga. *436 Dearing St.*
BARTKY, WALTER, Ph.D. (Chicago). Instr., Univ. of Chicago, Chicago, Ill.

BARTON, Asso. Prof. HELEN, Ph.D. (Johns Hopkins). North Carolina Coll. for Women, Greensboro, N. C.
BASOCO, M. A., M.S. (Chicago). Asst., Calif. Inst. of Tech., Pasadena, Calif.
BASSLER, KATHARINE R., A.M. (Bryn Mawr). Teacher, Bryn Mawr School for Girls, Baltimore, Md. *919 Cathedral St.*
BATCHELDER, Adj. Prof. P. M., Ph.D. (Harvard). Pure Math., Univ. of Texas, Austin, Tex. *808 W. 22nd St.*
BATEMAN, Prof. HARRY, Ph.D. (Johns Hopkins). Calif. Inst. of Tech., Pasadena, Calif. *R.F.D. 4, Box 163.*
BATEN, Asso. Prof. W. D., A.M. (Texas). Univ. of Tennessee, Knoxville, Tenn. *2032 Rose Ave.*
BATTIG, LEON, A.B. (Wisconsin). Instr. and Grad. Student, Oberlin Coll., Oberlin, Ohio. *178 N. Professor St.*
BAUDIN, Asso. Prof. M. C., A.M. (Ohio State). Romance Languages, Miami Univ., Oxford, Ohio. *402 E. Church.*
BAUER, Asso. Prof. G. N., Ph.D. (Columbia). Univ. of New Hampshire, Durham, N. H.
BAUER, L. A., Ph.D. (Berlin). Dir., Dept. of Terrestrial Magnetism, Washington, D. C. *36th St. and Broad Branch Rd.*
BAUMAN, Prof. J. A., Ph.D., D.D. (Muhlenberg). Emeritus, Muhlenberg Coll., Allentown, Pa. *399 Turner St.*
BAUR, Asst. Prof. P. E., A.B. (Baldwin-Wallace). Math. and Drawing, Baldwin-Wallace Coll., Berea, Ohio. *114 E. Grand St.*
BAYS, Prof. SÉVÉRIN. Faculté des Sciences, Univ. de Fribourg, Bethlehem, Fribourg, Switzerland.
BEAL, Asst. Prof. W. O., Ph.D. (Chicago). Astr., Univ. of Minnesota, Minneapolis, Minn.
BEAR, A. W., B.S. (Marquette). Instr., Marquette Univ., Milwaukee, Wis. *858 Scott St.*
BEATLEY, Asst. Prof. RALPH, A.M. (Columbia). Harvard Grad. School of Educ., Cambridge, Mass.
BEATTY, Asst. Prof. H. M., A.M. (Ohio State). Ohio State Univ., Columbus, Ohio. *200 Tibet Rd.*
BEATTY, Asst. Prof. SAMUEL. Univ. of Toronto, Toronto, Can.
BEATY, Asso. Prof. E. B., A.M. (California). Oregon Agric. Coll., Corvallis, Ore.
BEAUPRÉ, Prof. V.-E., B.A. (Manitoba). Calc. and Anal. Geom. École Polytech., and Univ. de Montreal, Montreal, Can. *228 St. Denis St.*
BECKWITH, Prof. ETHELWYNN R., Ph.D. (Radcliffe). Milwaukee-Downer Coll., Milwaukee, Wis.
BECKWITH, Prof. W. S., A.M. (Harvard). Georgia State Teachers Coll., Athens, Ga. *731 Cobb St.*
BEENKEN, MAY M., A.M. (Chicago). Fellow, Univ. of Chicago, Chicago, Ill. *R 8, Green Hall.*
BEETLE, Prof. R. D., Ph.D. (Princeton). Dartmouth Coll., Hanover, N. H.
BEILER, A. H., B.S. in E.E. (Cooper Union). Eng., Amer. Gas and Elec. Co., 30 Church St., New York, N. Y.
BEISEL, B. R., B.S. (Allegheny). Instr., Cornell Univ., Ithaca, N. Y. *521 E. State St.*
BELL, CLIFFORD, Ph.D. (California). Instr., Univ. of California at Los Angeles, Los Angeles, Calif. *2414 Burnside Ave.*
BELL, Prof. E. T., Ph.D. (Columbia). Calif. Inst. of Tech., Pasadena, Calif.
BELL, Prof. TALMON, A.B. (Sterling). Sterling Coll., Sterling, Kan.
BELLAMY, B. C., B.S., C.E. (Wyoming). Civil Eng., Laramie, Wyoming.
BENANDER, C. A., A.B. (Havard). Instr., Georgia School of Tech., Atlanta, Ga.
BENDER, H. A., Ph.D. (Illinois). Univ. of Illinois, Urbana, Ill. *403 N. Mathews.*
BENEDICT, Pres. H. Y., Ph.D. (Harvard). Univ. of Texas, Austin, Texas.
BENEDICT, Prof. SUZAN R., Ph.D. (Michigan). Smith Coll., Northampton, Mass. *12 Barrett Pl.*
BENNETT, Prof. A. A., Ph.D. (Princeton). Brown Univ., Providence, R. I.
BENNETT, Prof. J. L., A.M. (Kansas). Physics, Ottawa Univ., Ottawa, Kans.
BENNETT, T. L., Ph.D. (Illinois). Instr., Univ. of Illinois, Urbana, Ill.
BERGER, EDLA G., A.M. (Minnesota). Mathematician, Actuarial Dept., Equitable Life Assurance Soc., New York, N. Y. *14-11 161st St., Beechhurst, L. I., N. Y.*
BERGSTRESSER, C. A., A.M. (Lafayette), M.S. (Pennsylvania). Head of Dept., Jamaica High School, New York, N. Y. *156 Macon St. Brooklyn, N. Y.*
BERKELEY, L. M., A.M. (Virginia). 36 W. 91st St., New York, N. Y.
BERNSTEIN, Asso. Prof. B. A., Ph.D. (California). Univ. of California, Berkeley, Calif. *2785 Shasta Rd.*
BERRY, Prof. E. M., Ph.D. (Iowa). Lynchburg Coll., Lynchburg, Va. *504 Westwood Ave.*
BERRY, GRACE E., A.M. (Mt. Holyoke). Dean of Women, Pomona Coll., Claremont, Calif.
BERRY, Prof. W. J., A.M. (Harvard). Poly. Inst. of Brooklyn, Brooklyn, N. Y. *224 St. John's Pl.*
BERRY, W. J., M.S. (Colorado). Jr. Physicist, U. S. Bureau of Standards, Washington, D. C. *18 Seaton Pl. N. W.*
BERT, Prof. O. F. H., A.M. (Geneva). Washington and Jefferson Coll., Washington, Pa. *28 N. Lincoln St.*
BETANCOURT DEL VALLE, R., Commercial Eng. (Inst. Polytech. of Seville), B.C.S. (Lincoln-Jefferson Univ.). Accountant, Plazuela Sugar Co. and subsidiary companies, San Juan, P. R. *P. O. Box 1343.*
BETTINGER, A. K., A.M. (Wisconsin). Instr., Creighton Univ., Omaha, Nebr.

BETZ, WILLIAM, A.M. (Rochester). Vice-Prin., East High School, Rochester, N. Y. *652 Melville St.*
BIANCHI, Prof. L. Univ. of Pisa, Pisa, Italy. *Via Manzoni No. 3.*
BIBB, Asst. Prof. S. F., M.S. (Chicago). Armour Inst. of Tech., Chicago, Ill.
BICKFORD, JEANNETTE, A.M. (Radcliffe). Head of Dept., High School, Framingham, Mass. *The High School.*
BIGBEE, JOHN A., B.S. (Drury). High School, Little Rock, Ark. *3110 Battery St.*
BIGELOW, W. W., A.B. (Beloit). Rockton, Ill.
BIGGERSTAFF, JOHN, A.B. (Reed). Grad. Student, Stanford University. *250 Cowper St., Palo Alto, Calif.*
BILL, Prof E. G., Ph.D. (Yale). Dean of Freshmen, Dartmouth Coll., Hanover, N. H.
BINGLEY, Asso. Prof. G. A., A.M. (Princeton). St. John's Coll., Annapolis, Md. *240 Prince George St.*
BINNEY, Asst. Prof. J. H., A.M. (Texas). A. and M. Coll. of Texas, College Station, Tex. *Box 107, Main Bldg.*
BIRCHBY, W. N., A.M. (Colo. Coll.). 1500 Sinaloa Ave., Pasadena, Calif.
BIRCHENOUGH, Prof. HARRY, A.M. (Columbia). State Coll. for Teachers, Albany, N. Y. *687 Hudson Ave.*
BIRKHOFF, Prof. G. D., Ph.D. (Chicago). Harvard Univ., Cambridge, Mass. *984 Memorial Drive.*
BISHOP, DEAN F. L., Ph.D. (Chicago). Eng. and Mines, Univ. of Pittsburgh, Pittsburgh, Pa.
BIXBY, Brig. Gen. W. H., U. S. Army, retired, Washington, D. C. *1709 Lanier Pl., N. W.*
BLACK, FLORENCE, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans. *1300 La. St.*
BLACKBURN, E. P., A.B. (Oakland City Coll.). Teacher, High School, Mt. Vernon, Ind.
BLAIR, Prof. HAROLD, A.M. (Michigan). Western State Normal Coll., Kalamazoo, Mich. *1220 Academy St.*
BLAIR, Asst. Prof. LEORA. State Normal Coll., Natchitoches, La.
BLAIR, Asst. Prof. R. V., A.M. (Peabody). Vanderbilt Univ., Nashville, Tenn.
BLAIR, Miss VEVIA, A.M. (Columbia). Teacher, Horace Mann School, New York, N. Y. *430 W. 119th St.*
BLAU, L. W., A.M. (Texas). Instr., Physics, Univ. of Texas, Austin, Tex. *2419 Duval.*
BLICHFELDT, Prof. H. F., Ph.D. (Leipzig). Stanford University, Calif.
BLINCOE, Asso. Prof. J. W., A.M. (Virginia). Math. and Physics, Randolph-Macon Coll., Ashland, Va. *Box 365.*
BLISS, Prof. G. A., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
BLOM, Prof. E. C., A.M. (Missouri). Des Moines Univ., Des Moines, Ia. *Grad. Student, Columbia Univ. 1111 Amsterdam Ave., New York, N. Y.*
BLUE, Prof. A. H., M.S. (Iowa). Western Union Coll., Le Mars, Ia.
BLUMBERG, A. A. Instr., A. and M. Coll., College Station, Tex. *Box 149, Faculty Exchange.*
BLUMBERG, Prof. HENRY, Ph.D. (Göttingen). Ohio State Univ., Columbus, Ohio. *76 E. Blake Ave.*
BLUMENTHAL, L. M., Ph.D. (Johns Hopkins). Instr., Johns Hopkins Univ., Baltimore, Md.
BOEDER, PAUL, A.M. (Penna.). Instr., Univ. of Delaware, Newark, Del. *Dept. of Math.*
BOGARD, Prof. A., A.M. (Chicago). Coll. of St. Teresa, Winona, Minn. *1018 W. 7th St.*
BOLKS, STANLEY, M.S. (Iowa State Coll.). Instr., Purdue Univ., W. LaFayette, Ind. *139 S. Grant St.*
BOLTON, R. W., E.E. (Illinois). 2000 10th St., Glendale, Calif.
BOND, Prof. J. D., Ph.D. (Michigan). Univ. of Tennessee, Knoxville, Tenn.
BOND, Supt. O. J., LL.D. (South Carolina). The Citadel, The Milit. Coll. of South Carolina, Charleston, S. C.
BOND, W. M., A.M. (Columbia). Instr., Coll. of the City of N. Y., New York, N. Y. *420 W. 121st St.*
BOOTHROYD, Prof. S. L., M.S. (Colorado State Coll.). Astr. and Geodesy, Cornell Univ., Ithaca, New York.
BOREL, EMILE, Docteur ès Sc. (Paris). Prof. à la Faculté des Sciences de Paris, Paris, France. *32 rue du Bac, Paris VII.*
BORGER, Prof. R. L., Ph.D. (Chicago). Ohio Univ., Athens, Ohio. *Fenzel Apts.*
BORGMAN, W. M., Jr., B.S. (Michigan). Instr., Coll. of City of Detroit, Detroit, Mich.
BORTOLOTTI, Prof. ETTORE, Dottore in mat. (Bologna). Univ. of Bologna, Bologna, Italy. *Via Maggiore 100.*
BOWDEN, Prof. JOSEPH, Ph.D. (Yale). Adelphi Coll., Brooklyn, N. Y. *33 Exeter St.*
BOWEN, Asso. Prof. L. H. Furman Univ., Greenville, S. C.
BOWERS, Prof. W. R., A.M. (Columbia). Radford State Normal School, E. Radford, Va. *1028 Clement St.*
BOWERSOX, Asso. Prof. E. R., E.E. (Iowa). Physics and Math., State School of Mines, Univ. of Montana, Butte, Mont. *1117 Waukesha St.*
BOWLES, Asst. Prof. C. F., A.M. (Harvard). S. Dak. State School of Mines, Rapid City, S. Dak. *Box 377.*
BOYCE, M. G., M.S. (Chicago). Instr., Adelbert Coll., Western Reserve Univ., Cleveland, Ohio.
BOYD, DEAN P. P., Ph.D. (Cornell). Univ. of Kentucky, Lexington, Ky.

BRADLEY, Asso. Prof. H. C., B.S. (Mass. Inst. of Tech.). Mass. Inst. of Tech., Cambridge, Mass.
BRADSHAW, Prof. J. W., Ph.D. (Strassburg). Univ. of Michigan, Ann Arbor, Mich. *1304 Cambridge Rd.*
BRAHDY, MARIE, A.M. (Columbia). Teacher, High School, New York, N. Y. *55 Central Pk. W.*
BRAMBLE, Prof. C. C., Ph.D. (Johns Hopkins). Math. and Mech., Postgraduate School, U. S. Naval Acad., Annapolis, Md. *Fifth St.*
BRAND, Prof. LOUIS, Ph.D. (Harvard). Univ. of Cincinnati, Cincinnati, Ohio.
BRANDEBERRY, Asso. Prof. J. B., A.M. (Ohio State). Toledo Univ., Toledo, Ohio.
BRANDNER, F. A., M.S. (Chicago). Instr., Iowa State Coll., Ames, Ia. *2821 Boone St.*
BRATTON, Prof. W. A., A.B. (Williams). Dean and Prof., Whitman Coll., Walla Walla, Wash. *On leave 1927–1928, 417 W. 114th St., New York, N. Y.*
BRAY, Asst. Prof. H. E., Ph.D. (Rice). Rice Inst., Houston, Tex.
BRECKENRIDGE, W. E., A.M. (Yale). Asso., Teachers Coll., Columbia Univ., New York, N. Y. *21 Sycamore Ave., Mt. Vernon, N. Y.*
BREIT, GREGORY, Ph.D. (Johns Hopkins). Math. Physicist, Dept. of Terr. Magn., Washington, D. C. *36th St. and Broad Branch Rd.*
BRENKE, Prof. W. C., Ph.D. (Harvard). Univ. of Nebraska, Lincoln, Nebr. *1250 S. 21st St.*
BREWSTER, Asst. Prof. J. A., A.B. (Harvard). Coll. of the City of New York, New York, N. Y.
BREZLER, W. J., A.M. (Grove City). Instr., Univ. of Tennessee, Knoxville, Tenn. *1317 White Ave.*
BRIGHAM, Asst. Prof. L. A., A.M. (Boston Univ.). Astr. and Math., Boston Univ., Boston, Mass. *On leave of absence; Lick Observatory, Mt. Hamilton, Calif.*
BRINK, Asso. Prof. R. W., Ph.D. (Harvard). Univ. of Minnesota, Minneapolis, Minn. *55 Williams Ave. S. E.*
BRIXEY, J. C., A.M. (Oklahoma). Instr., Univ. of Oklahoma, Norman, Okla. *303 Univ. Blvd.*
BRODIE, Prof. W. M., M.E. (Va. Poly. Inst.); A.M. (Columbia). Va. Poly. Inst., Blacksburg, Va.
BROOKE, Prof. W. E., A.M. (Nebraska). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
BROWN, Asst. Prof. B. H., Ph.D. (Harvard). Dartmouth Coll., Hanover, N. H.
BROWN, E. C., A.M. (Maine). Instr., Worcester Poly. Inst., Worcester, Mass. *25 Schussler Rd.*
BROWN, Prof. E. W., A.M. (Yale). Yale Univ., New Haven, Conn. *116 Everit St.*
BROWN, Prof. H. S., M.S. (Lafayette). Hamilton Coll., Clinton, N. Y.
BROWN, LILIAN H., A.M. (Illinois). Teacher, La Salle-Peru Twp. H. S. and Jr. Coll., La Salle, Ill.
BROWN, Prof. LILLIAN O., A.M. (Columbia). Hood Coll., Frederick, Md.
BROWN, Prof. MYRTLE C., A.M. (Texas). Teachers Coll., Denton, Tex. *1415 W. Oak St.*
BROWN, O. E., M.S. (Iowa). Instr., Northwestern Univ., Evanston, Ill. *822 Foster St.*
BROWN, RUTH L., A.B. (Oberlin). Graduate Student, Oberlin Coll., Oberlin, Ohio. *93 E. Lorain St.*
BROWN, Asst. Prof. T. H., Ph.D. (Yale). 25 Meadow Way, Cambridge, Mass.
BROWNE, Asso. Prof. E. T., Ph.D. (Chicago). Univ. of North Carolina, Chapel Hill, N. C.
BRUCE, Prof. R. E., Ph.D. (Boston Univ.). Boston Univ., Boston, Mass.
BRYAN, Asso. Prof. N. R., Ph.D. (Columbia). Univ. of Maine, Orono, Me. *8 Juniper St.*
BRYANT, E. S., A.B. (Harvard). Head, Coll. Prep. Dept., High School, Everett, Mass. *53 Lexington St.*
BUCHANAN, Prof. DANIEL, Ph.D. (Chicago). Head of Dept., Univ. of British Columbia, Vancouver, B. C.
BUCHANAN, Prof. H. E., Ph.D. (Chicago). Tulane Univ., New Orleans, La.
BUCHANAN, Asso. Prof. MARGARET, Ph.D. (Bryn Mawr). West Virginia Univ., Morgantown, W. Va. *145 Waitman St.*
BULLARD, Prof. J. A., Ph.D. (Clark). U. S. Naval Acad., Annapolis, Md. *262 King George St.*
BULLITT, W. M., B.S. (Princeton), LL.B. (Louisville). 1710–1711 Inter Southern Bldg., Louisville, Ky.
BUMER, C. T., A.M. (Harvard). Ohio State Univ., Columbus, Ohio. *1081 Northwest Blvd.*
BUNYAN, L. H., A.M. (Wisconsin). Instr., Univ. of Wisconsin, Extension Div., Milwaukee, Wis. *137 2nd St.*
BURDICK, R. D., A.M. (Columbia). Instr., Coll. of the City of New York, New York, N. Y. *6083 Broadway.*
BURGESS, H. T., Ph.D. (Yale). 153 Court St., New Haven, Conn.
BURGESS, R. W., Ph.D. (Cornell). Statistician, Western Elec. Co., New York, N. Y. *195 Broadway.*
BURINGTON, R. S., A.M. (Ohio State). Instr., Case School of Appl. Sc., Cleveland, Ohio.
BURKE, J. G., A.M. (Mt. St. Mary's). Vice-Pres., Mt. St. Mary's Coll., Emmitsburg, Md.
BURKETT, Asst. Prof. F. J. H., A.M. (Pittsburg). Trinity Coll., Hartford, Conn.
BURLEY, J. F., Civil Engineer, Washington, D. C. *3010 Wisconsin Ave.*
BURNAM, Prof. J. E., A.M. (Texas). Simmons Coll., Abilene, Texas. *1766 Sandefer St.*
BURNEY, M. SUE, A.M. (Chicago). Instr., Jr. Coll., St. Joseph, Mo. *302 S. 11th St.*
BURROWS, W. R., B.S. (Chicago). In charge of grating, Wm. F. Klemp Co., Chicago, Ill. *3006 W. Marquette Rd.*

BURWELL, W. R., Ph.D. (Oxford Univ.). Macedonia, Ohio.
BUSHYAGER, G. R., A.M. (Penn. State). Instr., Penn. State Coll., State College, Pa. *University Club.*
BUSSEY, Asst. Dean and Prof. W. H., Ph.D. (Chicago). Univ. of Minnesota, Minneapolis, Minn. *106 Folwell Hall.*
BUTLER, Prof. L. G., A.M. (Oregon). Albany Coll., Albany, Ore.
BUTTERFIELD, Prof. A. D., M.S. (Worcester Poly. Inst.), A.M. (Columbia). Math. and Geodesy, Univ. of Vermont, Burlington, Vt. *31 S. Prospect St.*
BUXTON, Prof. W. H., A.M. (Oregon). Whitworth Coll., Spokane, Wash.
BYRD, SAM, M.S. (Arkansas). Teacher, Central High, Tulsa, Okla. *1532 Admiral Pl.*

CAIN, Prof. WILLIAM, A.M. (N. C. M. & P. I.). Emeritus, Univ. of North Carolina, Chapel Hill, N. C.
CAIRNS, Prof. W. D. Ph.D. (Göttingen). Oberlin Coll., Oberlin, Ohio.
CAJORI, Prof. FLORIAN, Ph.D. (Tulane). Univ. of Calif., Berkeley, Calif. *2844 Webster St.*
CALDERWOOD, H. F. Stockman, Glasgow, Mont.
CALDWELL, MINNIE W., A.M. (Missouri). Chowan Coll., Murfreesboro, N. C.
CALKINS, Prof. HELEN, A.M. (Columbia). Sweet Briar Coll., Sweet Briar, Va.
CALLAWAY, Mrs. THEODOSIA T., B.S. (Columbia). Stephens Coll., Columbia, Mo. *417 Hitt St.*
CAMP, Prof B. H., Ph.D. (Yale). Wesleyan Univ., Middletown Conn. *110 Mt. Vernon St.*
CAMP, Asso. Prof. C. C. Ph.D. (Cornell). Univ. of Nebraska, Lincoln, Nebr. *Dept. of Math.*
CAMPBELL Asso. Prof. A. D., Ph.D. (Cornell). Syracuse Univ., Syracuse, N. Y. *177 W. Ostrander Ave.*
CAMPBELL, Prof. D. F., Ph.D. (Harvard). Armour Inst. of Tech., Chicago, Ill. *1209 Hinman Ave., Evanston, Ill.*
CAMPBELL, G. A., Ph.D. (Harvard). Research Engineer, Amer. Tel. and Tel. Co., New York, N. Y. *Room 1657, 195 Broadway.*
CAMPBELL, H. A., A.M. (Creighton). Tech. High School, Omaha, Nebr. *Tech. High School, 32nd and Cuming.*
CAMPBELL, JESSIE R., A.B. (Syracuse). Instr., Hollywood High School, Los Angeles, Calif. *5400 Brynhurst.*
CAMPBELL, Prof. J. W., Ph.D. (Chicago). Univ. of Alberta, Edmonton South, Alberta, Can.
CAMPBELL, Asst. Prof. W. B., B.S. in M.E. (Penna.). Colgate Univ., Hamilton, N. Y.
CANDY, Prof. A. L. Acting Dean, Univ. of Nebraska, Lincoln, Nebr. *Station A.*
CANNING, JOSEPH, A.B. (Intermountain). 1519 Railroad Ave., Helena, Mont.
CAPARÓ, Prof. J. A., Ph.D. (Notre Dame). Notre Dame Univ., Notre Dame, Ind. *P. O. Box 54.*
CAPRON, Prof. P. A., A.M. (Harvard). U. S. Naval Acad., Annapolis, Md. *203 Duke of Gloucester St.*
CAREY, Asso. Prof. E. F. A., M.S. (California). Univ. of Montana, Missoula, Mont. *310 McLeod Ave.*
CARIS, Pres. A. G., A.M. (Defiance). Defiance Coll., Defiance, Ohio.
CARIS, Asst. Prof. P. A., Ph.D. (Pennsylvania). Univ. of Pennsylvania, Philadelphia, Pa. *1539 S. 58th St.*
CARIS, V. B., Ohio State Univ., Columbus, Ohio. *2435 Medary.*
CARLEN, MILDRED E., M.S. (Brown). Registrar, Grad. School, Brown Univ., Providence, R. I. *23 Exeter St.*
CARLSON, J. A., B.S. in Math. (Washington). Instr. in Sc., High School, Pendleton, Ore. *617 Thompson St.*
CARLSON, S. ELIZABETH, Ph.D. (Minnesota). Instr., Univ. of Minnesota, Minneapolis, Minn. *3020 14th Ave. S.*
CARMICHAEL, Prof. F. L., A.M. (Princeton). Math. and Statistics, Univ. of Denver, Denver, Colo. *1301 Sherman.*
CARMICHAEL, Prof. R. D., Ph.D. (Princeton). Univ. of Illinois, Urbana, Ill. *207 W. Washington Blvd.*
CARPENTER, Prof. D. R., A.M. (Princeton). Roanoke Coll., Salem, Va.
CARPENTER, R. H., A.M. (Kansas). Math. and Physics, Jr. Coll., Iola, Kans. *301 East St.*
CARR, Prof. E. L., A.M. (Chicago). Head of Dept., Union Univ., Jackson, Tenn.
CARR, Prof. F. E., Ph.D. (Chicago). Math. and Astr., Oberlin Coll., Oberlin, Ohio. *284 Forest St.*
CARROLL, Asst. Prof. I. S., A.M. (Columbia). Syracuse Univ., Syracuse, N. Y. *915 E. Colvin St.*
CARRUTH, Prof. W. M., A.B. (Cornell). Hamilton Coll., Clinton, N. Y.
CARSCALLEN, Asso. Prof. G. E., A.M. (Illinois). Wabash Coll., Crawfordsville, Ind.
CARTER, C. C., Atty. at Law, Bluffs, Ill. Honorary Life Member.
CARUS, E. H., Ph.D. (Chicago). La Salle, Ill.
CARUTHERS, R. W. E., B.S. in E.E. (Miss. A. and M. Coll.). Test man on company's test course, Gen. Elec. Co., Schenectady, N. Y. *1496 State St.*
CARVER, Prof. W. B., Ph.D. (Johns Hopkins). Cornell Univ., Ithaca, N. Y.
CARY, R. L., A.M. (Haverford). Woodbrook, Govans, Baltimore, Md.

CASTER, MARY E., A.M. (Columbia). Teacher, Eastside High School, Paterson, N. J. *119 Auburn St.*
CEDERBERG, Prof. W. E., Ph.D. (Wisconsin). Augustana Coll., Rock Island, Ill. *2542-22 1-2 Ave.*
CHACE, ARNOLD B., Sc.D. Chancellor, Brown Univ., Providence, R. I. *99 Power St.*
CHALMERS, LUCILLE M., A.M. (Michigan). Instr., Coll. of City of Detroit, Detroit, Mich.
CHAMBERS, Prof. G. G., Ph.D. (Pennsylvania). Univ. of Pa., Lansdowne, Pa. *79 Drexel Ave.*
CHANDLER, Asst. Prof. H. W., M.S. (Iowa). Univ. of Florida, Gainesville, Fla.
CHANEY, J. G., A.B. (Southwestern Univ.). Instr., High School, Lubbock, Tex. *2302 16th St.*
CHAPMAN, MARGARET F., A.M. (Wisconsin). State Coll., Mankato, Minn.
CHARLES, Prof. R. L., A.M. (Lehigh). Physics, Franklin & Marshall Coll., Lancaster, Pa. *510 Race St.*
CHASE, L. R. Instr., Rogers High School, Newport, R. I. *Boulevard Terr.*
CHENEY, W. F., JR., Ph. D. (Mass. Inst. of Tech)., Instr., Tufts Coll., Medford, Mass. *45 Sawyer Ave.*
CHEPMELL, Maj. C. H. 69 Wilbury Crescent, Hove, Sussex, England.
CHITTENDEN, Prof. E. W., Ph.D. (Chicago). Univ. of Iowa, Iowa City, Iowa. *221 Physics Bldg.*
CHRISTIAN, A. L., A.B. (Baylor). Penelope, Tex.
CHURCH, E. F., C.E. (Syracuse). Parish, N. Y.
CLACK, Prof. R. W., A.M. (Grinnell). Math. and Astr., Alma Coll., Alma, Mich. *209 W. Downie St.*
CLARK, Asst. Prof. A. G., A.M. (Colorado). Colorado Agric. Coll., Fort Collins, Colo.
CLARK, J. R., Ph.D. (Columbia). The Lincoln School of Teachers Coll., New York, N. Y. *425 W. 123rd St.*
CLARK, R. F., A.B. (Williams), Pd.B. (Albany State Normal). Dewitt Clinton High School, New York, N. Y. *194 Christie Hts. St., Leonia, N. J.*
CLARKE, Prof. E. H., Ph.D. (Chicago). Math. and Astr., Hiram Coll., Hiram, Ohio.
CLARKE, J. A., A.M. (Princeton). West Phila. High School for Boys, Philadelphia, Pa. *Devon, Pa.*
CLARKE, Sister M. BORGIA, A.M. (Catholic Univ.). Prof., Loretto Coll., Webster Groves, Mo.
CLARKE, MARY E., M.S. (Kentucky). Teacher, Sr. High School, Lexington, Ky. *117 Woodland Ave.*
CLAWSON, Prof. J. W., M.A. (New Brunswick). Ursinus Coll., Collegeville, Pa.
CLAYTON, Prof. J. L., A.M. Freshman Math., Georgetown Coll., Georgetown, Ky.
CLELAND, Prof. W. E., Ph.D. (Princeton). Geneva Coll., Beaver Falls, Pa.
CLEMENTS, Asso. Prof. G. R., Ph.D. (Harvard). U. S. Naval Acad., Annapolis, Md. *7 Thompson St.*
COAR, Prof H. L., Ph.D. (Illinois). Math. and Astr., Marietta Coll., Marietta, Ohio.
COBB, Prof. H. E., A.M. (Wesleyan). Lewis Inst., Chicago, Ill.
COBLE, Prof. A. B., Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md.
COE, Asst. Prof. C. J., A.M. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *524 Linden St.*
COFFIN, J. G., Ph.D. (Clark). Math. Physics. Genl. Baking Corp., New York, N. Y. *17 Elm St., Garden City, L. I., N. Y.*
COFFIN, Prof. L. M., A.M. (Michigan). Coe Coll., Cedar Rapids, Ia. *1027 2nd Ave.*
COHEN, Prof. ABRAHAM, Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md.
COHEN, Asst. Prof. TERESA, Ph.D. (Johns Hopkins). Penna. State Coll., State College, Pa. *512 W. Beaver Ave.*
COKER, Prof. E. C., A.B. (Virginia). Astr. and Math., Univ. of South Carolina, Columbia, S. C. *1730 College St.*
COLAW, J. M., A.M. (Dickinson). High School, Monterey, Va.
COLE, LENA R., A.M. (Missouri). Central Normal Coll., Danville, Ind.
COLEMAN, Prof. J. B., A.M. (Columbia). Univ. of South Carolina, Columbia, S. C.
COLEMAN, Prof. R. H., A.B. (Charleston). Coll. of Charleston, Charleston, S. C.
COLLIER, Asst. Prof. MYRTIE, B.S. (Chicago). Univ. of California at Los Angeles, Calif. *320½ N. Alexandria Ave.*
COLLITON, J. W., C.E., E.M. (Lafayette). Head of Dept. of Math., High School, Trenton, N. J. *223 Highland Ave.*
COLPITTS, Asso. Prof. JULIA T., Ph.D. (Cornell). Iowa State Coll., Ames, Ia. *29 Cranford Apts.*
COLWELL, Prof. R. C., Ph.D. (Princeton). Physics, West Virginia Univ., Morgantown, W. Va. *332 Demain Ave.*
COLYER, Prof. E. E., A.M. (Kansas). State Teachers Coll., Hays, Kans. *406 W. 4th St.*
COMSTOCK, Prof. C. E., A.M. (Knox). Bradley Poly. Inst., Peoria, Ill.
CONDIT, Prof. I. S., A.M. (Parsons). State Teachers Coll., Cedar Falls, Ia. *115 E. 11th St.*
CONGDON, Asso. Prof. A. R., A.M. (Nebraska). Pedagogy of Math., Univ. of Nebraska, Lincoln, Nebr. *Station A.*
CONKLING, R. P., A.B. (Cornell). Head of Dept., Newark Tech. School and Central High School, Newark, N. J. *Newark Tech. School.*
CONKWRIGHT, N. B., Ph.D. (Illinois). Instr., Univ. of Iowa, Iowa City, Ia. *127 Physics Bldg.*
CONWELL, G. M., Ph.D. (Princeton). Master in Math., St. Paul's School, Concord, N. H.

CONWELL, Prof. H. H., M.S. (Kansas). Beloit Coll., Beloit, Wis. *1621 Emerson St.*
COOK, S. E., A.M. (Bates). Tech. High School and Jr. Coll., Springfield, Mass. *Tech. High School.*
COOLEY, H. R., A.M. (Middlebury). Instr., New York Univ., New York, N. Y. *32 Waverly Pl.*
COOLIDGE, Prof. J. L., Ph.D. (Bonn). Harvard Univ., Cambridge 38, Mass. *27 Fayerweather St.*
COOPER, Adj. Prof. A. E., E.E., A.M. (Texas). Applied Math., Univ. of Texas, Austin, Tex. *Rice Ct., Rio Grande St.*
COPE, T. F., M.S. (Tulane). Instr., Univ. of Chicago, Chicago, Ill.
COPELAND, Asst. Prof. LENNIE P., Ph.D. (Penna.). Wellesley Coll., Wellesley, Mass. *14 Waban St.*
COPP, P. T., A.M. (Ohio State). Instr., Purdue Univ., W. LaFayette, Ind. *300 Sylvia St.*
CORBIN, Prof. C. E., A.M. (Northwestern). Coll. of the Pacific, Stockton, Calif. *117 W. Euclid Ave.*
COREY, S. A. 1079 23d St., Des Moines, Ia.
COSBY, Prof. BYRON, A.M. (Missouri). Prof. and Bus. Adm., State Teachers Coll., Kirksville, Mo.
COULTRAP, Prof. M. W., A.M. (Ohio Wesleyan). North-Western Coll., Naperville, Ill. *95 W. Franklin St.*
COURT, Asso. Prof. N. A., D.Sc. (Ghent, Belgium). Univ. of Oklahoma, Norman, Okla.
COWLEY, Asso. Prof. ELIZABETH B., Ph.D. (Columbia). Vassar Coll., Poughkeepsie, N. Y. *On leave of absence, 913 Arch St., N. S., Pittsburgh, Pa.*
COX, Prof. E. F., Ph.D. (Cornell). Physics and Math., West Virginia Coll. Inst., Institute, W. Va.
CRAGWALL, Prof. J. A., M.S. (Vanderbilt). Head of Dept., Wabash Coll., Crawfordsville, Ind. *Kennedy Pl.*
CRAIG, C. C., Ph.D. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *3020 Angell Hall.*
CRAMLET, C. M., Ph.D. (Washington). Natl. Research Fellow, Princeton Univ., Princeton, N. J. *47 Maple St.*
CRANE, Asso. Prof. RUFUS, A.M. (Ohio State). Math. and Eng., Ohio Wesleyan Univ., Delaware, Ohio. *39 Montrose Ave.*
CRATHORNE, Asso. Prof. A. R., Ph.D. (Göttingen). Univ. of Illinois, Urbana, Ill. *802 Penna. Ave.*
CRAWLEY, Prof. E. S., Ph.D. (Pennsylvania). Univ. of Pennsylvania, Philadelphia, Pa.
CRENSHAW, Prof. B. H., M.E. (Ala. Poly.). Alabama Poly. Inst., Auburn, Ala.
CRESSE, Prof. G. H., Ph.D. (Chicago). Univ. of Arizona, Tucson, Ariz. *On leave for 1927–1928; Judenstr. 37, Göttingen, Germany.*
CROCKETT, Prof. C. W., C.E. (Rensselaer). Math. and Astr., Rensselaer Poly. Inst., Troy, N. Y. *221 Stow Ave.*
CROMWELL, J. W., Jr., A.M. (Dartmouth). M St. High School, Washington, D. C. *1815 13th St. N. W.*
CROOM, Asst. Prof. A. S. Washburn Coll., Topeka, Kans. *1836 Strong Ave.*
CROWE, Asst. Prof. S. E., A.B. (Ohio State). Michigan Agric. Coll., East Lansing, Mich. *137 Univ. Drive.*
CRUM, C. W. R., M.D. 322 N. Charles St., Baltimore, Md.
CRUM, Prof. W. L., Ph.D. (Yale). Business Statistics, Stanford University, Calif.
CULMER, ORPHA A., A.M. (Michigan). State Normal School, Florence, Ala. *217 Oneal Ave.*
CUMMING, Adj. Prof. FORREST, A.M. (Georgia). Univ. of Georgia, Athens, Ga.
CUMMINGS, E. A., A.M. (Chicago). Teacher, North High School, Denver, Colo. *782 Monroe.*
CUMMINGS, Asso. Prof. LOUISE D., Ph.D. Vassar Coll., Poughkeepsie, N. Y.
CUNHA, Prof. PEDRO JOSÉ DA. Math. Analysis, Faculty of Sc., Lisbon Univ., and Rector of the Univ., Lisbon, Portugal. *Rua de S. Bento, No. 706.*
CURRIER, Asst. Prof. C. H., A.M. (Brown). Brown Univ., Providence, R. I.
CURTIS, Prof. H. B., Ph.D. (Cornell). Lake Forest Coll., Lake Forest, Ill. *College Hall.*
CURTISS, Prof. D. R., Ph.D. (Harvard). Northwestern Univ., Evanston, Ill. *2023 Sherman Ave.*
CUSICK, Pres. P. F. Canisius Coll., Buffalo, N. Y. *Main St. and Jefferson Ave.*
CUTTING, L. H., B.S. (Chicago). Teacher, Westport High School, Kansas City, Mo. *3917 Summit St.*

DABOLL, JEANNETTE G., A.B. (Mt. Holyoke). Teacher, Northfield Sem., East Northfield, Mass.
DADOURIAN, Prof. H. M., Ph.D. (Yale). Applied Math., Trinity Coll., Hartford, Conn. *125 Vernon St.*
DAHLENE, Prof. OSCAR, M.S. (Kansas). Eng. and Math., Univ. of Alabama, University, Ala.
DALAKER, Prof. H. H., Ph.D. (Cornell). Math. and Mech., Coll. of Eng. and Arch., Univ. of Minnesota, Minneapolis, Minn. *523 Walnut St., S.E.*
DALE, JULIA, Ph.D. Miss. Delta State Coll., Cleveland, Miss.
DALTON, Prof. J. P., D.Sc. (St. Andrews; Cape). Univ. of the Witwatersrand, Johannesburg, S. Afr. *P. O. Box 1176.*
DANCER, C. W., A.M. (Ohio State). Instr., Univ. of Wisconsin, Madison, Wis. *314 N. Allen St.*
DANIELLS, Asst. Prof. MARIAN E., M.S. (Iowa State). Iowa State Coll., Ames, Ia.
DANTZIG, Asst. Prof. TOBIAS, Ph.D. (Indiana). Univ. of Maryland. *1662 Park Rd., Washington, D. C.*

DAPPERT, J. W., Civil Eng., Lock Box 141, Taylorville, Ill.
DARLING, F. W., A.B. (Cornell). Asso. Mathematician, U. S. Coast and Geod. Surv., Washington, D. C.
DARNELL, ALBERTUS, Ph.B. (Michigan). Detroit Jr. Coll., Detroit, Mich. *1116 Edison Ave.*
DASPIT, Mrs. A. P., A.B. (La. State Univ.). Instr., La. State Univ., Baton Rouge, La. *759 Boyd Ave.*
DAUS, Asst. Prof. P. H., Ph.D. (California). Univ. of California at Los Angeles, Calif. *855 N. Vermont Ave.*
DAVIS, C. H. Travelers Ins. Co., Hartford, Conn.
DAVIS, Asst. Prof. D. R., Ph.D. (Chicago). Univ. of Oregon, Eugene, Ore. *Bartle Ct.*
DAVIS, H. A., A.M. (West Va.). 304 Elmwood Ave., Ithaca, N. Y.
DAVIS, Asst. Prof. H. T. Univ. of Indiana, Bloomington, Ind. *712 E. 2nd St.*
DAVIS, Asso. Prof. J. E., A.M. (Wisconsin). Drexel Inst., Philadelphia, Pa.
DAVIS, Prof. J. M., A.M. (Kentucky). Univ. of Kentucky, Lexington, Ky. *340 Madison Place.*
DAVIS, J. W., A.M. (Yale). Jr. Master, High School for Boys, Dorchester, Mass. *17 Wrentham St.*
DAVISSON, Prof. S. C., Sc.D. (Tübingen). Indiana Univ., Bloomington, Ind.
DAY, EDNA L., A.M. (Northwestern). Instr., Wesleyan Coll., Macon, Ga.
DEAN, ALICE C., A.M. (Rice). Acting Libn. and Fellow in Math., Rice Inst., Houston, Texas.
DEARMAN, Prof. D. S., A.M. (Vanderbilt). Kentucky Wesleyan Coll., Winchester, Ky.
DECHERD, Adj. Prof. MARY E., A. M. (Texas). Univ. of Texas, Austin, Texas. *2313 Nueces St.*
DECKER, Prof. F. F., Ph.D. (Syracuse). Syracuse Univ., Syracuse, N. Y. *312 Marshall St.*
DECLEENE, Rev. L. A. V., A.M. (Catholic Univ.). Prof., St. Norbert's Coll., West De Pere, Wis.
DE COU, Prof. E. E., M.S. (Chicago). Univ. of Oregon, Eugene, Ore. *929 Hilyard St.*
DEDERICK, L. S., Ph.D. (Harvard). Mathematician, Aberdeen Proving Ground, Aberdeen, Md.
DE LA GARZA, E. Box 304, Brownsville, Tex.
DE LONG, Prof. I. M., A.M. (Simpson). Retired, Univ. of Colorado, Boulder, Colo. *1341 Broadway.*
DE LURY, Prof. A. T., M.A. (Toronto). Univ. of Toronto, Toronto, Ont. Can.
DEMING, Prof. R. M., B.C.E. (Iowa State). Upper Iowa Univ., Fayette, Iowa.
DENMAN, L. H., B.S. (Denison). Head of Dept., Shurtleff Coll., Alton, Ill. *2068 Walnut St.*
DENNISON, C. H. Grad. Mass. Inst. of Tech. Chemist, Amer. Rubber Co., Cambridge, Mass. *183 Norfolk St., Wollaston, Mass.*
DENNY, Prof. A. K., M.S. (Lincoln). Prof. and Registrar, Lincoln Coll., Lincoln, Ill. *221 Oglesby Ave.*
DENTON, Asst. Prof. W. W., Ph.D. (Illinois). Univ. of Michigan, Ann Arbor, Mich. *1014 Cornwell Pl.*
DICE, ELIZABETH, A.M. (Texas). Teacher, High School, Dallas, Texas.
DICK, Prof. F. J. Theosophical Univ., Point Loma, Calif.
DICKINSON, Prof. C. N. Hollins Coll., Hollins, Va.
DICKSON, Prof. L. E., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill. *5535 University Ave.*
DICKSTEIN, Prof. S., Ph.D. Math. and History of Math., Univ. of Warsaw, Warsaw, Poland. *Marszalkowska St. 117.*
DIEDERICH, E. F., A.B. (Butler). 3434 Carrollton Ave., Indianapolis, Ind.
DILLINGHAM, Asso. Prof. ALEX. U. S. Naval Acad., Annapolis, Md. *12 Thompson St.*
DIMICK, Prof. C. E., A.M. (Penna.). U. S. Coast Guard Academy, New London, Conn. *Sales Ferry, Conn.*
DINWIDDIE, Pres. A. B., Ph.D. (Virginia). Pres. and Prof. of Math., Tulane Univ., New Orleans, La. *Station 20.*
DOAK, Asso. Prof. ELEANOR C., Ph.B. (Chicago). Mount Holyoke Coll., South Hadley, Mass.
DOAN, Asst. Prof. C. S., A.M. Purdue Univ., W. LaFayette, Ind. *415 Russell St.*
DOBBIN, Sister MARIOLA, A.M. (Wisconsin). Rosary Coll., River Forest, Ill.
DO BELL, Asst. Prof. H. A., A.M. (Syracuse). N. Y. State Coll. for Teachers, Albany, N. Y.
DODD, Prof. E. L., Ph.D. (Yale). Univ. of Texas, Austin, Texas. *3012 West Ave.*
DONAGHHO, Prof. J. S., A.M. (Marietta). Univ. of Hawaii, Honolulu, T. H. *961 Alewa Drive.*
DONAHUE, Asso. Prof. J. E., A.M. (Harvard). Univ. of Vermont, Burlington, Vt. *Essex Junction, Vt.*
DORROH, J. L., A.M. (Texas). Part-time instr., Pure Math., Univ. of Texas, Austin, Tex. *Univ. Station.*
DOSTAL, B. F., A.M. (Indiana). Instr., Univ. of Florida, Gainesville, Fla.
DOTTERER, Prof. J. E., A.M. (Illinois). Head of Dept. of Math., Manchester Coll., North Manchester, Ind.
DOUGHERTY, LUCY T., A.M. (Kansas). Instr., Jr. Coll., Kansas City, Kans. *Gould Hotel.*
DOUGHTY, ANNIE W., A.M. (Radcliffe). Teacher, Dana Hall School, Wellesley, Mass.
DOUGLAS, J. L., A.M. (Davidson). Davidson Coll., Davidson, N. C.
DOUGLASS, R. D., A.M. (Maine). Instr., Mass. Inst. of Tech., Cambridge, Mass. *76 Drew Rd., Belmont, Mass.*
DOUSHKESS, Asst. Prof. V. H., A.M. (Lafayette). Lafayette Coll., Easton, Pa. *216 McCartney St.*

DOWLING, Asso. Prof. L. W., Ph.D. (Clark). Univ. of Wisconsin, Madison, Wis.
DOWNEY, W. F., Ed.M. (Harvard). Headmaster, English High School, Boston, Mass. *77 Hammond Rd., Belmont, Mass.*
DOWNING, Asso. Prof. H. H., M.S. (Kentucky). Univ. of Kentucky, Lexington, Ky. *138 State St.*
DRESDEN, Prof. ARNOLD, Ph.D. (Chicago). Swarthmore Coll., Swarthmore, Pa.
DREW, J. W., A.M. (Cornell). Head of Dept., Jr. Coll. of Storer Coll., Harpers Ferry, W. Va. *Box 202.*
DRIVER, D. D., A.M. (Nebraska). Instr., Hesston Coll., Hesston, Kans.
DROKE, Prof. G. W., A.M. (Arkansas). Math. and Astr., Univ. of Arkansas, Fayetteville, Ark.
DUBE, Prof. L. H., Ph.D. (Gregorian Univ. Rome). Higher Math., Ottawa Univ., Ottawa, Can.
DUERKSEN, J. A., A.B. (Bethel). Asst. Mathematician, Coast and Geod. Survey, Washington, D. C. *3134 Monroe St. N.E.*
DUNCAN, BERTHA K., A.M. (Texas). Instr. in Educ., Univ. of Texas, Austin, Tex. *Woman's Bldg.*
D'UNGER, V. C., A.M. (Little Rock). Pyramid Life Ins. Co., Little Rock, Ark. *115 W. 15th St.*
DUNKEL, Asso. Prof. OTTO, Ph.D. (Harvard). Washington Univ., St. Louis, Mo.
DU PASQUIER, L. G., Ph.D. (Zurich). Prof. ordinaire de math. super., Univ. de Neuchâtel, Neuchâtel, Switzerland. *33 Sablons.*
DURELL, FLETCHER, Ph.D. (Princeton). Emeritus, formerly Head of Dept., Lawrenceville School, Lawrenceville, N. J. *Belleplain, N. J.*
DUREN, W. L., Jr., A.B. (Tulane). Tulane Univ., New Orleans, La.
DURFEE, Asst. Prof. W. H., M.C.E. (Harvard). Hobart Coll., Geneva, N. Y. *403 Pultney St.*
DURFEE, Prof. W. P., Ph.D. (Johns Hopkins). Hobart Coll., Geneva, N. Y. *639 Main St.*
DURHAM, ANNIE D., A.B. (Tex. Presbyter. Coll.). Instr., Texas Presbyterian Coll., Milford, Tex.
DUSTHEIMER, Prof. O. L., A.M. (Clark), M.S. (Michigan). Baldwin-Wallace Coll., Berea, Ohio. *272 Beach St.*
DYE, L. A., A.M. (Rochester). Grad. Student, Cornell Univ., Ithaca, N. Y. *24 Burkard Pl., Rochester, N. Y.*

EAGLES, Prof. T. R., A.M. (North Carolina). Howard Coll., Birmingham, Ala. *9016 2nd Ave. S.*
EARHART, FRANC C., B.S. (Lenox). Lenox Coll., Hopkinton, Ia.
EARL, J. M., A.B. (Carleton). Asst., Univ. of Minnesota, Minneapolis, Minn. *2023 Second Ave S.*
EARLE, Prof. M. D., A.M. (Furman). Furman Univ., Greenville, S. C.
ECHOLS, Col. C. P. Prof., U. S. Milit. Acad., West Point, N. Y.
ECHOLS, Prof. W. H., C.E. (Virginia). Univ. of Virginia, University, Va.
ECKERSLEY, J. O., C.E. (Cooper Union). Ch. Engr., Dept. of Markets, New York, N. Y. *4269. White Plains Ave.*
EDINGTON, Asso. Prof. W. E., Ph.D. (Illinois). Purdue Univ., W. Lafayette, Ind. *633 Russell St.*
EDMONDSON, AUBREY, B.S. in C.E. (Miss. A. and M. Coll.). Instr., Miss. A. and M. Coll., Starkville, Miss. *Box 2.*
EDMONDSON, Prof. T. W., Ph.D. (Clark). New York Univ., New York, N. Y. *Univ. Heights.*
EDWARDS, Asst. Prof. P. D., A.M. (Indiana). Ball Teachers Coll., Muncie, Ind. *407 Riverside Ave.*
EELLS, Asso. Prof. W. C., Ph.D. (Stanford). Educ., Stanford University, Calif. *Box 2607.*
EICHELBERGER, J. H., B.S. (Chicago). 821 S. Oak Ave., Freeport, Ill.
EIESLAND, Prof. J. A., Ph.D. (Johns Hopkins). West Virginia Univ., Morgantown, W. Va.
EIPPER, D. C., A.B. (Harvard). Head of Dept. of Math. and Physics, Berkshire School, Sheffield, Mass.
EISELE, CAROLYN, A.M. (Hunter). Instr., Hunter Coll., New York, N. Y. *419 E. 157th St.*
EISENHART, Dean L. P., Ph.D. (Johns Hopkins). Princeton Univ., Princeton, N. J. *22 Alexander St.*
ELLIOTT, Prof. W. W., Ph. D. (Cornell). Duke Univ., Durham, N. C. *Box 533.*
ELSTON, J. S., A.B. (Cornell). Asst. Actuary, Travelers Ins. Co., Hartford, Conn.
ELY, Prof. J. A. St. John's Univ., Shanghai, China.
EMCH, Prof. ARNOLD, Ph.D. (Kansas). Univ. of Illinois, Urbana, Ill. *1002 S. Orchard St.*
EMERY, STEPHEN, A.M. (Boston Univ.). Head of Dept., Erasmus Hall High School, Brooklyn, N. Y. *317 E. 21st St.*
EMMONS, Prof. C. W., A.M. (Illinois). Simpson Coll., Indianola, Ia.
EMMONS, Prof. L. C., A.M. (Harvard). Math. Statistics, Michigan State Coll., East Lansing, Mich.
ENGLISH, HARRY, A.B. (Johns Hopkins), LL.M. (Columbia Univ.). Head of Dept., High Schools, Washington, D. C. *2907 P St., N. W.*
ENRIQUES, Prof. FEDERIGO. Univ. of Bologna, Bologna, Italy.
ENTZ, G. G., B.S. (Columbia). 1352 Poinsettia Pl., Hollywood, Calif.
EPPERSON, C. A., A.M. (Missouri). Axtell, Mo.
EPPES, Prof. J. B., A.M. (Johns Hopkins). U. S. Naval Acad., Annapolis, Md.
ERICKSON, Asst. Prof. E. E., M.S. (Iowa). Miami Univ., Oxford, Ohio.
ERICSON, HENRY, B.S. (Chicago). Instr., High School, Milwaukee, Wis. *3414 Juneau St.*

ERWIN, Prof. J. T., A.M. (Vanderbilt). George Washington Univ., Washington, D. C. *2023 G St., N. W.*

ESCOTT, E. B., M.S. (Chicago). Walton School of Commerce, Chicago, Ill. *1019 S. East Ave., Oak Park, Ill.*

ESHLEMAN, Asst. Prof. J. D., Ph.D. (Chicago). Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*

ESTES, Asst. Prof. J. G., A.M. (Illinois). Texas Christian Univ., Fort Worth, Tex.

ESTY, Prof. T. C., A.M. (Amherst). Amherst Coll., Amherst, Mass.

ETTLINGER, Prof. H. J., Ph.D. (Harvard). Pure Math., Univ. of Texas, Austin, Tex. *3110 Harris Park Ave.*

EVANS, Prof. G. C., Ph.D. (Harvard). Pure Math., Rice Inst., Houston, Tex.

EVANS, G. W., A.B. (Harvard). 107 Ocean St., Lynn, Mass.

EVANS, Prof. H. B., Ph.D. (Pennsylvania). Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*

EVANS, P. H. Chief Actuary, Northwestern Mut. Life Ins. Co., Milwaukee, Wis. *210 Wisconsin Ave.*

EVANS, P. L., A.B. (Baker). Instr., Baker Univ., Baldwin, Kans.

EVERETT, Prof. H. S., Ph.D. (Chicago). Extension Prof., Univ. of Chicago, Chicago, Ill. *5545 Woodlawn Ave.*

EVERETT, Prof. J. P., A.M. (Michigan). Western State Normal School, Kalamazoo, Mich. *907 W. South St.*

EVERETT, Asso. Prof. J. R., A.M. (Wisconsin). Colo. School of Mines, Golden, Colo.

EVERETT, OTIS, A.B. (Harvard), LL.B. (Fordham). 57 Broadway, New York, N. Y.

EVERS, Asst. Prof. CORNELIUS, M.S. (Michigan State). Eng. Math., Albion Coll., Albion, Mich. *505 Fitch St.*

EVERSULL, BESS M., Ph.D. (Cincinnati). Instr., Smith Coll., Northampton, Mass. *277 Crescent St.*

EWING, W. M., A.M. (Rice). Rice Inst., Houston, Tex.

FAIRCHILD, Prof. J. T., A.M., C.E. (Ohio Northern); Ph.M. (Carnegie Inst.). Ohio Northern Univ., Ada, Ohio.

FARNER, Prin. E. F., A.M. (Kansas). Jr. Coll. and High School, Parsons, Kans. *2730 Clark.*

FARNUM, FAY, Ph.D. (Cornell). Instr., Washington Square Coll., New York Univ., New York, N. Y. *Washington Square.*

FAUGHT, Prof. J. B., Ph.D. (Penna.). Yankton Coll., Yankton, S. D.

FEDERICO, P. J., A.M. (George Washington). Asst. Examiner, U. S. Patent Office; Instr., George Washington Univ., Washington, D. C. *1417 Newton St. N.W.*

FEEMSTER, Prof. H. C., A.M. (Nebraska). York Coll., York, Nebr.

FEHN, Asso. Prof. A. R., Ph.B. (Baldwin-Wallace). Centre Coll., Danville, Ky. *337 W. Broadway.*

FEHR, Dr. HENRI. Editor L'Enseignment de Math., Geneva, Switzerland. *110 Florissant.*

FELTGES, EDNA M., A.M. (Wisconsin). Math., Lindblom High School, Chicago, Ill. *4611½ Drexel Blvd.*

FENNER, BEATRICE A., A.M. (Stanford). Orland, Glenn Co., Calif.

FENTON, FRANCES G., M.S. (Denver). Teacher, Acctg. and Bookkeeping, North High School, Denver, Colo. *2246 Eudora St.*

FERGUSON, Prof. C. E., A.M. (Missouri). Stephen F. Austin State Teachers Coll., Nacogdoches, Tex.

FERGUSON, Miss ZOE, B.S. (Missouri). Math. and Physics, Crane Jr. High School, Chicago, Ill. *50 Aldine Ave.*

FERRY, Pres. F. C., Ph.D. (Clark). Hamilton Coll., Clinton, N. Y.

FIELD, FLORENCE E., A.M. (Michigan). Teacher, High School, Jackson, Mich. *304 S. Hannah, Albion, Mich.*

FIELD, Prof. FLOYD, A.M. (Harvard). Georgia School of Tech., Atlanta, Ga. *Route 1, Decatur, Ga.*

FIELD, Prof. PETER, Ph.D. (Cornell). Univ. of Michigan, Ann Arbor, Mich. *904 Olivia Ave.*

FIELD, S. E., A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *Edgewood Hills, R. F. D. 3.*

FIELDS, Prof. J. C., Ph.D. (Johns Hopkins). Univ. of Toronto, Toronto, Can.

FINDLAY, Prof. WILLIAM, Ph.D. (Chicago). McMaster Univ., Toronto, Can. *153 Westminster Ave.*

FINE, Prof. H. B., Ph.D. (Leipzig). Princeton Univ., Princeton, N. J. *Library Pl.*

FINKEL, Prof. B. F., Ph.D. (Penna.). Math. and Physics, Drury Coll., Springfield, Mo. Honorary Life Member. *1227 Clay St.*

FINLEY, Prof. G. W., A.M. (Kansas State). Colo. State Teachers Coll., Greeley, Colo. *1933 9th Ave.*

FISCHER, C. H., B.S. (Washington). Instr., Beloit Coll., Beloit, Wis.

FISHER, Prof. IRVING, Ph.D. (Yale). Pol. Econ., Yale Univ., New Haven, Conn. *460 Prospect St.*

FISKE, Prof. T. S., Ph.D. (Columbia). Columbia Univ., New York, N. Y.

FITCH, ANNIE L. MACKINNON, (Mrs. EDWARD FITCH), Ph.D. (Cornell). Clinton, N. Y.

Fitch, Philip, A.M. Asst. Prin., North High School, Denver, Colo. *3647 Clay St.*
Fite, Prof. W. B., Ph.D. (Cornell). Columbia Univ., New York, N. Y.
Fitterer, Prof. J. C., C.E. (Colorado). Prof. of Civil and Irrig. Eng., Univ. of Wyoming, Laramie, Wyo. *511 S. 11th St.*
Flagg, Asst. Prof. Elinor B., M.S. (Illinois). Ill. State Normal Univ., Normal, Ill. *Fell Hall.*
Flanders, D. A., Ph.D. (Penna.). Natl. Research Fellow, Princeton Univ., Princeton, N. J. *9 Aiken Ave., Princeton, N. J.*
Flanders, Asst. Prof. R. L., B.S. (Norwich). Oklahoma A. and M. Coll., Stillwater, Okla. *212 Elm St.*
Fleet, Prof. R. R., Ph.D. (Heidelberg, Germany). William Jewell Coll., Liberty, Mo.
Fleisher, Asst. Prof. Edward, M.S. (New York Univ.). Coll. of the City of New York. *1068 Park Pl., Brooklyn, N. Y.*
Fleisig, A. J. Instr., St. Procopius Coll., Lisle, Ill.
Fleming, Asst. Prof. Annie W., A.M. (California). Iowa State Coll., Ames, Ia. *719 Douglas Ave.*
Flynn, J. D., A.B. (Trinity). Statistician, Goodwin-Beach & Co., Investment Brokers, Hartford, Conn.
Foberg, J. A., B.S. (Illinois). State Dir. of Math. Instruction, Camp Hill, Pa.
Focke, Dean T. M., Ph.D. (Göttingen). Case School of Appl. Sc., Cleveland, Ohio.
Foraker, Prof. F. A., M.S. (Ohio Northern). Univ. of Pittsburgh. *1313 Macon Ave., Swissvale, Pa.*
Ford, Asst. Prof. L. R., Ph.D. (Harvard). Rice Inst., Houston, Tex.
Ford, Prof. W. B., Ph.D. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *204 Mason Hall.*
Forsyth, Asst. Prof. C. H., Ph.D. (Michigan). Dartmouth Coll., Hanover, N. H.
Fort, Prof. Tomlinson, Ph.D. (Harvard). Lehigh Univ., Bethlehem, Pa.
Foster, R. M., B.S. (Harvard). Dept. of Dev. and Research, Amer. Tel. and Tel. Co., New York, N. Y. *195 Broadway.*
Fowlkes, Asso. Prof. J. G., Ph.D. (Columbia). Educ., Univ. of Wisconsin, Madison, Wis.
Fox, A. H., A.M. (Oberlin). Grad. Student, Harvard Univ., Cambridge, Mass. *52 Perkins Hall.*
Fox, Prof. Hally, B.S. (Miss. A. & M.). Mississippi A. & M. Coll., A. & M. College, Miss.
Fox, J. F. Sr. Eng. Draughtsman, U. S. Bureau of Standards, Washington, D. C. *2440 20th St. N. Y.*
Frankel, E. T., B.S. (C. C. N. Y.). Statistician, 3569 Broadway, New York, N. Y.
Frankenbush, Bertha E., A.M. (Tulane). Teacher, High School, New Orleans, La. *1126 2nd St.*
Frankish, Ellen H., A.B. (Nebraska). Head of Dept. of Math., North High School, Omaha, Neb. *3011 N. 45th St.*
Franklin, Asst. Prof. Philip, Ph.D. (Princeton). Mass. Inst. of Tech., Cambridge, Mass. *312 Pleasant St., Belmont, Mass.*
Frary, R. W., B.S. (Va. Milit. Inst.). Center Hill, Fla.
Freas, Elizabeth, A.B. (Lake Erie). Monticello Sem., Godfrey, Ill.
Fréchet, Maurice. Prof. d'Analyse Supér., Univ. de Strasbourg, Strasbourg, France. *4 Rue Wencker III.*
Frecheville, George, M.A. (Oxford; Cambridge). Research, Agric. Econ. Research Inst., Oxford, England. *83 High St.*
Freeman, Gladys H., A.M. (Chicago). Instr., No. Illinois State Teachers Coll., De Kalb, Ill. *248 Augusta Ave.*
Freeman, Neda B., A.M. (Boston Univ.). Teacher, Kent State Normal Coll., Kent, Ohio. *325 E. Main St.*
Frumveller, Prof. A. F. Univ. of Detroit, Detroit, Mich. *651 E. Jefferson Ave.*
Fry, T. C., Ph.D. (Wisconsin). Bell Tel. Laboratories, New York, N. Y. *463 West St.*
Fubini, Prof. Guido. Prof. di Analisi Mat., R. Scuola di Ingegneria, Turin, Italy. *Via Pietro Micca 12.*
Fulmer, Asst. Prof. H. K., A.M. (Columbia). Georgia School of Tech., Atlanta, Ga.
Funk, Asst. Prof. J. C., A.M. (Columbia). Jr. Coll., Santa Maria, Calif.
Funkhouser, Asst. Prof. H. G., A.M. (Columbia). Washington and Lee Univ., Lexington, Va. *Box 465.*

Gaba, Prof. M. G., Ph.D. (Chicago). Univ. of Nebraska, Lincoln, Nebr. *1425 S. 22d St.*
Gafafer, W. M., A.M. (Columbia). Instr., Barnard Coll., Columbia Univ. *Leonia, N. J.*
Gage, W. H., M.A. (Univ. of B..C.). Instr., Victoria Coll., Victoria, B. C. *Dept. of Math.*
Gaines, Prof. R. E., A.M. (Furman). Richmond Coll., Richmond College, Va.
Gale, Prof. A. S., Ph.D. (Yale). Univ. of Rochester, Rochester, N. Y. *11 Thayer St.*
Galloway, Prof. C. R., B.M.E. (Kentucky). Idaho Tech. Inst., Pocatello, Ida. *332 S. 10th Ave.*
Garabedian, Asso. Prof. C. A., Ph.D. (Harvard). Univ. of Cincinnati, Cincinnati, Ohio. *3655 Middleton Ave.*
Garabedian, H. L., A.M. (Harvard). Instr., Northwestern Univ., Evanston, Ill. *Dept. of Math.*
Gardiner, J. A., A.B. (Penn. State). Instr., Howard High School, Wilmington, Del. *1305 Tatnall St.*

GARDNER, Prof. R. W., A.M. (Boston). Head of Dept. of Math., Eastern Nazarene Coll., Wollaston, Mass.
GARLOUGH, Prof. C. D., A.M. (Illinois). Prof. and Registrar, Wheaton Coll., Wheaton, Ill. *911 N. Main St.*
GARNER, C. L., B.S. in C.E. (N. C. State Coll.). Hydrogr. and Geod. Eng., U. S. Coast and Geod. Surv., Washington, D. C.
GARRETSON, W. V. N., Ph.D. (Michigan). Head of Dept., Ouachita Coll., Arkadelphia, Ark.
GARRETT, Prof. W. H., A.M. (Illinois Coll.). Vice-Pres. and Prof., Math. and Astr., Baker Univ., Baldwin City, Kans.
GARRISON, Prof. D. M., Grad. U. S. Naval Acad. Head of Dept., St. John's Coll., Annapolis, Md. *236 Prince George St.*
GARVER, RAYMOND, Ph.D. Instr., Univ. of Rochester, Rochester, N. Y.
GAULDEN, AMELIA, A.B. Teacher, High School, Slidell, La.
GAUT, Asst. Prof. A. E., M.S. (Chicago). Bradley Poly. Inst., Peoria, Ill.
GAVER, H. H., A.M. (Virginia). Headmaster, Urban Milit. Acad., Los Angeles, Calif. *637 N. Wilcox Ave.*
GAY, Asst. Prof. H. J., A.M. (Clark). Worcester Poly. Inst., Worcester, Mass. *7 Belvidere Ave.*
GAYDEN, Prof. W. J., A.M. (Baylor). Math. and Eng., Howard Payne Coll., Brownwood, Tex. *900 Booker St.*
GAYLORD, H. D., A.M. (Harvard). Mathematics Master, Browne and Nichols School, Cambridge, Mass. *20 Charlesgate, Boston, Mass.*
GAYLORD, Asst. Prof. LESLIE J., M.S. (Chicago). Agnes Scott Coll., Decatur, Ga.
GECKELER, Prof. O. T., A.B. (Indiana). Head of Dept., Carnegie Inst. of Tech., Pittsburgh, Pa.
GEHMAN, Asst. Prof. H. M., Ph.D. (Penna.). Yale Univ., New Haven, Conn. *599 Whalley Ave.*
GEORGES, J. S., Ph.D. (Chicago). Instr., School of Educ., Univ. of Chicago, Chicago, Ill. *School of Educ.*
GÉRARDIN, ANDRÉ. Editor *Sphinx-Oedipe*, Nancy, France. *32 Quai Claude le Lorrain.*
GERST, Rev. F. J., Ph.D. (Johns Hopkins). Prof. Math., St. Louis Univ., St. Louis, Mo.
GHORMLEY, Asst. Prof. L. O., A.M. (Chicago). Univ. of Tennessee. *On leave of absence, Grad. School, Univ. of Chicago ,Chicago, Ill.; 5315 Drexel Ave.*
GIBBENS, Asst. Prof. GLADYS E. C., Ph.D. (Chicago). Univ of Minnesota, Minneapolis, Minn. *122 Folwell Hall.*
GIBSON, EMMA M. Drury Coll., Springfield, Mo. *1189 Clay Ave.*
GIBSON, FALKA M., A.B. (California). Teacher, High School, San Francisco, Calif. *2034 Capp St., Oakland, Calif.*
GIBSON, Prof. J. L., Ph.D. (Vienna). Univ. of Utah, Salt Lake City, Utah.
GILL, B. P., A.M. (Columbia). Instr., Coll. of the City of New York, New York, N. Y. *41 St. Nicholas Terr.*
GILLESPIE, Prof. D. C., Ph.D. (Göttingen). Cornell Univ., Ithaca, N. Y. *Cayuga Hts.*
GILLESPIE, Prof. WILLIAM, Ph.D. (Chicago). Princeton Univ., Princeton, N. J. *Grad. College.*
GILLEY, C. A., A.B. (Texas). Alpine, Tex.
GILMAN, Asst. Prof. R. E., Ph.D. (Princeton). Brown Univ., Providence, R. I. *44 E. Manning St.*
GINGRICH, Prof. C. H., Ph.D. (Chicago). Math. and Astr., Carleton Coll., Northfield, Minn.
GINNINGS, R. M., M.S. (Chicago). Head of Dept. of Math., Western Ill. State Teachers Coll. Macomb, Ill. *314 N. Ward St.*
GLAZIER, Asst. Prof. HARRIET E., A.M. (Chicago). Univ. of California at Los Angeles, Calif. *1307 Lucile Ave.*
GLENN, Prof. O. E., Ph.D. (Pennsylvania). Univ. of Pa., Philadelphia, Pa. *127 McKinley Ave., Lansdowne, Pa.*
GLOVER, Prof. B. C., A.M. (Chicago). Otterbein Coll., Westerville, Ohio.
GLOVER, Prof. J. W., Ph.D. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *620 Oxford Rd.*
GOLD, J. S., B.S. (Bucknell). Instr.. Bucknell Univ., Lewisburg Pa. *Box 191.*
GOLDBERG MICHAEL, B.S. in E.E. (Penna.). Bureau of Ordnance, Navy Dept., Washington, D. C. *623 19th St. N.W.*
GORE, G. D., M.S. (Chicago). Instr., S. Dakota State Coll., Brookings, S. Dak.
GORRELL, Asso. Prof. G. W., A.M. (Ohio State). Univ. of Denver, Denver, Colo. *2236 S. Milwaukee St.*
GOSSARD, Prof. H. C., Ph.D. (Johns Hopkins). Dean of Men and Prof. of Math., Nebr. Wesleyan Univ., Lincoln, Nebr. *2841·N. 51st St.*
GOULD, ALICE B., A.B. (Bryn Mawr). Care W. W. Vaughan, Exchange Bldg., Boston, Mass.
GOUWENS, Asso. Prof. CORNELIUS, Ph.D. (Chicago). Iowa State Coll., Ames, Ia. *109 Hyland Ave.*
GRABER, Prof. M. E., Ph.D. (Iowa). Physics, Morningside Coll., Sioux City, Ia.
GRAESSER, Asst. Prof. R. F., Ph.D. (Illinois). Univ. of Arizona, Tucson, Ariz. *918 E. 6th St.*
GRAHAM, MARIA D., B.S. (Teachers Coll., Columbia Univ.). Head of Dept. of Math., Teachers Training School, Greenville, N. C.
GRAHAM, Asst. Prof. P. H., A.M. (Virginia). Washington Square Coll., New York Univ., New York, N. Y. *136 Brewster Ave., Flushing, N. Y.*

GRANT, ALICE A., A.M. (Brown). Box 41, Red Deer, Alb., Can.
GRANT, Prof. E. D., Ph.D. (Chicago). Math. and Registrar, Earlham Coll., Richmond, Ind.
GRANVILLE, W. A., Ph.D. (Yale). Educ. Director, U. S. National Life and Casualty Co., Chicago, Ill. *513 Aldine Ave.*
GRAUSTEIN, Asso. Prof. W. C., Ph.D. (Bonn). Harvard Univ., Cambridge, Mass. *32 Shepard St.*
GRAVATT, Prof. T. E., M.S. (Rutgers). Pennsylvania State Coll., State College, Pa.
GRAVES, Asso. Prof. G. H., Ph.D. (Columbia). Purdue Univ., W. LaFayette, Ind. *829 Main St.*
GRAVES, Asst. Prof. L. M., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill. *6048 Ingleside Ave.*
GREEN, Prof. R. L., A.M. (Indiana). Emeritus, Stanford University, Calif.
GREENLEAF, Asst. Prof. H. E. H., A.M. (Boston). DePauw Univ., Greencastle, Ind. *1024 College Ave.*
GRIFFIN, Prof. F. L., Ph.D. (Chicago). Reed Coll., Portland, Ore.
GROVE, Prof. C. C., Ph.D. (Johns Hopkins). Polytech. Inst. of Brooklyn. *143 Milburn Ave., Baldwin, L. I., N. Y.*
GROVE, Asso. Prof. V. G., Ph.D. (Chicago). Michigan State Coll., East Lansing, Mich. *212 W. Grand River Ave.*
GUILBEAU, LUCYE, A.B. (La. State). Teacher, St. Gabriel High School, Baton Rouge, La. *911 West St.*
GUMMER, Prof. C. F., Ph.D. (Chicago). Queen's Univ., Kingston, Ont. *143 Collingwood St.*
GUMMERE, Prof. H. V., A.M. (Harvard). Drexel Inst., Philadelphia, Pa. *Sevilla Ct. Apts., Bala, Cynwyd, Pa.*
GUNDERSEN, Prof. CARL, Ph.D. (Columbia). Oklahoma A. and M. Coll., Stillwater, Okla.
GWINN, Asst. Prof. I. J., M.S. (Iowa). Morningside Coll., Sioux City, Ia.
GWINNER, Prof. HARRY, M.E. (Maryland). Univ. of Maryland. *1207 W. Mulberry St., Baltimore, Md.*

HACKNEY, LILLIAN, A.B. (W. Virginia). Marshall Coll., Huntington, W. Va.
HADAMARD, Prof. JAQUES, LL.D. (Yale). École Polytech. et Coll. de France, Paris, France. *25 rue Humboldt, Paris, XIVe.*
HADLEY, Prof. LAURENCE, Ph.D. (Mich.). Purdue Univ., W. LaFayette, Ind. *121 Lutz Ave.*
HADLEY, Prof. S. M., Ph.D. (Wisconsin). Penn. Coll., Oskaloosa, Ia. *427 N. 3rd St.*
HADLOCK, E. H. Instr., Univ. of Maine, Orono, Me. *38 Oak St.*
HAERTTER, L. D., A.M. (Columbia). John Burroughs School, Clayton, Mo.
HALDEMAN, C. B. Ross, Butler Co., Ohio.
HALL, F. C., A.M. (Columbia). Instr., Rutgers Univ., New Brunswick, N. J. *60 Cedar Ave., Highland Park.*
HALL, Prof. H. L., A.B. (Indiana). N. W. State Teachers Coll., Alva, Okla. *923 7th Ave.*
HALL, RUTH H., A.B. (Brown). Rosemary Hall, Greenwich, Conn.
HALL, Prof. W. S., M.E., M.S. (Lafayette). Lafayette Coll., Easton, Pa. *College Campus.*
HALLAM, W. A., A.M. (Johns Hopkins). Student Asst., Johns Hopkins Univ., Baltimore, Md. *Box 494, J. H. U.*
HALPERIN, Prof. HILLEL, E.E. (Liége), A.M. (Columbia). A. and M. Coll. of Texas, College Station, Tex.
HAMILTON, W. M., A.M. (Michigan). Asst., U. S. Nautical Almanac Office, U. S. Naval Observ., Washington, D. C.
HAMMOND, Prof. E. S., Ph.D. (Princeton). Bowdoin Coll., Brunswick, Me. *84 Federal St.*
HAMPTON, LAURENCE, A.M. (Nebraska). Instr., Univ. of Oklahoma, Norman, Okla. *Dept. of Math.*
HANAWALT, Prof. F. W., A.M. (DePauw). Math. and Astr., Coll. of Puget Sound, Tacoma, Wash. *826 N. Steele St.*
HANCOCK, CLARA L., A.M. (Iowa). Teacher, Jr. Coll., Virginia, Minn.
HANCOCK, Prof. HARRIS, Ph.D. (Berlin). Univ. of Cincinnati, Cincinnati, Ohio.
HANNA, Prof. U. S., Ph.D. (Penna.). Indiana Univ., Bloomington, Ind. *828 Atwater Ave.*
HANSON, H. O., A.B. (Columbia). Mut. Life Ins. Co., Corona, L. I., N. Y. *2730 Ditmars Blvd.*
HANTHORN, EMMA E., A.B. (Nebraska). Teacher, State Teachers Coll., Kearney, Nebr. *714 W. 23rd St.*
HARDIN, Prof. J. A., A.M. (Chicago). Dean, Centenary Coll. of La., Shreveport, La.
HARDING, Prof. A. M., Ph.D. (Chicago). Math. and Astr., and Dir., Genl. Extension Service, Univ. of Arkansas, Fayetteville, Ark. *537 Leverett St.*
HARDING, HOWARD, B.M.E. (Michigan). With Rochester Gas and Elec. Corp., Rochester, N. Y. *29 Kingston St.*
HARDY, Prof. G. H. Savilian Prof. of Geometry, Oxford Univ., Oxford, Eng. *New College.*
HARDY, Prof. J. G., Ph.D. (Johns Hopkins). Williams Coll., Williamstown, Mass.
HARGETT, Prof. A. J., A.M. (Transylvania). Texas Christian Univ., Fort Worth, Tex.
HARMOUNT, G. P., A.M. (Ohio State). Head of Dept., East High School, Columbus, Ohio. *2290 Indianola Ave.*

HARPER, H. D. Instr., Murray Hill Voc. School, New York, N. Y. *3549 70th St., Jackson Hts.*
HARRELL, Asso. Prof. J. W., A.M. (Brown). Pure Math., Baylor Univ., Waco, Tex.
HARRINGTON, Asst. Prof. C. E., M.E. (Cornell), M.S. (Buffalo). Asst. to the Dean, Univ. of Buffalo, Buffalo, N. Y. *52 Winter St.*
HARRIS, Asso. Prof. ISABEL, A.M. (Columbia). Westhampton Coll., University of Richmond, Va.
HARRIS, MARGARET E., A.M. (Teachers Coll., Columbia). Chair of Math., Grenada Coll., Grenada, Miss.
HARRY, S. C., A.B. (Johns Hopkins). Head of Dept., City Coll., Baltimore, Md. *1528 Linden Ave.*
HARSHBARGER, FRANCES, A.M. (West Virginia). Asst., Univ. of Ill., Urbana, Ill. *Dept. of Math.*
HARSHBARGER, Prof. W. A., B.S., Sc.D. Washburn Coll., Topeka, Kans. *1401 College Ave.*
HART, Asso. Prof. BERTHA I., A.M. (Cornell). Western Maryland Coll., Westminster, Md.
HART, Prof. J. N., C.E. (Maine), M.S. (Chicago). Univ. of Maine, Orono, Me.
HART, Asso. Prof. R. W., A.M. (Illinois). Math. and Appl. Mech., State Teachers Coll., Pittsburg, Kans.
HART, Prof. W. L., Ph.D. (Chicago). Univ. of Minnesota, Minneapolis, Minn.
HART, Asso. Prof. W. W., A.B. (Chicago). Univ. of Wisconsin, Madison, Wis. *R.F.D. 7.*
HARTER, Prof. G. A., Ph.D. (St. John's). Delaware Coll., Newark, Del.
HARTIG, Asst. Prof. H. E., Ph.D. (Minnesota). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
HARTNELL, GEORGE. Observer in charge, Cheltenham Magn. Observ., Cheltenham, Md.
HARTSFIELD, J. M., B.S. (Rice Inst.). Southwestern Bell Tel. Co., Dallas, Tex. *4942 Reiger Ave.*
HARTWICK, Prof. F. C., M.S. (Fribourg, Suisse). Univ. of Dayton, Dayton, Ohio.
HASEMAN, Prof. CHARLES, Ph.D. (Göttingen). Math. and Mech., Univ. of Nevada, Reno, Nev.
HASEMAN, MARY G., Ph. D. (Bryn Mawr). Head of Dept., Jr. Coll. of Connecticut, Bridgeport, Conn.
HASKELL, Prof. M. W., Ph.D. (Göttingen). Univ. of California, Berkeley, Calif. *P. O. Box 3.*
HASKELL, R. N., B.S. (Chicago). 619 S. A St., Elwood, Ind.
HASSLER, Prof. J. O., Ph.D. (Chicago). Univ. of Oklahoma, Norman, Okla. *425 Lahoma Ave.*
HATCH, Asst. Prof. D. A., Eng. of Mines (Lafayette). Lafayette Coll., Easton, Pa. *705 High St.*
HATCHER, Asst. Prof. T. W., M.E. (Va. Poly. Inst.), M.S. (Iowa State Coll.). Virginia Polytech. Inst., Blacksburg, Va. *Box 476.*
HATHAWAY, Prof. A. S., B.S. (Cornell). Retired, Rose Poly. Inst. *Boerne, Tex.*
HATFIELD, CHARLES, A.M. (Tennessee). Head of Dept., Lincoln Memorial Univ., Harrogate, Tenn. *Box 59.*
HAWKES, Dean H. E., Ph.D. (Yale). Columbia Univ., New York, N. Y.
HAWKES, Prof. JULIA M., Ph.D. (Michigan). Math. and Astr., Doane Coll., Crete, Nebr.
HAYASHI, Prof. TSURUICHI. M.Sc. (Tokyo Imper. Univ.), D.Sc. (Government). Honorary Pres. Math. Assoc. of Japan for Sec. Educ. Tohoku Imperial Univ., Sendai, Japan. *Kamoncho 10.*
HAZARD, Asso. Prof. C. T., A.M. (Indiana). Purdue Univ., W. LaFayette, Ind. *344 N. Western Ave.*
HAZELTINE, Asso. Prof. B. A., B.S. (Tufts). Dean of Men, Middlebury Coll., Middlebury, Vt. *38 South St.*
HAZLETT, Asst. Prof. OLIVE C., Ph.D. (Chicago). Univ. of Illinois, Urbana, Ill.
HEDRICK, Prof. E. R., Ph.D. (Göttingen). Univ. of California at Los Angeles, Calif.
HEIN, R. G., A.B. (Wisconsin). Teacher, High School, Waupun, Wis.
HELWIG, MARY E. Teacher, High School, Kansas City, Kans. *736 State Ave.*
HEMKE, P. E., Ph.D. (Johns Hopkins). Post Graduate School, U. S. Naval Acad., Annapolis, Md.
HENDERSON, Prof. ARCHIBALD, Ph.D. (North Carolina; Chicago). Head of Dept., Univ. of North Carolina, Chapel Hill, N. C.
HENDERSON, ROBERT, B.A. (Toronto). Actuary, Equitable Life Assur. Soc., New York, N. Y. *393 Seventh Ave.*
HENDRICKS, A. F., B.S. (Valparaiso). Pres. Emeritus, Mayfield Coll., Marble Hill, Mo. *Seminary Hill, Tex.*
HENNEL, Asso. Prof. CORA B., Ph.D. (Indiana). Indiana Univ., Bloomington, Ind. *822 Atwater Ave.*
HENROTEAU, F. C., Dr. of Physics and Math. (Brussels). Head of Astrophys. Dept., Dominion Observ., Ottawa, Ca.
HENRY, ETTA M., B.S. (Michigan). Teacher, High School, Kenova, W. Va.
HENRY, Prof. T. B., A.M. (Kansas). Highland Coll., Highland, Kans.
HEREN, Asst. Prof. MABEL M., M.S. (Northwestern). Knox Coll., Galesburg, Ill.
HERRON, Prof. C. L., A.M. (Hillsdale). Hillsdale Coll., Hillsdale, Mich. *188 Hillsdale St.*
HESS, Asso. Prof. G. W., Ph.D. (Michigan). Howard Coll., Birmingham, Ala. *8009 Berney Ave.*
HESSELTINE, EVELYN, A.M. (Nebraska). Teacher, Normal School, Spearfish, S. Dak.
HICKEY, DEBORAH M., A.M. Rice Inst., Houston, Texas. *511 Harvard St.*
HICKOX, Prof. E. J., A.M. (Columbia). Math. and Physics, Internatl. Y. M. C. A. Coll., Springfield, Mass.
HICKS, H. C., M.S. (Chicago). Teaching Fellow, Calif. Inst. of Tech., Pasadena, Calif.

HICKSON, A. O., A.M. (Brown). Instr., Brown Univ., Providence, R. I. *95 Carrington Ave.*
HIGGINS, Prof. W. S., M.E.E. (Harvard). Univ. of Florida, Gainesville, Fla.
HIGH, M. DE TURK, A.B. (Franklin and Marshall). Head of Dept. and Dean of Men, Central State Normal School, Lock Haven, Pa. *Normal School.*
HIGHTOWER, Prof. RUBY U., Ph.D. (Missouri). Shorter Coll., Rome, Ga.
HILDEBRANDT, MARTHA, Ph.D. (Chicago). Teacher, Proviso High School, Maywood, Ill. *109 27th Ave., Bellewood, Ill.*
HILDEBRANDT, Prof. T. H., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich. *1930 Cambridge Rd.*
HILL, Prof. A. L., A.B. (Doane). Math. and Phys., State Teachers Coll., Peru, Nebr.
HILL, Asst. Prof. M. A., Jr., A.M. (North Carolina). Univ. of North Carolina, Chapel Hill, N. C. *On leave of absence 1927–28 Univ. of Chicago, 1124 E. 56th St., Chicago, Ill.*
HILL, P. R., M.S. (Georgia). Instr., Univ. of Georgia, Athens, Ga. *512 N. Jackson St.*
HILL, Asso. Prof. W. H., A.M. (Colorado). State Teachers Coll., Pittsburg, Kans. *1621 S. Walnut.*
HILLARD, Asso. Prof. C. R., A.M. (Columbia). Wheaton Coll., Wheaton, Ill.
HILLE, Asst. Prof. C. E., Ph.D. (Stockholm, Sweden). Princeton Univ., Princeton, N. J. *174 Prospect Ave.*
HIMWICH, A. A., M.D. (New York Univ.). Physician, New York, N. Y. *1871 Madison Ave.*
HINSCH, Asso. Prof. V. B., E.M. (Missouri School of Mines). School of Mines and Metallurgy, Rolla, Mo.
HIRSCH, BLANCHE, B.S. (Columbia). Prin., Alcuin Prep. School, New York, N. Y. *48 W. 86th St.*
HIRSCHLER, Prof. E. J., M.S. (Chicago). Bluffton Coll., Bluffton, Ohio. *On leave of absence 1927–1928, Yale Univ.*
HITCHCOCK, Prof. R. R., A.M. (Northwestern). Univ. of N. Dakota, University, N. D.
HOARE, Prof. A. J., A.M (Michigan). Fairmount Coll., Wichita, Kans. *1717 Holyoke Ave.*
HOBART, C. M., A.M. (Illinois). Atwood, Ill. (?)
HOBBS, Prof. A. W., Ph.D. Univ. of North Carolina, Chapel Hill, N. C.
HODGDON, F. C., A.B. (Tufts). Educ. Publisher, Ginn and Co., New York, N. Y. *70 Fifth Ave.*
HODGE, Asst. Prof. F. H., A.M. (Boston). Purdue Univ., W. LaFayette, Ind. *820 N. Main St.*
HODGKINS, Dean H. L., Ph.D. (George Washington). Prof. of Math. and Dean of the Univ., George Washington Univ., Washington, D. C.
HOLBROOK, W. C. N. E. Sales Mgr., Henry Holt and Co., Boston, Mass. *6 Park S*
HOLDER, Prof. F. J., Ph.D. (Yale). Mercer Univ., Macon, Ga. *912 College St.*
HOLGATE, Prof. T. F., Ph.D. (Clark). Northwestern Univ., Evanston, Ill. *617 Library Pl.*
HOLLCROFT, Prof. T. R., Ph.D. (Cornell). Wells Coll., Aurora, N. Y.
HOLLIS, ELINOR V., M.S. (Chicago). 985 Memorial Dr., Cambridge, Mass.
HOLMES, C. F., M.S. (Howard). Instr., State Normal School, Elizabeth City, N. C.
HOLMES, HELMA L., A.M. (Nebraska). Instr., Pure Math., Univ. of Texas, Austin, Tex. *1908 Univ. Ave.*
HOLMES, J. T. Orleans, Ill.
HOLROYD, INA E., B.S. (Kans. State Agric. Coll.; Kans. State Teachers Coll.). Instr., State Agric. Coll., Manhattan, Kans.
HOLT, Asst. Prof. H. K., A.M. (Ohio State). Penna. Milit. Coll., Chester, Pa.
HOPKINS, FANNIE, A. B. (Franklin). Asst., Univ. of Wisconsin, Madison, Wis. *902 Grant St.*
HOPKINS, Asso. Prof. L. A., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich.
HORN, MARVEL C., A.M. (Ohio State). Teacher, High School, Columbus, Ohio. *69 Sherman Ave.*
HORNE, Prof. C. E., Ph.D. (Chicago). Univ. of Porto Rico, Rio Pedras, P. R.
HORSBURGH, E. M., D.Sc. (Edinburgh). Lecturer in Technical Math. and Eng. Dyn., Univ. of Edinburgh, Edinburgh, Scotland. *11 Granville Terr.*
HORSFALL, I. O., A.B. (Utah). Head of Dept., Latter-Day Saints Univ., Salt Lake City, Utah, *On leave of absence 1927–1928, Univ. of Chicago. 5542 Kimbark Ave.*
HORTON, ANNIE B., A.M. (Texas). Head of Dept., High School, Edinburgh, Tex. *Faculty Club.*
HORTON, Adj. Prof. GOLDIE P., Ph.D. (Texas). Pure Math., Univ. of Texas, Austin, Tex. *504 W. 32nd St.*
HOSFORD, Asso. Prof. H. M., Ph.D. (Illinois). Southern Methodist Univ., Dallas, Tex. *Box 342, S. M. U.*
HOSHAUER, J. C., B.S. in Educ. (Bucknell). Head of Dept., High School, Williamsport, Pa.
HOSKINS, Prof. L. M., C.E. (Wisconsin). Emeritus, Stanford Univ., Palo Alto, Calif. *1240 Waverly St.*
HOUSEHOLDER, Asso. Prof. F. C., A.B. (Kansas). N. Dakota Agric. Coll., Fargo, N. D. *1129 College St.*
HOWARD, Prof. C. M., E. Mines (Ala. Poly. Inst.). Texas Woman's Coll., Fort Worth, Tex. *3318 Ave. I.*
HOWARD, Prof. W. E., Ph.D. (Indiana). Math. and Astr., Univ. of Tulsa, Tulsa, Okla.
HOWE, Asst. Prof. ANNA M., Ph.D. Sophie Newcomb Memorial Coll., New Orleans, La. *67 Audubon Blvd.*

HOWE, Prof. G. K., B.S. (Worcester Poly. Inst.). Atlanta Univ., Atlanta Ga. *626 Beckwith St. S.W.*
HOWIE, Prof. J. M., A.M. (Columbia). Nebraska Wesleyan Univ., Lincoln, Nebr. *5218 Walker Ave.*
HOWLAND, Prof. L. A., Ph.D. (Munich). Fisk Prof. of Math., Wesleyan Univ., Middletown, Conn. *34 Howe Ave.*
HOYT, R. S., M.S. (Princeton). Engr., Dept. of Development and Res., Amer. Tel. and Tel. Co., New York, N. Y. *195 Broadway.*
HUBBS, Asst. Prof. H. N., A.M. (Rochester). Hobart Coll., Geneva, N. Y. *87 Madison St.*
HUBER, Asst. Prof. C. M., Ph.D. (Illinois). Rutgers Univ., New Brunswick, N. J.
HUBERT, Asst. Prof. W. G., Sc.D. (New York Univ.). Coll. of the City of New York. *125 Lee Ave., Yonkers, N. Y.*
HUFFER, Asso. Prof. R. C., A.M. (Illinois). Math. and Astr., Beloit Coll., Beloit, Wis. *1302 Emerson St.*
HUGHES, H. K., A.M. (Iowa). Fellow, Univ. of Michigan, Ann Arbor, Mich. *337 E. William St.*
HUGHES, Asso. Prof. JEWELL C., Ph.D. (Chicago). Univ. of Arkansas, Fayetteville, Ark.
HULBURT, Prof. L. S., Ph.D. (Johns Hopkins). Emeritus, Johns Hopkins Univ., Baltimore, Md.
HULL, Asso. Prof. DANIEL, M.S. (Notre Dame). Math. and Physics, Univ. of Notre Dame, Notre Dame, Ind.
HUME, Chancellor ALFRED, D.Sc. (Vanderbilt). Univ. of Mississippi, University P. O., Miss.
HUNT, Asst. Prof. G. H., C.E. (Cornell). Univ. of California at Los Angeles, Calif. *855 N. Vermont Ave.*
HUNT, Prof. MILDRED, Ph.D. (Chicago). Illinois Wesleyan Univ., Bloomington, Ill. *1211 Fell Ave.*
HUNTER, H. E., M.S. (Illinois). State Teachers Coll., Silver City, N. Mex. *316 7th St.*
HUNTINGTON, A. H., A.M. (Columbia). Asst. Prin., Beaumont High School, St. Louis, Mo.
HUNTINGTON, Prof. E. V., Ph.D. (Strassburg). Mech., Harvard Univ., Cambridge, Mass. *48 Highland St.*
HURWITZ, Prof. W. A., Ph.D. (Göttingen). Cornell Univ., Ithaca, N. Y. *White Hall 8.*
HUSSEY, L. W., A.M. (Harvard). N. S. S. C., Union Coll., Schenectady, N. Y.
HUTCHERSON, Asso. Prof. W. R., A.M. (Kentucky). Math. and Astr., Berea Coll., Berea, Ky.
HUTCHINS, MABEL, A.B. (Blue Mountain). Head of Dept., Blue Mountain Coll., Blue Mountain, Miss.
HUTCHINSON, Prof. C. A., A.M. (Wittenberg). Eng. Math., Univ. of Colorado, Boulder, Colo. *837 15th St.*
HYDE, Asso. Prof. EMMA, A.M. (Chicago). State Agric. Coll., Manhattan, Kans.

ILOFF, Prof. P. M., A.M. (Michigan). State Teachers Coll., Chico, Calif. *1329 Esplanade.*
INGOLD, Prof. BYRON, A.M. (Central Wesl.). Culver-Stockton Coll., Canton, Mo.
INGOLD, Prof. LOUIS, Ph.D. (Chicago). Univ. of Missouri, Columbia, Mo. *8 Hamilton Way.*
INGRAHAM, Prof. M. H., Ph.D. (Chicago). Univ. of Wisconsin, Madison, Wis.
IRWIN, Asso. Prof. FRANK, Ph.D. (Harvard). Univ. of California, Berkeley, Calif. *2921 Regent St.*
ISAACS, Prof. C. A., A.M. (Columbia). Head of Dept. of Math., State Coll. of Washington, Pullman, Wash.
ISHAM, F. H., A.M. (Teachers Coll., Columbia). Teacher, State Normal School, Potsdam, N. Y. *34 Bay St.*

JABLONOWER, JOSEPH, Pd.M. (N. Y. Univ.). Teacher, Ethical Culture School, New York, N. Y. *33 Central Pk. W.*
JACKSON, Prof. DUNHAM, Ph.D. (Göttingen). Univ. of Minnesota, Minneapolis, Minn. *119 Folwell Hall.*
JACKSON, FRANCES H., A.M. (Columbia). 5909 Chestnut St., Philadelphia, Pa.
JACKSON, Asso. Prof. J. B., A.M. (Columbia). Univ. of South Carolina, Columbia, S. C. *227 S. Waccamaw Ave.*
JACKSON, MARY F., B.S. (Nebraska). Teacher, High School, Lincoln, Nebr. *1546 E St.*
JACKSON, ROSA L., A.M. Acting Instr., Stanford Univ. *353 Lytton Ave., Palo Alto, Calif.*
JACOBS, JESSIE M., Ph.D. (Illinois). Instr., Univ. of Texas, Austin, Tex.
JAEGER, C. G., Ph.D. (Missouri). Instr., Univ. of Missouri, Columbia, Mo. *Eng. Bldg.*
JAMES, Asso. Prof. GLENN, Ph.D. (Columbia). Univ. of California at Los Angeles, Calif.
JAMISON, Prof. G. H., A.M. (Chicago). State Teachers Coll., Kirksville, Mo. *Box 116*
JANES, Asst. Prof. W. C., A.M. (Nebraska). State Agric. Coll., Manhattan, Kans.
JARRETT, ETHEL L., A.B. (Cornell). Chicago Latin School for Girls, Chicago, Ill. *59-69 Scott St.*
JEFFERS, H. M., Ph.D. (California). Astron., Lick Observ., Mt. Hamilton, Calif.
JEFFERY, R. L., A.M. (Cornell). Instr., Cornell Univ., Ithaca, N. Y. *516 Univ. Ave.*
JEKEL, A. H., A.B. (Michigan). 951 Lincoln Pl., Boulder, Colo.
JENKINS, W. A., A.M. (Michigan). With L. A. Glover and Co., Chicago, Ill. *7243 Bennett Ave.*
JENSEN, Asst. Prof. C. M., Ph.D. (Minnesota). Macalester Coll., St. Paul, Minn. *3315 17th Ave. S., Minneapolis, Minn.*

JERBERT, A. R., Ph.D. (Washington). Assoc., Univ. of Washington, Seattle, Wash. *707 E. 55th.*

JEROME, Prof. JESSIE M., A.M. (Hiram). Hiram Coll., Hiram, Ohio.

JIMENEZ, Rev. FELIPE, B.S. (Notre Dame). Colegio de San Juan de Letran, Manila, P. I. *200 Beaterio St.*

JOFFE, S. A., M.S. (New York Univ.). Asst. Actuary, Mut. Life Ins. Co., New York, N. Y. *32 Nassau St.*

JOHN, F. W., M.E. (Cornell). Instr., Washington Square Coll., New York Univ. *182 Nepperhan Ave., Yonkers, N. Y.*

JOHNSON, ABIGAIL E., B.S. (Columbia). Teacher, High School, Morristown, N. J. *Box 234, Morris Plains, N. J.*

JOHNSON, Prof. B. F., A.M. State Normal School, Cape Girardeau, Mo.

JOHNSON, Prof. C. L., B.S. (Ore. Agric.). Head of Dept., Oregon Agric. Coll., Corvallis, Ore. *1001 Jefferson St.*

JOHNSON, E. H., A.B. (DePauw). Westfield, Ind.

JOHNSON, Prof. E. N., A.M. (Drake), M.S. (Kansas). Butler Coll., Indianapolis, Ind. *304 Downey Ave.*

JOHNSON, Acting Asst. Prof. MARIE M., M.S. (Iowa). Oberlin Coll., Oberlin, Ohio. *116 Elm St.*

JOHNSON, MYRA I., A.M. (Columbia). 420 W. 119th St., New York, N. Y.

JOHNSON, Prof. R. A., Ph.D. (Harvard). Hunter Coll., New York, N. Y.

JOHNSON, Asst. Prof. R. P., A.M. (Virginia; Harvard). Carnegie Institute of Tech., Pittsburgh, Pa. *5916 Douglass Ave., Squirrel Hill.*

JOHNSON, W. W. Instr. in Appl. Math., Y. M. C. A. Night Schools, Cleveland, Ohio. *1078 E. 99th St.*

JOHNSTON, F. E., Ph.D. (Illinois). Instr., Univ. of Illinois, Urbana, Ill. *911 W. High St.*

JOHNSTON, Asst. Prof. K. P., B.A., B.S. (Queen's). Queen's Univ., Kingston, Ont., Can. *Dept. of Math.*

JOHNSTON ,Asso. Prof. L. S., A.M. (Missouri). Penn. State Coll., State College, Pa. *10 Liberal Arts Bldg.*

JOHNSTON, NELLIE M., A.B. (Kearney State Teachers Coll.). Warren Twp. High School, Gurnee, Ill.

JONES, AGNES A., Teacher, Norwin High School, Irwin, Pa.

JONES, B. W., A.M. (Harvard). Grad. School, Univ. of Chicago, Chicago, Ill. *49 Hitchcock Hall.*

JONES, Prof. E. H., A.M. (Harvard). Southern Methodist Univ., Dallas, Tex.

JONES, Prof. J. S. W., D.Sc. (Washington Coll.). Washington Coll., Chestertown, Md. *404 Washington Ave.*

JONES, OLIVE M., A.M. (Columbia). Head of Dept. of Math., Queen's Coll., Charlotte, N. C.

JONES, P. C., B.S. (Mass. Inst. of Tech.). Div. of Public., Bell Tel. Labs., New York, N. Y. *463 West St.*

JONES, R. W., B.S. (Delaware). Instr., Univ. of Delaware, Newark, Del. *204 W. 17th St., Wilmington, Del.*

JONES, S. I., B.S. (N. Tex. Normal Coll.; Lipscomb Coll.). Asst. Treas., Life and Casualty Ins. Co., Nashville, Tenn.

JORDAN, Asst. Prof. E. E., M.A. (Dalhousie). Univ. of British Columbia. *4195 15th Ave. W., Point Grey, B. C., Can.*

JORDAN, Asso. Prof. H. E., Ph.D. (Chicago). Univ. of Kansas, Lawrence, Kans. *1600 Kentucky St.*

JORDAN, Asso. Prof. M. F., A.M. (Maine). Astr., Univ. of Maine, Orono, Me.

JORDAN, WILLIAM, A.M. (DePauw). Head of Dept. of Math., Oakland City Coll., Oakland City, Ind.

JURDAK, Prof. M. H., M.A. (Amer. Univ. of Beirut). Amer. Coll. of Beirut, Syria.

JUSTICE, Asst. Prof. H. K., C.E. (Cincinnati). Univ. of Cincinnati, Cincinnati, Ohio. *4767 High Ridge, Price Hill.*

JUSTIN, E. M., M.S. (Case School). Case School of Appl. Sc., Cleveland, Ohio.

KARAPETOFF, Prof. VALADIMIR, M.M.E. (Leningrad). Elec. Eng., Cornell Univ., Ithaca, N. Y. *520 E. Buffalo St.*

KARPINSKI, Prof. L. C., Ph.D. (Strassburg). Univ. of Michigan, Ann Arbor, Mich. *1315 Cambridge Rd.*

KARR, LOIS, A.M. (Wisconsin). Instr., Lindenwood Coll., St. Charles, Mo.

KASNER, Prof. EDWARD, Ph.D. (Columbia). Columbia Univ., New York, N. Y. *22 W. 119th St.*

KAUFMAN, H. P., M.E. (Polytech. Inst. of Brooklyn). Jr. Mathematician, Coast and Geod. Survey, Washington, D. C. *4918 9th. N. W.*

KAZARINOFF, K., Agrégé (Moscow). Instr., Univ. of Michigan, Ann Arbor, Mich. *1515 Cambridge Rd.*

KEARNEY, Prof. DORA E., A.M. (Minnesota). Teaching Dept., State Teachers Coll., Cedar Falls, Ia. *1210 W. 22nd St.*

KEITH, Asst. Prof. MARY N., A.B. (Wellesley). Dean of Women, Univ. of Redlands, Redlands, Calif.

KELLOGG, Prof. O. D., Ph.D. (Göttingen). Harvard Univ., Cambridge 38, Mass. *20 Craigie St.*
KELLS, Asst. Prof. L. M., Ph.D. (Columbia). U. S. Naval Acad., Annapolis, Md. *22 Thompson St.*
KEMPNER, Prof. A. J., Ph.D. (Göttingen). Head of the Dept., Univ. of Colorado, Boulder, Colo.
KENDALL, Asst. Prof. CLARIBEL, Ph.D. (Chicago). Univ. of Colorado, Boulder, Colo. *1305 Euclid Ave.*
KENNEDY, E. C., E.M. (Texas). Box 254, Ranger, Tex.
KENNEDY, KATHRYN M., A.B. (Indiana State Normal). Normal Training High School, Terre Haute, Ind.
KENNELLY, Prof. A. E., Sc.D. (Pittsburgh; Toulouse). Elec. Eng., Harvard Univ., Cambridge, Mass.
KERSTEN, H. J., M.S. in E.E. (Wisconsin), A.M. (Chicago). Instr., Coll. of Eng. and Commerce, Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math., Coll. of Eng. and Commerce.*
KEULEGAN, G. H., A.M. (Ohio State). Asso. Physicist, Bur. of Standards, Washington, D. C. *Room 311, N. W. Bldg.*
KIEFER, Prof. E. C., M.S. (Michigan). James Millikin Univ., Decatur, Ill. *1299 W. Macon St.*
KIEFFER, NORA A., A.M. (Columbia). Cumberland Valley State Teachers School, Shippensburg, Pa.
KILLEN, Asst. Prof. C. G., A.B. (La. State Normal). La. State Normal Coll., Natchitoches, La.
KILLEBREW, C. D., M.S. (Ala. Poly.). Alabama Poly. Tech., Auburn, Ala.
KILPATRICK, Miss SYDNEY M., A.B. (La. State). Teacher, Bolton High School, Alexandria, La.
KIMBALL, Prof. R. S., A.M. (Brown). Dir., Citizenship Training Assoc., Columbus, Ohio. *301 Parkview Bldg.*
KIMBALL, T. C., A.M. (Princeton). Teacher, Lawrenceville School, Lawrenceville, N. J. *Box 207.*
KINDLE, Prof. J. H., A.M. (Ohio State). Univ. of Cincinnati, Cincinnati, Ohio.
KING, EULA WEEKS (Mrs. H. L.), Ph.D. (Missouri). 3228 Lafayette Ave., St. Louis, Mo.
KINGERY, Prof. D. N., A.M. Macalester Coll., St. Paul, Minn. *135 Amherst St.*
KINGSTON, Prof. H. R., Ph.D. (Chicago). Math. and Astr., Univ. of Western Ontario, London, Ont.
KINNEY, J. M., Ph.D. (Chicago). Instr., Crane Jr. Coll., Chicago, Ill. *8058 Bennett Ave.*
KIRCHNER, Prof. W. H., B.S. (Worcester Poly.). Drawing and Descr. Geom., Univ. of Minnesota, Minneapolis, Minn. *208 Main Eng. Bldg.*
KLAUBER, L. M., A.B. (Stanford). San Diego Cons. Gas and Elec. Co., San Diego, Calif. *233 W. Juniper St.*
KLINE, Asst. Prof. J. R., Ph.D. (Penna.). Univ. of Pennsylvania, Philadelphia, Pa. *Box 6, College Hall.*
KLIPPLE, E. C., A.B. (Texas). Instr., Jr. College, San Antonio, Tex.
KNAPP, Prof. G. A., A.M. (Hamilton). Maryville Coll., Maryville, Tenn.
KNEBELMAN, Asst. Prof. M. S., M.S. (Lehigh). Lehigh Univ. *On leave of absence 1927–1928, Grad. School, Princeton Univ., Princeton, N. J. 9 Aiken Ave.*
KNEDLER, P. A., A.B. (Muhlenberg). Instr., High School, Crawford, N. J. *16 Berkeley Pl.*
KNEPPER, Prof. MARTHA M., A.M. (Missouri). State Teachers Coll., Cape Girardeau, Mo. *909 College Hill.*
KNIGHT, ELIZABETH E., A.M. (Illinois). Instr., State Normal School, Milwaukee, Wis. *627 Downer Ave.*
KNIGHT, KATE K., A.B. (E. Central Teachers Coll.). Instr., E. C. T. C., Ada, Okla.
KNIGHT, Asst. Prof. L. C., Ph.B. (Wooster). The Coll. of Wooster, Wooster, Ohio.
KNISELY, ALEXANDER. Columbia City, Ind.
KNOX, Prof. J. J., M.S. (Chicago). Dak. Wesleyan Univ., Mitchell, S. Dak. *1219 W. Univ. Blvd.*
KOCH, E. H., Jr., B.S. in E.E. (Penna.). Brooklyn Tech. High School. *208 N. Maple Ave., East Orange, N. J.*
KOKOMOOR, Asst. Prof. F. W., Ph.D. (Michigan). Univ. of Florida, Gainesville, Fla.
KONANTZ, Prof. EMMA L., A.M. (Ohio Wesleyan). Peking Univ., Kwei Chia Chiang, Peking, China. *On leave of absence 1927-28, 2506 3rd Ave. S., Minneapolis, Minn.*
KOVARIK, Prof. A. F., Ph.D. (Minnesota). Physics, Yale Univ., New Haven, Conn. *Sloane Lab.*
KRATHWOHL, Asso. Prof. W. C., Ph.D. (Chicago). Armour Inst. of Tech., Chicago, Ill.
KREINES, G. A., B.S. (New York Univ.). Math., Columbia Univ. *1001 Lafayette Ave., Brooklyn, N. Y.*
KRETH, DANIEL, C. E. Surveyor and Civil Engr., Wellman, Ia.
KRYLOFF, Prof. NICOLAS, Doctor of Math. *honoris causa* (Kieff). U. R. S. S., Kieff, Ukraine. *Box N 135, Postpack.*
KUHN, Prof. H. W., Ph.D. (Cornell). Ohio State Univ., Columbus, Ohio. *1145 Fairview Ave.*
KUNTE, HELEN L., A.M. (Columbia). Instr., Hunter Coll., New York, N. Y. *Park Ave. and 68th St.*
KURZIN, W. H., M.S. (Chicago). Teacher, Crane Jr. Coll., Chicago, Ill. *1631 W. Roosevelt Rd.*
KUSNER, J. H., A.B. (Penna.). Instr., Univ. of Florida, Gainesville, Fla. *Dept. of Math.*
KYES, Prof. D. H., Ph.D. (Boston Univ.). Math. and Physics, Dak. Wesleyan Univ., Mitchell, S. Dak. (?) *193 Mackubin St., St. Paul, Minn.*

LABOCCETTA, Ing. LETTERIO, C.E. (Naples). Via Due Macelli 31, Rome (7), Italy.
LAMBERT, W. D., A.M. (Harvard). Mathematician, U. S. Coast and Geod. Surv., Washington, D. C.
LAMPEN, Prof. A. E., A.M. (Michigan). Hope Coll., Holland, Mich. *86 E. 14th St.*
LAMPLAND, C. O., A.M. (Indiana). Astronomer, Lowell Observ., Flagstaff, Ariz.
LAMPSON, Asst. Prof. K. W., Ph.D. (Chicago). Lehigh Univ., Bethlehem, Pa.
LANDIS, Prof. W. W., A.M. (Dickinson). Dickinson Coll., Carlisle, Pa.
LANDRY, Prof. A. E., Ph.D. (Johns Hopkins). Catholic Univ. of America, Washington, D. C. *3624 13th St., Brookland, D. C.*
LANE, Asso. Prof. E. P., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
LANGE, LUISE, Ph.D. (Göttingen). 5643 Dorchester Ave., Chicago, Ill.
LANGER, Prof. R. E., Ph.D. (Harvard). Univ. of Wisconsin, Madison, Wis. *521 S. Randall Ave.*
LANGMAN, HARRY, Ph.D. (Columbia). Prop., Solar Chem. Mfg. Co., Brooklyn, N. Y. *6919 Burchell St., Arverne, L. I., N. Y.*
LAO, CHI-CHIANG, A.B. (Yale in China). Instr., Coll. of Yale in China, Changsha, Hunan, China.
LAREW, Prof. GILLIE A., Ph.D. (Chicago). Randolph-Macon Woman's Coll., Lynchburg, Va.
LARKIN, E. J., B.S. (Cooper Union). Instr., Cooper Union, New York, N. Y. *P. O. Box 595, Westbury, L. I., N. Y.*
LARSON, Asst. Prof. OLGA, A.M. (Missouri). Fla. State Coll. for Women. *Box 84, Apopka, Fla.*
LASLEY, Prof. J. W., Jr., Ph.D. (Chicago). Pure Math., Univ. of North Carolina, Chapel Hill, N. C. *322 Rosemary Lane.*
LATIMER, Prof. C. G., Ph.D. (Chicago). Univ. of Kentucky, Lexington, Ky. *Dept. of Math.*
LATSHAW, ELMER. 1839 N. 60th St., West Philadelphia, Pa.
LAURENTINE, Sister MARIE, A.B. (Trinity Coll., Washington). Instr., Emmanuel Coll., Boston, Mass.
LAVES, Asso. Prof. KURT, Ph.D. (Berlin). Univ. of Chicago, Chicago, Ill. *5611 Kenwood Ave.*
LEARY, A. F., Ed.M. (Boston Coll.). Jr. Master, English High School, Boston, Mass. *English High School, Montgomery St.*
LEAVENS, D. H., A.M. (Yale). Treas. Coll. of Yale in China, Changsha, China. *202 Broadway, Norwich, Conn.*
LEE, G. FARLEY, A.M. (Columbia). Union Springs, Ala.
LEFSCHETZ, Asso. Prof. SOLOMON, Ph.D. (Clark). Princeton Univ., Princeton, N. J.
LEHMAN, Prof. D. A., A.M. (Western Reserve). Goshen Coll., Goshen, Ind.
LEHMANN, C. H., M.E. (Cooper Union). Tel. Engineer, N. Y. Tel. Co; Instr., Night Schools of Eng., Cooper Union. *1502 Pacific St., Brooklyn, N. Y.*
LEHMER, Prof. D. N., Ph.D. (Chicago). Univ. of California, Berkeley, Calif. *2736 Regent St.*
LEIB, Prof. D. D., Ph.D. (Johns Hopkins). Connecticut Coll., New London, Conn.
LEIPER, Prof. C. L., Grad. U. S. Naval Acad. U. S. Naval Acad., Annapolis, Md.
LEMON, H. B., A.M. (Michigan). Ins. Dept., Chamber of Commerce of U. S., Washington, D. C. *2440 20th St. N. W.*
LENNES, Prof. N. J., Ph.D. (Chicago). Univ. of Montana, Missoula, Mont.
LEONARD, Prof. H. B., Ph.D. (Colorado). Univ. of Arizona, Tucson, Ariz.
LEONARD, KATHARINE, A.M. (Vermont). Head of Dept., State Teachers Coll., Moorhead, Minn.
LEONARD, RUTH E., M.S. (Chicago). Box 224, Manhattan, Kans.
LERCH, F. S., A.B. (Lehigh). Instr., Univ. of Delaware, Newark, Del. *146 W. Main St.*
LESTER, Prof. O. C., Ph.D. (Yale). Dean of Grad. School and Prof. of Physics, Univ. of Colorado, Boulder, Colo. *1061 Eleventh St.*
LE STOURGEON, Asso. Prof. ELIZABETH, Ph.D. (Chicago). Univ. of Kentucky, Lexington, Ky.
LEVY, Asst. Prof. HARRY, Ph.D. (Princeton). Univ. of Illinois, Urbana, Ill.
LEVY, Asst. Prof. SOPHIA H., Ph.D. (California). Univ. of California, Berkeley, Calif. *475 Wheeler Hall.*
LEWANDOWSKI, STEPHEN, B.S. (Marquette). Teacher of Alg., Kilbourn Pre-Voc. School, Milwaukee, Wis. *4 Wright St.*
LEWIS, Prof. ANNA D., Ph.D. (Carleton). Math. and Physics, Lake Erie Coll., Painesville, Ohio.
LEWIS, A. J., A.M. (Denver). Instr., Univ. of Denver. *Wheatridge, Colo.*
LEWIS, Asso. Prof. C. F., M.S. (K.S.A.C.). State Agric. Coll., Manhattan, Kans.
LEWIS, Asso. Prof. F. A., A.M. (Alabama). Univ. of Alabama, University, Ala. *Box 1444.*
LEWIS, Prof. FLORENCE P., Ph.D. (Hopkins). Goucher Coll., Baltimore, Md.
LIBBY, B. B., A.M. (Wisconsin). Lowell High School, San Francisco, Calif. *1350 27th Ave.*
LIBMAN, E. E., Ph.D. (Illinois). Natl. Research Fellow, Univ. of Illinois, Urbana, Ill. *805 S. Lynn St.*
LIGHT, Prof. G. H., Ph.D. (Yale). Univ. of Colorado, Boulder, Colo. *801 Base Line.*
LINARES, ENRIQUE, B.S. in E.E. (Univ. of Santa Clara). Actg. Chief Eng., Central Board of Roads, Panama City, R. P. *P. O. Box 540.*
LINDEMANN, Prof. C. A., A.M. (Bucknell). Pure Math. and Secy. of Faculty, Bucknell Univ., Lewisburg, Pa.

LINDQUIST, Prof. THEODORE, Ph.D. (Chicago). Mich. State Normal Coll., Ypsilanti, Mich. *103 Elm St.*

LINDSEY, Prof. LOUIS, Ph.D. (Syracuse). Appl. Math., Syracuse Univ., Syracuse, N. Y. *201 Scottholm Blvd.*

LINEHAN, Asso. Prof. P. H., Ph.D. (Columbia). Coll. of the City of New York, New York, N. Y. *346 Convent Ave.*

LINFIELD, Asso. Prof. B. Z., Ph.D. (Harvard). Univ. of Virginia, University, Va.

LING, Dean G. H., Ph.D. (Columbia). Univ. of Saskatchewan, Saskatoon, Can.

LINHART, G. A., Ph.D. (Yale). Teacher, Jr. Coll., Riverside, Calif. *Junior College.*

LINTON, M. ALBERT, A.M. (Haverford). Vice-Pres., Actuary, The Provident Mut. Life Ins. Co., Philadelphia, Pa. *409 Chesnut St.*

LITTAUER, S. B., Ch. E. (Rensselaer). Hunter Coll., New York, N. Y. *150 Claremont Ave.*

LIVINGSTON, Asst. Prof. G. R., A.M. (California). State Teachers Coll., San Diego, Calif.

LOCKE, L. L., A.M. (Grove City Coll.). Teacher, Brooklyn Tr. School for Teachers, Brooklyn, N. Y. *950 St. Johns Place.*

LOCKWOOD, E. C., A.M. (Brown). Instr., American Coll., Madura, S. India.

LODWICK, DECA, Ph.B. (Iowa). Head of Dept. of Math., High School, Long Beach, Calif. *4112 Colorado St.*

LOEWEN, Prof. O. B., A.M. (Kansas). Ottawa Univ., Ottawa, Kans.

LOGSDON, Asst. Prof. Mrs. MAYME I., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.

LOH, ZUNG-NYI, A.M. (Cornell). Nansiang, Kiangsu, China.

LONG, Asst. Prof. FLORENCE, M.S. (Illinois). Earlham Coll., Earlham, Ind.

LONG, Asst. Prof. T. R., A.M. (Rochester). Univ. of Rochester, Rochester, N. Y. *75 Cypress St.*

LONG, Prof. W. F., A.B. (F. and M.). Math. and Astr., Franklin and Marshall Coll., Lancaster, Pa.

LONGENECKER, J. V., M.S. (Iowa). Actuary, Farmers and Bankers Life Ins. Co., Wichita, Kans. *701 Beacon Bldg.*

LONGLEY, Prof. W. R., Ph.D. (Chicago). Yale Univ., New Haven, Conn.

LONGMIRE, Mrs. BIRDIE E., A.B. (Univ. of Nashville). Teacher, Bolton High School, Alexandria, La. *2104 White St.*

LORD, MARGARET, A.B. (Mt. Holyoke). 198 Jackson St., Lawrence, Mass.

LOVE, Prof. C. E., Ph.D. (Michigan). Univ. of Michigan, Ann Arbor, Mich.

LOVETT, Pres. E. O., Ph.D. (Virginia; Leipzig). Rice Inst., Houston, Tex.

LOVITT, Prof. W. V., Ph.D. (Chicago). Colorado Coll., Colorado Springs, Colo.

LOWENSTEIN, L. L., A.B. (Cornell). Grad. Student, Cornell Univ., Ithaca N. Y. *425 Wyckoff Ave.*

LOWRY, G. L., A.M. (Bucknell). Vice-Prin., High School, Olean, N. Y. *124 Laurens St.*

LUBBEN, Adj. Prof. R. G., Ph.D. (Texas). Pure Math., Univ. of Texas, Austin, Texas. *708 W. 22½ St.*

LUBIN, C. I., Ch.E. (Cincinnati). Grad. School, Harvard Univ., Cambridge, Mass. *Conant 36.*

LUBY, WILLIAM, A.M. (Kansas). Jr. Coll., Kansas City, Mo. *5411 Rockhill Rd.*

LUCEY, HENRY, A.B. (Amherst). Public Latin School, Boston, Mass. *101 Fernwood Rd.*

LUCK, Prof. J. J., Ph.D. (Virginia). Univ. of Virginia, University, Va. *Colonnade Club.*

LUFKIN, H. M., Ph.D. (Penna.) Instr., Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*

LUNN, Prof. A. C., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.

LUTZ, H. L., A.M. (Columbia). Instr., Cooper Union, New York, N. Y. *55 W. 184th St.*

LUTZ, Asst. Prof. JUNA M., A.M. (Chicago). Butler Coll., Indianapolis, Ind. *2524 College Ave.*

LYLE, Adj. Prof. G. A., M.S. (Lehigh). Texas Tech. Coll., Lubbock, Tex.

LYMAN, Prof. E. A., A.B. (Michigan). Michigan State Normal Coll., Ypsilanti, Mich. *126 N. Washington St.*

LYONS, Asst. Prof. W. H., A.M. (Denver). State Agric. Coll., Manhattan, Kans. *820 Moro St.*

LYTLE, Asst. Prof. E. B., Ph.D. (Yale). Teaching of Math., Univ. of Illinois, Urbana, Ill. *903 Busey Ave.*

MACCREADIE, Asst. Prof. W. T., A.M. (Harvard). Bucknell Univ., Lewisburg, Pa. *1425 W. Market St.*

MACDONALD, MARY H., A.M. (Bryn Mawr). Teacher, Physics and Geom., Shipley School, Bryn Mawr, Pa. *124 Coulter Ave., Ardmore, Pa.*

MACDONALD, Prof. S. L., M.S. (Columbia). Math. and Astr., State Agric. Coll., Fort Collins. Colo.

MACDONALD, Prof. W. E., A.M. (Harvard). Canton Christian Coll., Canton, China.

MACDUFFEE, Asst. Prof. C. C., Ph.D. (Chicago). Ohio State Univ., Columbus, Ohio. *366 Oakland Park Ave.*

MACGREGOR, F. S., A.B. (Harvard). Mgr. Textbook Dept., Harper and Bros., New York, N. Y. *49 E., 33rd St.*

MACKIE, Asso. Prof. E. L., Ph.D. (Chicago). Univ. of North Carolina, Chapel Hill, N. C. *9 Cobb Terr.*

MacMillan, Prof. W. D., Ph.D. (Chicago). Astr., Univ. of Chicago, Chicago, Ill.
MacNeish, Asst. Prof. H. F., Ph.D. (Chicago). Coll., of the City of New York. *Scarsdale, N. Y.*
MacNutt, Barry, M.S. (Lehigh). Gloucester, Mass.
McAlister, Prof. E. H., A.M. (Oregon). Mech. and Astr., Univ. of Oregon, Eugene, Ore. *334 Pearl St.*
McCain, Prof. Gertrude I., Ph.D. (Indiana). Westminster Coll., New Wilmington, Pa.
McCarthy, E. D., A.B. (Cornell). Instr., Penn. State Coll., State College, Pa. *Univ. Club.*
McCarty, A. L., A.M. (California). Instr., Lowell High School, San Francisco, Calif. *2137 Parker St., Berkeley, Calif.*
McCarty, M. E., A.M. (Michigan). Head of Dept., Director of Extension Dept., State Normal School, Fairmont, W. Va. *1108 Fenimore St.*
McClellan, Ada A., B.S. (Chicago). Teacher, High School, Long Beach, Calif. *313 N. New Hamp. Ave., Los Angeles, Calif.*
McClenon, Prof. R. B., Ph.D. (Yale). Grinnell Coll., Grinnell, Ia. *1512 Fourth Ave.*
McCormick, Prof. Clarence, A.M. (Clark). Southwestern State Teachers Coll., Weatherford, Okla. *On leave 1927–1928, attending Teachers Coll., Columbia Univ.*
McCue, Dean M. J., C.E., M.S. (Notre Dame). Coll. of Eng., Univ. of Notre Dame, Notre Dame, Ind.
McDill, Prof. R. M., A.M. (Indiana). Hastings Coll., Hastings, Nebr. *119 E. 7th St.*
McDonald, Mrs. Emma Whiton (Mrs. J. H. McDonald), A.M. (California). 769 Vincente Ave., Berkeley, Calif.
McDonough, Laura F., A.B. (Penna.). Teacher, Moravian Coll. for Women, Bethlehem, Pa.
McEwen, Prof. G. F., Ph.D. (Stanford). Physical Oceanography, Scripps Inst., Univ. of California. *Box 68, La Jolla, Calif.*
McEwen, W. H., A.M. (Minnesota). Instr., Regina Coll., Regina, Sask., Canada.
McFarland, Dora, A.M. (Oklahoma). Instr., Univ. of Oklahoma, Norman, Okla.
McFarland, Elsie J., Ph.D. (California). Instr., Univ. of Oklahoma, Norman, Okla. *Faculty Exchange.*
McGavock, Asst. Prof. Martha P., A.M. (Chicago). Rockford Coll., Rockford, Ill.
McGaw, Prof. F. M., A.M. (Phila.). Math. and Eng. Drawg., Cornell Coll., Mt. Vernon, Ia.
McGinley, Margaret F., A.M. (Colorado). Teacher, State Preparatory School, Boulder, Colo. *1045 Lincoln Pl.*
McKee, J. R. Instructing Federal Students, College Station, Tex.
McKelvey, Asso. Prof. J. V., Ph.D. (Cornell). State Coll., Ames, Ia. *2117 Graeber St.*
McKelvey, Martha M. (Mrs. J. V.), M.S. (Iowa). State Coll., Ames, Ia. *2117 Graeber St.*
McKinney, Prof. T. E., Ph.D. (Chicago). Math. and Astr., Univ. of South Dakota, Vermilion, S. D. *222 N. Univ. St.*
McKnight, Nannie J., A.M. (George Washington). Teacher, Western High School, Washington, D. C. *3037 Dumbarton Ave.*
McLatchey, Nina M., A.M. (Kansas). Teacher, High School, Topeka, Kans. *917 Morris Ave.*
McMackin, F. J., Ph.D. (Columbia). Instr., Cooper Union Inst. of Tech., New York, N. Y. *49 Vreeland Ave., Rutherford, N. J.*
McMillan, Mary B., A.M. (Wisconsin). Wisconsin Rapids, Wis.
McNatt, J. Q. Div. Engr., Colo. Fuel and Iron Co., Canon City, Colo. *910 Harding Ave.*
McNelly, Sister Mary of the Presentation, A.M. (Catholic Univ.). Prof., Our Lady of the Lake Coll., San Antonio, Tex.
McShane, E. J., B.E., B.S. (Tulane). Instr., Tulane Univ., New Orleans, La. *1426 Touro St.*
McSweeny, Prof. A. A., A.M. (Montana). Head of Dept., John Tarleton Agric. Coll., Stephenville, Tex.
Maddox, Prof. A. C., A.M. (Columbia). La. State Normal Coll., Natchitoches, La.
Mahoney, J. O., M.S. (Vanderbilt). Dir., Dallas Evening Schools, Dallas, Tex. *1900 Crockett St.*
Maiden, W. M., A.M. (Virginia). Washington Square Coll., New York Univ., New York, N. Y. *Washington Sq. E.*
Maizlish, Asso. Prof. Israel, Ph.D. (Minnesota). Physics, Centenary Coll., Shreveport, La.
Maizlish, Mrs. Yetta V., A.B. (Centenary). Instr., Centenary Coll. *On leave of absence, Grad. School., Univ. of Iowa; 121 E. Fairchild St., Iowa City, Ia.*
Majella, Bro., A.M. (Catholic Univ.). St. John's Prep. School, Danvers, Mass.
Male, C. T., M.C.E. (Union). Instr., Union Coll. *Niskayuna, Schenec. Co., N. Y.*
Mallory, V. S., A.M. (Columbia). Dir. Dept. of Math., High School, East Orange, N. J. *351 Springdale Ave.*
Manchester, Prof. R. E., A.M. (Michigan). State Teachers Coll., Kent, Ohio.
Maney, Prof. C. A., M.S. (Chicago). Prof. and Registrar, Transylvania Coll., Lexington, Ky.
Mangold, Prof. M. Cecilia, M.S. (Trinity Coll., Washington). Trinity Coll., Washington, D. C.
Manning, Dr. H. M. U. S. Public Health Service, Marcus Hook, Pa.
Manning, Asso. Prof. H. P., Ph.D. (Johns Hopkins). Retired Brown Univ., Providence, R. I. *106 Carrington Ave., E. Side Sta.*
Manson, Prof. E. S., M.S. (Mass. Inst. of Tech.). Astr., Ohio State Univ., Columbus, Ohio.

MANY, ANNA E., A.M. (Tulane). Counselor to Women, Sophie Newcomb Memorial Coll., New Orleans, La. *1325 Henry Clay Ave.*
MARKLEY, Prof. J. L., Ph.D. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *1816 Geddes.*
MARM, ANNA, A.M. (Univ. of Kansas). Instr., Univ. of Kansas, Lawrence, Kans. *741 N. 2d St. Lindsborg, Kans.*
MARQUARD, Asso. Prof. W. B., E.M. (Ohio State). Appl. Mech., Lafayette Coll., Easton, Pa. *828 Burke St.*
MARRIOTT, Asst. Prof. R. W., Ph.D. (Pennsylvania). Swarthmore Coll., Swarthmore, Pa.
MARRS, Prof. MARY E., A.B. (N. Tex. Normal Coll.). John Tarleton Agric. Coll., Stephenville, Tex. *Box 342.*
MARSH, J. A., A.M. (Amherst). Head of Dept., High School of Commerce, Boston, Mass. *56 Peter Parley Rd., Boston 30.*
MARSHALL, FRANCES L., B.S. (Ottawa). *4532 Washington St., Kansas City, Mo.*
MARSHALL, Prof. WILLIAM, Ph.D. (Zurich). Head of Dept., Purdue Univ., LaFayette, Ind.
MARTIN, Prof. EMILIE N., Ph.D. (Bryn Mawr). Mount Holyoke Coll., South Hadley, Mass.
MARTIN, MARY C., B.S. (Knox). *404 W. Stephenson St.*, Freeport, Ill.
MARTIN, Prof. T. A., A.M. (Ohio Wesleyan; Yale). Berea Coll., Berea, Ky.
MARTINSON, E. P., A.M. (Nebraska). Instr., Colo. School of Mines, Golden, Colo.
MARTYN, W. J., M.A. (New Zealand). Mathematical Master, Otago Boys High School, Dunedin, New Zealand.
MASKELL, Sister BLANCHE MARIE, A.M. (Fordham). Coll. of St. Elizabeth, Convent Station, N. J.
MASON, Prof. T. E., Ph.D. (Indiana). Purdue Univ., LaFayette, Ind. *103 Waldron St.*
MASON, W. E., M.S.E. (Michigan). Instr., Appl. Math., Univ. of California at Los Angeles, Calif. *855 N. Vermont Ave.*
MATHESON, Prof. J., M.A. (Queen's Univ.). Queen's Univ., Kingston, Ont., Can.
MATHEWS, JANE H., B.S. (Columbia). Teacher, High School, Pittsburgh, Pa. *King Edward Apts., Melwood and Bayard Sts.*
MATHEWS, Asso. Prof. R. M., Ph.D. (Illinois). West Virginia Univ., Morgantown, W. Va. *332 S. High St.*
MATHEWSON, Asst. Prof. L. C., Ph.D. (Illinois). Dartmouth Coll., Hanover, N. H.
MATHIAS, Prof. H. R., A.M. (Indiana). Indiana Central Coll., Indianapolis, Ind. *1421 Castle Ave.*
MATTESON, Asso. Prof. JANE L., Michigan State Normal Coll., Ypsilanti, Mich. *318 Cross St.*
MAUCH, MARGARET E., M.S. (Chicago). Instr., Randolph-Macon Woman's Coll., Lynchburg, Va. *R.M.W.C., Box 294.*
MAULDIN, JANICE W., A.M. (Chicago). Instr., Centenary Coll., Shreveport, La.
MAURUS, Prof. E. J., M.S. (Notre Dame). Univ. of Notre Dame, Notre Dame, Ind. *L. B. 75.*
MAY, Prof. LIDA B., A.M. (Texas). Kidd-Key Coll., Sherman, Texas.
MAYER, Asst. Prof. E. S., A.M., C.E. (Rensselaer). U. S. Naval Acad., Annapolis, Md.
MEACHAM, Prof. E. D., Ph.D. (Chicago). Univ. of Oklahoma, Norman, Okla. *529 Lahoma Ave.*
MEARS, FLORENCE M., Ph. D. (Cornell). Head of Dept., Alabama Coll., Montevallo, Ala. *Box 113.*
MEDER, A. E., Jr., A.M. (Columbia). Instr., N. J. Coll. for Women, New Brunswick, N. J. *424 St. Mark's Ave., Westfield, N. J.*
MEEK, KATE M. High School, Pasadena, Calif. *919 E. Calif. St.*
MELVILLE, Asso. Prof. C. E., A.B. (Northwestern). Clark Univ., Worcester, Mass.
MENDENHALL, Pres. W. O., Ph.D. (Michigan). Friends Univ., Wichita, Kans.
MENSENKAMP, L. E., A.M. (Illinois). Instr., High School, Freeport, Ill. *521 W. Cottonwood St.*
MENUET, Asso. Prof. R. L., B.E. (Tulane). Tulane Univ., New Orleans, La. *443 Lowerline St.*
MERCEDES, Sister M., M.S. (Notre Dame). Instr., Mary Manse Coll., Toledo, Ohio.
MERGENDAHL, Asst. Prof. T. E., M.S. (Tufts). Tufts Coll., Tufts College, Mass.
MERRILL, Prof. A. S., Ph.D. (Chicago). Univ. of Montana, Missoula, Mont. *541 Beckwith Ave.*
MERRILL, Prof. HELEN A., Ph.D. (Yale). Wellesley Coll., Wellesley, Mass.
MERRILL, J. E., M.S. (Case). Instr., Case School of Appl. Sc., Cleveland, Ohio. *Dept. of Math.*
MERRISS, A. A. 913 East 23d St. North, Portland, Ore.
MESSICK, Asst. Prof. C. A., A.M. (Kansas). Univ. of Florida, Gainesville, Fla. *1638 W. Mechanic.*
MESSICK, Prof. J. F., Ph.D. (Johns Hopkins). Emory Univ., Emory University, Ga.
MEYER, J. B., M.S. (Purdue). Head of Dept., State Teachers Coll., Valley City, N. D. *814 Fifth Ave.*
MEYER, Rev. J. H., S.J. 132 Baronne St., New Orleans, La.
MICHAL, Asst. Prof. A. D., Ph.D. (Rice). Ohio State Univ., Columbus, Ohio.
MICHIE, Prof. J. N., A.M. (Michigan). Texas Tech. Coll., Lubbock, Tex. *2215 16th St.*
MICKELSON, E. L., A.M. (Minnesota). Instr., Univ. of Wisconsin, Madison, Wis. *Dept. of Math.*
MICKLE, M. KATHERINE, A.B. (Alabama). Roanoke, Ala.
MIKAMI, YOSHIO. Reasearch Assoc., Imperial Acad., Tokyo, Japan. *7 Daini Mejiro Bunkamura, 1321 Shimo Ochiai.*
MIKESH, J. S. Lawrenceville School, Lawrenceville, N. J.

MILES, Asso. Prof. E. J., Ph.D. (Chicago). Yale Univ., New Haven, Conn. *87 Marvel Rd.*
MILLER, A. L., Ph.D. (Harvard). 25 Clinton Rd., Brookline 46, Mass.
MILLER, Prof. BESSIE I., Ph.D. (Johns Hopkins). Rockford Coll., Rockford, Ill.
MILLER, C. W., B.S. (Bucknell). Teacher, High School, Newburgh, N. Y. *21 Farrell St.*
MILLER, Prof. E. B., A.M. (Chicago). Illinois Coll., Jacksonville, Ill. *302 Woodland Pl.*
MILLER, F. H., M.S. (Cornell). Instr., Cooper Union. *8636 104th St., Richmond Hill, N. Y.*
MILLER, Prof. G. A., Ph.D. (Cumberland). Univ. of Illinois, Urbana, Ill. *1203 W. Illinois St.*
MILLER, Prof. I. L., A.M. (Indiana). State Coll. of A. and M. Arts, Brookings, S. D. *811 12th Ave.*
MILLER, Prof. J. A., Ph.D. (Chicago). Math. and Astr., Swarthmore Coll., Swarthmore Pa.
MILLER, Prof. J. J., A.M. (Oklahoma). Oklahoma Coll. for Women, Chickasha, Okla. *1421 Minn. Ave.*
MILLER, Prof. J. S., Sc.D. (Virginia). Emory and Henry Coll., Emory, Va.
MILLER, Asso. Prof. NORMAN, Ph.D. (Harvard). Queen's Univ., Kingston, Ont., Can.
MILLER, W. I., A.M. (Bucknell). Fellow, Univ. of Pittsburgh. *722 Broad St., Sewickley, Pa.*
MILLIKAN, C. B., Ph.B. (Yale). Teaching Fellow, Calif. Inst. of Tech., Pasadena, Calif. *300 Palmetto Dr.*
MILLINGTON, Asst. Prof. H. G., C.E. (Rensselaer). Coll. of Eng., Univ. of Vermont, Burlington, Vt. *224 Plattsburg Ave.*
MILLS, Prof. C. N., A.M. (Indiana). Illinois State Normal Univ., Normal, Ill. *602 N. School St.*
MILLS, VICENTE. Chief Surveyor, Bureau of Lands, Manila, P. I. *P. O. Box 2442.*
MILNE, Asst. Prof. T. H., M.A. (Toronto). Univ. of Manitoba, Winnipeg, Man., Can.
MILNE, Prof. W. E., Ph.D. (Harvard). Univ. of Oregon, Eugene, Ore.
MINISTER, J. E., B.S. (Ohio Univ.). Teacher, High School, Ingram, Pittsburgh, Pa. *128 Sampson Ave.*
MIRICK, G. R., A.M. (Michigan). Teacher, Lincoln School, Teachers Coll., New York, N. Y. *425 W. 123d St.*
MISER, NELLIE P. (Mrs. W. L.), A.B. (Huron Coll.). Teacher, Coll. Math., Ward-Belmont School, Nashville, Tenn. *1715 Ashwood Ave.*
MISER, Prof. W. L., Ph.D. (Chicago). Vanderbilt Univ., Nashville, Tenn. *1715 Ashwood Ave.*
MITCHELL, A. K., A.M. (Johns Hopkins). Grad. Student, Johns Hopkins Univ., Baltimore, Md. *Box 544, J. H. U.*
MITCHELL, Prof. B. E., Ph.D. (Columbia). Millsaps Coll., Jackson, Miss. *727 Arlington Ave.*
MITCHELL, Mrs. DECIMA E., M.A. (Alberta). Grad. Student, Univ. of Texas, Austin, Tex. *600 W. 32nd St.*
MITCHELL, H. F., B.S. (Chicago). Teacher, Booker T. Washington High School, Tulsa, Okla. *541 N. Detroit Ave.*
MITCHELL, Prof. H. H., Ph.D. (Princeton). Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
MITCHELL, Prof. U. G., Ph.D. (Princeton). Univ. of Kansas, Lawrence, Kans. *1313 Mass. St.*
MITRA, Prof. N. B., M.A. (Calcutta Univ.). Ewing Christian Coll., Allahabad City, U. P., India.
MIZE, CLARA P., M.S. (Chicago). Head of Dept., Martin Coll., Pulaski, Tenn. *Box 432.*
MODE, Asst. Prof. E. B., A.M. (Harvard). Boston Univ., Boston, Mass.
MODESITT, R. L., A.M. (Indiana). Teacher, Eastern Illinois State Teachers Coll., Charleston, Ill. *1021 Taylor St.*
MOENCH, Prof. L. W., A.B. (Macalester). Math. and Physics, Phalen Luth. Sem., St. Paul, Minn. *795 E. Maryland St.*
MOLINA, E. C. Tel. Engr., Tel. and Tel. Co., 195 Broadway, New York, N. Y.
MOLLOY, Sister M. CORONA, M.S. (Notre Dame). Mount St. Joseph Coll., Mount St. Joseph, Ohio.
MOODY, Prof. W. A., A.M. (Bowdoin). Bowdoin Coll., Brunswick, Me. *60 Federal St.*
MOORE, Asso. Prof. C. L. E., Ph.D. (Cornell). Mass. Inst. of Tech., Cambridge, Mass.
MOORE, Prof. C. N., Ph.D. (Harvard). Univ. of Cincinnati, Cincinnati, Ohio. *219 Woolper Ave.*
MOORE, Prof. E. H., Ph.D. (Yale). Univ. of Chicago, Chicago, Ill. *Hotel del Prado.*
MOORE, Asso. Prof. F. C., A.B. (Dartmouth). Mass. Agric. Coll., Amherst, Mass.
MOORE, G. E., M.S. (Illinois). Asst., Univ. of Illinois, Champaign, Ill. *712 W. Arlington Ct.*
MOORE, L. T., Ph.D. (Johns Hopkins). Instr., Yale Univ., New Haven, Conn. *25 Dwight St.*
MOORE, Prof. R. L., Ph.D. (Chicago). Pure Math., Univ. of Texas, Austin, Texas.
MOORE, T. W., Ph.D. (Yale). Natl. Research Fellow, Johns Hopkins Univ., Baltimore, Md.
MOORE, Prof. W. A., A.M. (Chicago). Birmingham Southern Coll., Birmingham, Ala.
MOOREFIELD, NANCY L., A.M. (Columbia). Chicora Coll., Columbia, S. C.
MOOTS, E. E., Ph.D. (Iowa). Chairman, Math. and Eng., Cornell Coll., Mt. Vernon, Ia. *824 Summit Ave.*
MORENO, Prof. H. C., Ph.D. (Clark). Applied Math., Stanford University, Calif.
MORENUS, Prof. EUGENIE M., Ph.D. (Columbia). Sweet Briar Coll., Sweet Briar, Va. *On leave 1927–1928.*
MORGAN, F. M., Ph.D. (Cornell). Asst. Dir., Clark School, Hanover, N. H.
MORGAN, W. D. 740 Curfew Ave., St. Paul, Minn.

MORIARTY, M. M. S., A.B. (Holy Cross). Instr., Holyoke High School, Holyoke, Mass.
MORIN, Sister M. PRUDENTIA, A.B. (Catholic Univ.). Teacher, Coll. of St. Scholastica, Duluth, Minn.
MORITZ, Prof. R. E., Ph.D. (Strassburg). Univ. of Washington, Seattle, Wash.
MORLEY, Prof. FRANK, A.M., Sc.D. (Cambridge, Eng.). Johns Hopkins Univ., Baltimore, Md. *2026 Park Ave.*
MORLEY, Prof. R. K., Ph.D. (Clark). Worcester Poly. Inst., Worcester, Mass. *43 Laconia Rd.*
MORRILL, W. K., A.M. (Johns Hopkins). 510 Rossiter Ave., Baltimore, Md.
MORRIS, Prof. C. C., A.M. (Harvard). Ohio State Univ., Columbus, Ohio.
MORRIS, E. B., Ph.B. (Sheffield Sc. School). Actuary, Life Dept., The Travelers Ins. Co., Hartford, Conn.
MORRIS, Prof. F. R., Ph.D. (California). State Teachers Coll., Fresno, Calif.
MORRIS, Asst. Prof. MAX, A.M. (Harvard). Case School of Appl. Sc., Cleveland, Ohio.
MORRIS, Prof. RICHARD, Ph.D. (Cornell). Rutgers Coll., New Brunswick, N. J. *12 Johnson St.*
MORROW, E. B., A.M. (Princeton). Headmaster, Gilman Country School, Roland Park, Md.
MORSE, Asst. Prof. D. S., Ph.D. (Cornell). Union Coll., Schenectady, N. Y. *1508 Nott St.*
MORSE, Asst. Prof. MARSTON, Ph.D. (Harvard). Harvard Univ., Cambridge, Mass. *14 Francis St.*
MORSE, P. M., A.M. (Princeton). Grad. Coll., Princeton Univ., Princeton, N. J.
MORSE, W. P., A.B. (Maine). Instr., Univ. of Maine, Orono, Me. *Delta Tau Delta House.*
MORTON, Asso. Prof. A. B., A.M. (Brown). Pure Math., Georgia School of Tech., Atlanta, Ga.
MOSKOVITZ, DAVID, M.S. (Carnegie Tech.). Instr., Carnegie Inst. of Tech., Pittsburgh, Pa. *5884 St., Hobart St.*
MOSSMAN, Asst. Prof. THIRZA A., A.M. (Chicago). State Agric. Coll., Manhattan, Kans.
MOULTON, Prof. E. J., Ph.D. (Chicago). Northwestern Univ., Evanston, Ill. *1114 Colfax St.*
MOULTON, Prof. F. R., Ph.D. (Chicago). Dir. Utilities Power and Light Corp. *227 S. La Salle St., Chicago, Ill.*
MOURAD, Lieut. Commander SALIH (Turkish Navy; retired). Prof. of Physics, Govt. Eng. Coll.; App. Math., Robert College, Constantinople, Turkey. *Robert College, Bebek.*
MOURSUND, A. F., A.M. Instr., Texas Tech. Coll., Lubbock, Tex. *Box 35, Tech. Sta.*
MOYLE, K. E., A.B. (Bucknell). High School, Somerville, N. J.
MUEHLMAN, Rev. Paul, A.M. (St. Louis Univ.). Loyola Univ., Chicago, Ill. *Sheridan Rd. and Loyola Ave.*
MUIR, Sir THOMAS, D.Sc. (Univ. of Cape Town). Late Supt.-Genl. of Educ., Cape Colony. *Elmcote, Sandown Rd., Rondebosch, S. Afr.*
MULLEMEISTER, Miss HERMANCE, Ph.D. (Utrecht, Holland). Instr., Univ. of Washington, Seattle, Wash. *Dept. of Math.*
MULLEN, Mrs. SELAH W., B.S. (Michigan). Teacher, Northeastern High School, Detroit, Mich.
MULLINGS, M. E., A.B. (Texas). Instr., Univ. of Cincinnati, Cincinnati, Ohio. *Dept of Math.*
MULLINS, Asso. Prof. G. W., Ph.D. (Columbia). Barnard Coll., Columbia Univ., New York, N. Y. *Barnard Coll., 119th and Broadway.*
MUNROE, Florence L., A.B. (Wellesley). Teacher, High School, Northampton, Mass. *5 Franklin St.*
MURNAGHAN, Asso. Prof. F. D., Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md. *702 Evesham Ave., Govans, Baltimore.*
MURRAY, C. A., A.M. (Texas). W. Texas State Teachers Coll., Canyon, Tex.
MURRAY, Prof. D. A., Ph.D. (Johns Hopkins). Appl. Math., McGill Univ., Montreal, Can.
MURRAY, Asst. Prof. F. H., Ph.D. (Harvard). Dalhousie Univ., Halifax, N. S.
MURTO, Miss ARRIA, B.S. (Missouri). Teacher, High School, Carthage, Mo. *914 Howard St.*
MUSSELMAN, Asso. J. R., Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md.
MYERS, Prof. G. W., Ph.D. (Munich). Teaching of Math. and Astr., Univ. of Chicago, Chicago, Ill. *6639 Kimbark Ave., Jackson Pk. Sta.*
MYERS, Prof. H. S., A.M. (Chicago). Southwestern Coll., Winfield, Kans. *1501 E. 3rd.*

NASSAU, Asso. Prof. J. J., Ph.D. (Syracuse). Math. and Astr., Case School of Appl. Sc., Cleveland, Ohio.
NAUER, A. R. Mech. Engr., 4634 Nebraska Ave., St. Louis, Mo.
NEELLEY, Asso. Prof. J. H., Ph.D. (Yale). Carnegie Inst. of Tech., Pittsburgh, Pa. *3415 Iowa St., Schenley Hts.*
NEFF, Prof. I. F., M.S. (Drake; Chicago). Drake Univ., Des Moines, Ia. *2801 Brattleboro Ave.*
NEIKIRK, Asst. Prof. L. I., Ph.D. (Penna.). Univ. of Washington, Seattle, Wash. *4723 Twenty-first Ave., N. E.*
NELSON, Prof. A. L., Ph.D. (Chicago). Coll. of City of Detroit, Detroit, Mich.
NELSON, C. A., Ph.D. (Chicago). Rutgers Univ., New Brunswick, N. J.
NELSON, J. E., A.B. (Texas). Dir., University Jr. Coll., San Antonio, Tex. *419 S. Alamo.*
NELSON, W. A., A.B. (Texas). Jr. Coll., Tyler, Tex. *Box 676.*

NELSON, Asso. Prof. W. K., M.S., E.E. (Colorado). Eng. Math., Univ. of Colorado, Boulder, Colo. *925 Grandview Ave.*
NESS, MARIE M., A.M. (Minnesota). Research Assoc., Univ. of Minnesota, Minneapolis, Minn. *1200 W. Franklin Ave.*
NEUBAUER, GRETA, A.B. (Wyoming). Instr., Univ. of Wyoming, Laramie, Wyo. *Ivinson Hall.*
NEWELL, M. J., A.M. (Michigan). Teacher, High School, Evanston, Ill. *2226 Hartzell.*
NEWKIRK, B. L., Ph.D. (Munich). Research Dept., Genl. Elec. Co., Schenectady, N. Y.
NEWLIN, Asst. Prof. R. L., M.S. (Chicago). Ohio Wesleyan Univ., Delaware, Ohio. *253 N. Washington St.*
NEWLIN, W. A., M.S. (California). Head of Dept. of Math., High School and Jr. Coll., Pasadena, Calif. *1679 Rose Villa St.*
NEWSON, C. V., A.B. (Emporia). Asst. Instr., Univ. of Michigan, Ann Arbor, Mich. *745 E. Univ. Ave.*
NEWSOM, Prof. MARY W., Ph.D. (Göttingen). Eureka Coll., Eureka, Ill.
NEWTON, F. E., Ph.B. (Yale). Head of Dept., Phillips Acad., Andover, Mass. *9 Salem St.*
NEWTON, G. T., A.B. (Texas). Head of Dept. of Math., High School, Cameron, Texas. *310 E. 15th St.*
NEWTON, Dean MARY L., A.M. (Columbia). All Saints Coll., Vicksburg, Miss.
NICHOLS, G. D., A.M. (Nebraska). Univ. of Arkansas, Fayetteville, Ark. *Dept. of Math.*
NICHOLS, Prof. I. C., Ph.D. (Michigan). Appl. Math., La. State Univ., Baton Rouge, La.
NICOLET, JUSTIN, C.E. 1641 Irving Pk. Blvd., Chicago, Ill.
NIXON, J. C., M.S. (Chicago). Univ. of Alabama, Tuscaloosa, Ala. *1523 8th St.*
NOBLE, Prof. C. A., Ph.D. (Göttingen). Univ. of California, Berkeley, Calif. *2224 Piedmont Ave.*
NORDGAARD, Prof. M. A., Ph.D. (Columbia). St. Olaf Coll., Northfield, Minn. *1406 St. Olaf Ave.*
NUTT, EDITH M., A.B. (Wellesley). Teacher, Math. and German, High School, Natick, Mass. *11 Union St.*
NYBERG, J. A., M.S. (Chicago). Instr., Hyde Park High School, Chicago, Ill. *1039 E. Marquette Rd.*
NYSWANDER, Asst. Prof. J. A., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich.

ODELL, LETITIA R., A.M. (Denver). Teacher, North High School, Denver, Colo. *3435 Eliot St.*
O'DONNELL, G. A., A.M. (Woodstock; Georgetown). Fairview, Concord Rd., Weston, Mass.
OERGEL, C. T., Diploma in Power Eng. (Drexel Evening School). Student, Penn. State Coll. *2020 S. 2nd St., Philadelphia, Pa.*
OGLESBY, E. J., A.M. (Virginia). Prof. and Administrative Chmn., Washington Sq. Coll., New York, N. Y. *202 S. 18th St., Sta. A, Flushing, N. Y.*
O'KEAN, HARRY, B.S. (Alfred). Instr., High School, Friendship, N. Y. *Box 948.*
OLDS, Asst. Prof. E. G., A.M. (Pittsburgh). Carnegie Inst. of Tech., Pittsburgh, Pa. *1424 Barnsdale St.*
OLDS, Prof. G. D., LL.D. (Rochester). Pres. Emeritus, Amherst Coll., Amherst, Mass. *4 Fayerweather St., Cambridge, Mass.*
OLSON, EMMA J., A.M. (Chicago). Instr., Northwestern Univ., Evanston, Ill. *724 Simpson St.*
OLSON, Asst. Prof. H. L., Ph.D. (Chicago). Michigan State Coll., East Lansing, Mich. *P. O. Box 561.*
OPP, J. E., A.M. (Nebraska). Supt. City Schools, Burwell, Nebr.
O'QUINN, R. L., A.B. (Louisiana). Instr., La. State Univ., Baton Rouge, La.
OSBORN, JESSE, Ph.D. (Cornell). Harris Teachers Coll., St. Louis, Mo.
OSBORNE, Prof. G. A. Emeritus, Mass. Inst. of Tech. *10 Commonwealth Ave., Boston, Mass.*
OSGOOD, Prof. W. F., Ph.D. (Erlangen). Harvard Univ., Cambridge, Mass. *74 Avon Hill St.*
O'SHAUGHNESSY, Prof. LOUIS, Ph.D. (Penna.). Appl. Math., Virginia Poly. Inst., Blacksburg, Va. *Box 177.*
OTT, Prof. W. P., Ph.D. (Chicago). Univ. of Alabama, University, Ala.
OVERMAN, Prof. J. R., A.M. (Columbia). Head of Dept. of Math., State Normal Coll., Bowling Green, Ohio.
OWENS, Prof. F. W., Ph.D. (Chicago). Head of Dept., Penna. State Coll., State College, Pa. *402 S. Allen St.*
OWENS, HELEN B. (Mrs. F. W.), Ph.D. (Cornell). 402 S. Allen St., State College, Pa.

PAASWELL, GEORGE, C. E. (Cornell). Consulting Engr., New York, N. Y. *1950 Andrews Ave., Bronx.*
PACKER, MARGARET C., A.M. (Brown). Bell Tel. Labs., 463 West St., New York, N.Y. *Room 847.*
PALMER, Prof. C. I., A.B. (Michigan). Armour Inst. of Tech., Chicago, Ill. *6440 Greenwood Ave.*
PALMIÉ, Prof. ANNA H., Ph.B. (Cornell). Coll. for Women, Western Reserve Univ., Cleveland, Ohio.
PARADISO, L. J., A.M. (Ohio State). Instr., Lehigh Univ., Bethlehem, Pa.

Parker, W. P., A.M. Union Christian Coll., Pyengyang, Korea. *On leave, 448 Clairmont Ave., Decatur, Ga.*
Parkinson, Asst. Prof. G. A., A.M. (Ohio State). Univ. of Wisconsin, Extension Div., Milwaukee, Wis. *137 2nd St.*
Parson, S. F. Head of Dept., No. Ill. State Normal School, De Kalb, Ill. *305 College Ave.*
Partridge, E. A., Ph.D. (Penna.). Science, West Phila. High School for Boys, Philadelphia, Pa. *48th and Walnut Sts.*
Passano, Asso. Prof. L. M., A.B. (Johns Hopkins). Mass. Inst. of Tech., Cambridge, Mass.
Patten, Prof. W. E., C.E. (Cornell). Hydraulic Eng., Tangshan Univ., Tangshan, N. China.
Patterson, Asso. Prof. B. C., Ph.D. (Johns Hopkins). Hamilton Coll., Clinton, N. Y. *Dwight Ave.*
Pattillo, Dean N. A., Ph.D. (Johns Hopkins). Randolph-Macon Woman's Coll., Lynchburg, Va.
Patton, Bess, M.S. (Chicago). Teacher, Girls High School, Atlanta, Ga. *1287 North Ave. N. E.*
Paul, W. R. Teacher, Silver City, Ia.
Paula, Sister Mary, M.S. (Notre Dame). Prof., Marygrove Coll., Detroit, Mich.
Paxton, Asso. Prof. E. K., A.M. (Columbia). Washington and Lee Univ., Lexington, Va.
Payne, C. K. Instr., Washington Sq. Coll., New York Univ., New York, N. Y. *100 Washington Sq. E.*
Pearce, Prof. W. H., A.M. (Michigan). Central State Normal School, Mt. Pleasant, Mich.
Pedersen, Asso. Prof. F. M., Sc.D. (New York Univ.). Coll. of the City of New York, New York, N. Y. *520 W. 114th St.*
Peele, Prof. D.D., A.M. (Chicago). Columbia College, Columbia, S. C.
Pegram, Prof. Annie M., A.M. (Duke). Greensboro Coll., Greensboro, N. C.
Pehrson, Asst. Prof. E. W. Univ. of Utah, Salt Lake City, Utah.
Pelletier, Prof. Arthur. Higher Alg., École Poly., Montreal, Can. *5043 St. André St.*
Penn, S. S. 834 Hewitt Pl., New York, N. Y.
Penrod, Prof. E. B., M.M.E. (Cornell). Physics, Hillsdale Coll., Hillsdale, Mich. *143 Oak St.*
Perkins, F. W., A.M. (Harvard). Grad. Student, Harvard Univ., Cambridge 38, Mass. *249 Huron Ave.*
Perkins, Prof. L. R., A.M. (Tufts). Middlebury Coll., Middlebury, Vt.
Perry, Dean Ruth C., A.B. (Wellesley). Dean of Women, Asst. Prof., Univ. of Chattanooga, Chattanooga, Tenn.
Peters, J. W., A.B. (Johns Hopkins). Carroll Rd. and Wirt Ave., Arlington, Baltimore, Md.
Peterson, J. K., A.B. (Vanderbilt). Instr., Harvard Univ., Cambridge, Mass. *46 Conant Hall.*
Peterson, O. J., A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *411 N. State St.*
Peterson, R. E., B.S. (Grove City Coll.). Instr., Culver Milit. Acad., Culver, Ind.
Pettersen, C. A., Ph.B. (Northwestern). Asst. Prin., Schurz High School, Chicago, Ill. *3922 Lowell Ave.*
Pettit, Prof. H. P., Ph.D. (Illinois). Marquette Univ., Milwaukee, Wis. *1664 Bartlett Ave.*
Petty, Actg. Pres. L. E., A.M. (Peabody). Texas Presbyterian Coll., Milford, Tex.
Phalen, Prof. H. R., Ph.D. (Chicago). St. Stephen's Coll., Annandale-on-Hudson, N. Y.
Philips, Prof. A. W., A.M. (Chicago). State Teachers Coll. of Emporia, Emporia, Kans.
Phillips, Rev. E. C., Ph.D. (Johns Hopkins). Dir., Georgetown Coll. Observ., Washington, D. C.
Phillips, Asso. Prof. H. B., Ph.D. (Johns Hopkins). Mass. Inst. of Tech., Cambridge, Mass.
Pickett, Jonathan. Prin., High School, Benavides, Tex.
Pierce, Prof. Jesse, M.S. (Chicago). Heidelberg Univ., Tiffin, Ohio.
Pierce, Asso. Prof. T. A., Ph.D. (California). Univ. of Nebraska, Lincoln, Nebr. *1811 Pepper Ave.*
Pierson, A. D., A.M. (Missouri). Jr. Coll., Kansas City, Mo. *437 S. Denver.*
Pi Mu Epsilon Fraternity, Secy. of, Syracuse Univ., Syracuse, N. Y. *Box 13, Faculty Post-Office, Hall of Languages.*
Pincherle, Prof. Salvatore, Sc.D. (Bologna). Univ. of Bologna, Bologna, Italy.
Pitman, Asst. Prof. J. H., A.M. (Swarthmore). Math. and Astr., Swarthmore Coll., Swarthmore, Pa. *328 Vassar Ave.*
Plant, Prof. L. C., M.S. (Chicago). Michigan State Coll., East Lansing, Mich.
Plapp, E. Marie, M.S. (Chicago). 4140 N. Keeler Ave., Chicago, Ill.
Plymale, R. B., A.M. (Mercer). Instr., Fr. Math., Mercer Univ., Macon, Ga.
Pobanz, J. F., Ph.D. (California). Jr. Coll., Modesto, Calif. *408 Magnolia Ave.*
Podolsky, Boris, A.M. (So. California). Teaching Fellow, Calif. Inst. of Tech., Pasadena, Calif. *394½ S. Michigan.*
Pollard, Asst. Prof. H. S., M.S. (Iowa). Miami Univ., Oxford, Ohio.
Pooler, L. G., A.M. (Columbia). Physics, Columbia Univ., New York, N. Y.
Poor, Asso. Prof. V. C., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich. *930 Packard St.*
Poritsky, Hillel, Ph.D. (Cornell). Natl. Research Fellow, Harvard Univ., Cambridge, Mass. *22 Shaler Lane.*
Porter, Asst. Prof. C. S., A.M. (Clark). Amherst Coll., Amherst, Mass.

PORTER, Prof. M. B., Ph.D. (Harvard). Univ. of Texas, Austin, Tex. *310 E. 21st St.*
PORTER, T. I., A.B. (Missouri). Instr., State Agric. Coll., Manhattan, Kans.
PORTER, Asso. Prof. W. L., A.B. (Howard Coll.). A. and M. Coll. of Texas, College Station, Tex.
POST, E. L., Ph.D. (Columbia). Teacher, Brooklyn Tech. High School. *560 W. 144th St., New York, N. Y.*
POUND, Asst. Prof. V. E., Ph.D. (Toronto). Univ. of Buffalo, Buffalo, N. Y. *122 Berkshire Ave.*
POUNDER, Asst. Prof. I. R., M.A. (Toronto). Univ. of Toronto, Toronto, Ont.
POWELL, J. E., M.S. (Ohio State). Instr., Michigan State Coll., East Lansing, Mich. *240 Delta St.*
POWELSON, INEZ D., A.M. (California). Teacher, Bakersfield, Calif. *Box 458.*
PRATER, A. W., A.M. (California at Los Angeles). Asst., Univ. of California at Los Angeles. *1247 Allen Ave., Glendale, Calif.*
PRESTON, AMY F., A.M. (Columbia). Teacher, Jr. High School, Columbus, Ohio. *1445 E. Main St.*
PRICE, G. B., A.B. (Miss. Coll.). Grad. Student and Instr., Harvard Univ., Cambridge, Mass. *36A Conant Hall.*
PRIDE, Asst. Prof. H. H., Ph.D. (New York Univ.). New York Univ., New York, N. Y. *Univ. Hts.*
PRIESTER, Asst. Prof. G. C., M.S. (Minnesota). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn. *Coll. of Eng. and Archit.*
PUGH, G. T., Ph.D. (Vanderbilt). Head of Dept. of Math., Winthrop Coll., Rock Hill, S. C.
PURCELL, E. J. Instr., Math. and Physics, Evans School, Tucson, Ariz.
PUTNAM, R. G., Ph.D. (Chicago). Instr., New York Univ., New York, N. Y. *Bridge and Church Sts., Tarrytown, N. Y.*
PUTNAM, Prof. T. M., Ph.D. (Chicago). Univ. of California, Berkeley, Calif.
PYKE, Prof. A. J., B.A. (Toronto). Univ. of Saskatchewan, Saskatoon, Sask., Canada.

QUARLES, LOUIS, A.B. (Michigan). Lawyer, Milwaukee, Wis. *Lines, Spooner and Quarles, 68 Wisconsin St.*
QUIGLEY, MARY J., A.B. (Radcliffe). Instr., Teachers Coll. of the City of Boston. *332 Savin Hill Ave., Dorchester, Mass.*
QUILTY, PATRICK, C.E. (Cooper). Borough Eng., Dept. Water Supply G. and E., New York, N. Y. *601 W. 163rd St.*

RAGSDALE, Prof. VIRGINIA, Ph.D. N. C. Coll. for Women, Greensboro, N. C. *Jamestown, N. C.*
RAIFORD, T. E., A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *1512 Grange Ave.*
RAINICH, Asst. Prof. G. Y., Master in Pure Math. (Kazan). Univ. of Michigan, Ann Arbor, Mich. *618 Lawrence.*
RAMAGE, C. J., Ph.D. (Grove City). Lawyer, Saluda, S. C.
RAMBO, Asso. Prof. SUSAN M., Ph.D. (Michigan). Smith Coll., Northampton, Mass. *12 Barrett Pl.*
RAMLER, Prof. O. J., Ph.D. (Catholic Univ.). Catholic Univ., Washington, D. C. *12 Girard St. N. E.*
RAMSDELL, Prof. G. E., A.M. (Harvard). Bates Coll., Lewiston, Me. *40 Mountain Ave.*
RAMSEY, Prof. ARTHUR, A.B. (Grove City). Grove City Coll., Grove City, Pa. *Box 31.*
RAMSEY, MARGARET, B.S. (Linfield). Instr., Linfield Coll., McMinnville, Ore. *321 N. D St.*
RANKIN, J. M. Instr., Coll. of Idaho, Caldwell, Ida. *1810 Ash St.*
RANKIN, Prof. W. W., Jr., A.M. (North Carolina). Duke Univ., Durham, N. C.
RANSOM, Prof. W. R., A.M. (Tufts; Harvard). Tufts Coll., Tufts College, Mass. *29 Sawyer Ave.*
RANUM, Prof. ARTHUR, Ph.D. (Chicago). Cornell Univ., Ithaca, N. Y. *3 Central Ave.*
RASEL, D. M., B.S. (W. and J.). Instr., Appl. Math., Washington and Jefferson Coll., Washington, Pa.
RASOR, Prof. S. E., A.M., M.S. (Chicago). Ohio State Univ., Columbus, Ohio. *1594 Neil Ave.*
RAU, Prof. A. G., Ph.D. (Lehigh). Moravian Coll., Bethlehem, Pa. *38 W. Market St.*
RAWLINS, Asst. Prof. C. H., Jr., Ph.D. (Johns Hopkins). Postgrad. School, U. S. Naval Acad., Annapolis, Md. *13 Franklin St.*
RAYNOR, Asso. Prof. G. E., Ph.D. (Princeton). Univ. of Oklahoma, Norman, Okla. *602 Santa Fe Ave.*
REA, Asst. Prof. P. L., A.M. (Marietta). Marietta Coll., Marietta, Ohio.
REAGAN, C. A., B.S. (Moores Hill). Friends Univ., Wichita, Kans.
REAVES, Prof. CAROLINE M., A.M. (Oklahoma). Coker Coll., Hartsville, S. C.
REAVES, Dean S. W., Ph.D. (Chicago). Arts and Sciences, Univ. of Oklahoma, Norman, Okla. *527 Chautauqua Ave.*
RECHARD, Asso. Prof. O. H., A.M. (Gettysburg). Univ. of Wyoming, Laramie, Wyo. *269 N. 8th St.*
REDDEN, Asso. Prof. J. E., M.S. (Iowa). John Tarleton Agric. Coll., Stephenville, Tex. *Box 568.*
REDDICK, Prof. H. W., Ph.D. (Columbia). Cooper Union, New York, N. Y.
REDDITT, Asst. Prof. B. H., A.M. (Johns Hopkins). Kenyon Coll., Gambier, Ohio. *On leave 1927–1928: 2006 Maryland Ave., Baltimore, Md.*
REECE, Prof. R. H., A.M. (Colorado). Math. and Physics, N. Mexico School of Mines, Socorro, N. M.

REED, Asso. Prof. F. W., Ph.D. (Virginia). Ohio Univ., Athens, Ohio.
REED, Prof. L. J., Ph.D. (Penna.). Biometry and Vital Statistics, Johns Hopkins Univ., Baltimore, Md.
REES, Asst. Prof. C. J., A.M. (Chicago). Univ. of Delaware, Newark, Del.
REES, Asso. Prof. E. L., C.E. (Kentucky), A.M. (Chicago). Univ. of Kentucky, Lexington, Ky. *200 Univ. Ave.*
REES, P. K., A.M. (Texas). Instr., Texas Tech. Coll., Lubbock, Tex. *Box 71.*
REES, W. A., A.M. (Texas). Asst., A. and M. Coll.; Allen Acad., Bryan, Tex.
REEVE, Prof. W. D., Ph.D. (Minnesota). Teachers Coll., Columbia Univ., New York, N. Y.
REGER, JULIA M., M.S. (Iowa). Munic. Univ. of Akron, Akron, Ohio. *495 E. Buchtel Ave.*
REID, Prof. L. W., Ph.D. (Göttingen). Haverford Coll., Haverford, Pa.
REID, W. T., A.M. (Texas). Grad. Student, Univ. of Texas, Austin, Tex. *2811 Guadalupe St.*
REILLY, Asso. Prof. J. F., Ph.D. (Iowa). Univ. of Iowa, Iowa City, Ia. *Room 212, Physics Bldg.*
REINERT, MARGUERITE, A.M. (California). Instr., Pasadena High School and Jr. Coll. *529 S. Hancock St., Los Angeles, Calif.*
REINSCH, Asso. Prof. B. P., Ph.D. (Illinois). School of Eng., So. Methodist Univ., Dallas, Tex.
REMICK, Prof. B. L., Ph.M. (Cornell Coll.). State Agric. Coll., Manhattan, Kans. *613 Houston St.*
RENNER, THERESA M., M.S. (Illinois). Instr., Blackburn Univ., Carlinville, Ill.
REUSSER, FRED, M.S. (Iowa). Head of Dept. of Math., Buena Vista Coll., Storm Lake, Ia.
REUTERDAHL, Pres. ARVID. Ramsey Inst. of Tech., St. Paul, Minn. *222 Otis Ave.*
REYNOLDS, Asso. Prof. C. N., Ph.D. (Harvard). West Virginia Univ., Morgantown, W. Va. *317 Brockway Ave.*
REYNOLDS, Prof. J. B., Ph.D. (Moravian). Math. and Theor. Mech., Lehigh Univ., Bethlehem, Pa. *721 W. Broad St.*
REYNOLDS, LENA E., A.M. (California). Head of Dept., Jr. Coll., Fullerton, Calif.
RHODES, C. E., A.B. (Cornell). Instr., Univ. of Cincinnati, Cincinnati, Ohio. *99 W. McMillan St.*
RICE, Prof. HARRIS, A.M. (Harvard). Worcester Poly. Inst., Worcester, Mass.
RICE, H. L., A.B. (Iowa). Teacher, Math. and Educ., Jr. Coll., Webster City, Ia. *717½ 1st St.*
RICE, J. M., E.E. (Lehigh). Eng. Asst., Bell Tel. Co. of Pa. *25 Bank St., Bridgeton, N. J.*
RICE, Asso. Prof. J. N., Ph.D. (Catholic Univ.). Catholic Univ., Washington, D. C. *3326 Thirteenth St., N. E.*
RICE, L. H., A.B. (Syracuse). Instr., Mass. Inst. of Tech., Cambridge, Mass.
RICHARDS, W. A., A.M. (Chicago). Morton Jr. Coll., Cicero, Ill. *209 Washington Blvd., Oak Park, Ill.*
RICHARDSON, Prof. A. V., M.A. (Cambridge, England). Bishops Coll., Lenoxville, Quebec, Canada.
RICHARDSON, Prof. C. H., A.M. (Illinois). Georgetown Coll., Georgetown, Ky.
RICHARDSON, Asst. Prof. M. R., A.M. (Trinity). Georgia School of Tech., Atlanta, Ga. *763 Greenwood Ave.*
RICHARDSON, Dean R. G. D., Ph.D. (Yale). Grad. School, Brown Univ., Providence, R. I.
RICHERT, Prof. D. H., A.B. (Oberlin). Bethel Coll., Newton, Kans.
RICHESON, Asst. Prof. A. W., A.M. (Johns Hopkins). Univ. of Maryland, Baltimore, Md. *1210 St. Paul St.*
RICHMOND, SUSAN V., A.B. (Randolph-Macon Woman's Coll.). Teacher, Western High School, Washington, D. C. *1645 31st St., N. W.*
RICKARD, Asst. Prof. HORTENSE, A.M. (Ohio State). Ohio State Univ., Columbus, Ohio. *79 S. Beechwold Rd.*
RIDER, Asso. Prof. P. R., Ph.D. (Yale). Washington Univ., St. Louis, Mo.
RIESBECK, LAURA, B.S. in Educ. (Ohio Univ.). Teacher, High School, Zanesville, Ohio. *1001 Greenwood Ave.*
RIETZ, Prof. H. L., Ph.D. (Cornell). Univ. of Iowa, Iowa City, Ia.
RIGGS, Prof. N. C., M.S. (Harvard). Mechanics, Carnegie Inst. of Tech., Pittsburgh, Pa.
RISLEY, Prof. W. J., Sr., A.M. (Illinois; Harvard). Colorado School of Mines, Golden, Colo.
RISSELMAN, W. C., A.M. (Minnesota). Teaching Asst., Univ. of Minnesota, Minneapolis, Minn. *1704 Brook Ave. S. E.*
RITT, Asso. Prof. J. F., Ph.D. (Columbia). Columbia Univ., New York, N. Y.
RITTENHOUSE, W. H. 4133 N. 9th St., Philadelphia, Pa.
ROBB, J. M., A.M. (Michigan). Head of Dept., High School, Everett, Wash. *2617 Virginia Ave.*
ROBBINS, ALICE V., Ph.M. (Chicago). Teacher, State Teachers Coll., Mankato, Minn. *418 S. 4th St.*
ROBBINS, Asst. Prof. C. K., A.M. (Harvard). Purdue Univ., W. LaFayette, Ind. *418 Vine.*
ROBERT, Asso. Prof. H. M., Jr., A.M. (Yale). U. S. Naval Acad., Annapolis, Md. *53 Southgate Ave.*
ROBERTS, J. H., A.B. (Texas). Instr., Univ. of Texas, Austin, Tex. *1912 Speedway.*
ROBERTS, Prof. MARIA M., B.L. (Iowa State). Dean, Jr. Coll., and Prof., Iowa State Coll., Ames, Ia. *219 Ash Ave.*

ROBERTS, WALTER, B.S. (Penna.). Head of Dept., West Phila. High School, Philadelphia, Pa. *3419 N. 22nd St.*
ROBINSON, FANNIE H., A.B. (Smith). 37 N. Walnut St., East Orange, N. J.
ROBINSON, Asso. Prof. G. E., B.A. (Dalhousie). Univ. of British Columbia, Vancouver, B. C. *852 Thurlow St.*
ROBINSON, G. EDNA, A.M. (Missouri). Ozark Wesleyan Coll., Carthage, Mo.
ROBINSON, Prof. H. A., A.M. (Johns Hopkins). Agnes Scott Coll., Decatur, Ga.
ROBINSON, Adj. Prof. L. V., A.M. (Texas). Texas Tech. Coll., Lubbock, Tex.
ROBINSON, Mrs. W. F., A.M. (North Carolina). Head of Dept., Mars Hill Coll., Mars Hill, N. C.
ROBISON, Asst. Prof. G. M., Ph.D. (Cornell). Duke Univ., Durham, N. C.
RODGERS, Dean T. G., A.M. (Wisconsin). New Mexico Normal Univ., E. Las Vegas, N. Mex. *1018 Fourth St.*
ROE, Prof. E. D., Jr., Ph.D. (Erlangen). John Raymond French Prof. of Math., Dir. of the Observatory, Research Professor, Syracuse Univ., Syracuse, N. Y. *123 W. Ostrander Ave.*
ROESER, H. M., M.S. (Geo. Washington Univ.). Engineer, Physicist, U. S. Bur. of Standards, Washington, D. C.
ROEVER, Prof. W. H., Ph.D. (Harvard). Pure Math., Washington Univ., St. Louis, Mo.
ROGERS, B. A., M.S. (Chicago). Phys., Western Elec. Co., Chicago, Ill. *2233 S. 53rd Ave., Cicero, Ill.*
ROGERS, Dean J. C., B.S. (Earlham). Dean and Prof., Piedmont Coll., Demorest, Ga.
ROMAN, IRWIN, Ph.D. (Chicago). Mathematical Physicist, Geophysical Research Corp., New York, N. Y. *2021 E. 12th St., Tulsa, Okla.*
ROOD, Miss MARION B., M.S. (Michigan). 1014 Church St., Ann Arbor, Mich.
ROOT, Prof. R. E., Ph.D. (Chicago). U. S. Naval Acad., Annapolis, Md. *7 Franklin St.*
ROSANOFF, LILLIAN, Ph.D. (Clark). 70 Morningside Dr., New York, N. Y.
ROSEBRUGH, Prof. T. R., M.A. (Toronto). Elec. Eng., Univ. of Toronto, Toronto, Can. *92 Walmer Rd.*
ROSENBACH, Asst. Prof. J. B., M.S. (Illinois). Carnegie Inst. of Tech., Pittsburgh, Pa. *5936 Alderson St., Suite No. 5.*
ROSENBAUM, JOSEPH, Ph.D. (Cornell). Instr., The Milford School, Milford, Conn.
ROSENGARTEN, Prof. GEORGE, Ph.D. (Penna.). W. Phila. Boys High School, Philadelphia, Pa. *414 Edgewood Rd., Upper Darby P. O.*
ROSS, G. A., A.M. (George Washington). Teacher, Central High School, Washington, D. C.
ROTH, W. E., A.M. (Wisconsin). Instr., Univ. of Wisconsin, Madison, Wis. *319 N. Brooks.*
ROTHERMEL, FLORENCE. Teacher, West Phila. High School, Philadelphia, Pa. *320 S. 42nd St.*
ROTHFUSS, W. J., A.B. (Illinois). Instr., Tulane Univ., New Orleans, La. *7416 Burthe St.*
ROTHROCK, Prof. D. A., Ph.D. (Leipzig). Dean, Indiana Univ., Bloomington, Ind. *1000 Atwater Ave.*
ROUSE, Asst. Prof. L. J., Ph.D. (Michigan). Univ. of Michigan, Ann Arbor, Mich. *1137 Michigan Ave.*
ROWE, Prof. J. E., Ph.D. (Johns Hopkins). Head of Dept., and Dir. of Extension, Coll. of William and Mary, Williamsburg, Va.
ROWLAND, Prof. S. A. Ohio Wesleyan Univ., Delaware, Ohio. *49 Oak Hill Ave.*
RUMBLE, Prof. D., A.M. (Harvard). Emory Univ., Emory University, Ga.
RUMNEY, ETHEL A., M.S. (Chicago). Instr., State Teachers Coll. of Emporia, Emporia, Kans. *1006 Market St.*
RUNDSTROM, Prof. INEZ. Gustavus Adolphus Coll., St. Peter, Minn.
RUNGE, Asst. Prof. LULU L., A.M. (Wisconsin). Univ. of Nebraska, Lincoln, Nebr.
RUNNING, Prof. T. R., Ph.D. (Wisconsin). Univ. of Michigan, Ann Arbor, Mich. *1019 Michigan Ave.*
RUPP, C. A., A.M. (Harvard). 5525 Woodlawn Ave., Chicago, Ill.
RUPP, Asst. Prof. E. C., M.S. (Denison). Denison Univ., Granville, Ohio.
RUSK, Prof. W. J., M.A. (Toronto). Math. and Astr., Grinnell Coll., Grinnell, Ia. *1415 Park St.*
RUSSELL, BEULAH, A.M. (Chicago). Williamsburg, Va.
RUSSELL, Prof. W. P., A.M. (Cumberland). Pomona Coll., Claremont, Calif.
RUTHERFORD, H. M. Computer, Geophysical Research Corp., Tulsa, Okla. *P. O. Box 2022.*
RUTLEDGE, Asst. Prof. GEORGE, Ph.D. (Illinois). Mass. Inst. of Tech. *29 Bellevue, Rd. Belmont, Mass.*
RYNO, K. JEANETTE, A.M. (Tennessee). Math. and Sc., Centenary Coll., Cleveland, Tenn.

SABIN, MARY S., A.M. (Denver). Teacher, East High School, Denver, Colo.
SAFFORD, Prof. F. H., Ph.D. (Harvard). Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
SAGEN, G. O., A.M. (California). Kern Co. High School, and Jr. College., Bakersfield, Calif.
SALKOVER, Asst. Prof. MEYER, Ph.D. (Yale). Univ. of Cincinnati, Cincinnati, Ohio.
SALVOSA, Prof. L. R., Grad. U. S. Milit. Acad.; M.S. (Michigan). Forest Eng., Univ. of the Philippines. *432 Maynard St., Ann Arbor, Mich.*

SANCHEZ, RAFAEL, B.S. (Porto Rico). Instr., Univ. of Porto Rico, Mayaguez, Porto Rico. *P. O. 231.*

SANDERS, BERNICE, A.M. (Ohio State). Auditor, The Supreme Life and Casualty Co., Columbus, Ohio. *1005 E. Long St.*

SANDERS, Prof. S. T. La. State Univ., Baton Rouge, La. *714 Mills Ave.*

SANFORD, VERA, Ph. D. (Columbia). Teacher, The Lincoln School of Teachers Coll., New York, N. Y. *The Lincoln School, 425 W. 123rd St.*

SAUNDERS, FAITH, A.M. (Missouri). Maysville, De Kalb Co., Mo.

SAUREL, Prof. PAUL, Sc.D. (Bordeaux, France). Coll. of the City of New York, New York, N. Y.

SAUTÉ, GEORGE, A.M. (Brown). C. R. B. Fellow. 41 Boulevard Bischoffsheim, Brussels, Belgium.

SAVARY, C. M., C.E. Chief of Geodesy and Cartography, Dept. of Lands and Forests, Quebec, P. Q. *586 St. Foye Road, Quebec City, P. Q.*

SCAMMON, Prof. R. E., Ph.D. (Harvard). Anatomy, Univ. of Minnesota, Minneapolis, Minn. *Inst. of Anatomy, Univ. of Minn.*

SCARBOROUGH, Asst. Prof. J. B., Ph.D. (Johns Hopkins). U. S. Naval Acad., Annapolis, Md. *P. O. Box 332.*

SCARBOROUGH, Prof. J. H., Ph.D. (Vanderbilt). State Teachers Coll., Warrensburg, Mo.

SCHARF, Rev. C. G., A.B. (St. Cyril), B.S. (Chicago). Treas. and Instr. in Sc., St. Cyril High School, Chicago, Ill. *6413 Dante Ave.*

SCHMALL, C. N., A.B. (C. C. N. Y.). Publ. School No. 69, Borough of Manhattan, New York, N. Y. *103 W. 118th St.*

SCHMIEDEL, OSCAR, A.M. (Bethany, W. Va.). Atascadero, Calif.

SCHOONMAKER, HAZEL E., Ph.D. (Cornell). Resident Doctor, Cornell Univ., Ithaca, N. Y. *319 Dryden Rd.*

SCHOONOVER, R. H., A.B. (Indiana). Instr., Detroit Jr. Coll. *606 Hawthorne Ave., Royal Oak, Mich.*

SCHORLING, Asso. Prof. RALEIGH, Ph.D. School of Educ., Univ. of Michigan, Ann Arbor, Mich. *School of Educ.*

SCHOTTENFELS, IDA M., A.M. (Chicago). 5521 Kimbark Ave., Chicago, Ill.

SCHREIBER, E. W., A.M. (Chicago). Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *503 Monroe St.*

SCHUB, PINCUS, formerly student in Turkey. Instr., Hebrew School, Newark, N. J. *179 E. Broadway, New York, N. Y.*

SCHUYLER, ELMER, M.S. (Lafayette). Instr., Bay Ridge High School, Brooklyn, N. Y. *87 Seventy-first St.*

SCHWEITZER, A. R., Ph.D. (Chicago). 452 Oakdale Ave., Chicago, Ill.

SCOTT, Prin. F. A., A.M. (Columbia). High School, Paris, Ky. *122 Duncan Ave.*

SEARLE, MARY, B.S. (Wellesley). 1658 N. W. 3rd St., Miami, Fla.

SEE, Prof. T. J. J., Ph.D. (Berlin). Dir., Naval Observatory, Mare Island, Calif.

SEIDLIN, Prof. JOSEPH, M.S. (Columbia). Alfred Univ., Alfred, N. Y.

SEIVERLING, I. F., A.M. (Columbia). Head of Dept., State Normal School, Millersville, Pa.

SELLEW, Prof. G. T., Ph.D. (Yale). Knox Coll., Galesburg, Ill.

SENSENIG, Prof. WAYNE, Ph.D. (Penna.). 309 Bangor Rd., Cynwyd, Mont. Co., Pa.

SEUBERT, G. A., B.S. (Pittsburgh). 309 Weldon St., Latrobe, Pa.

SEWELL, Pres. J. S., Grad. U. S. Milit. Acad. Alabama Marble Co., Birmingham, Ala. *2801 Mountain Ave.*

SEWELL, Prof. S. M., A.M. (Texas), M.S. (Chicago). Southwest Texas State Normal Coll. San Marcos, Texas. *500 N. Guadalupe St.*

SHANNON, Prof. J. I., A.M. (St. Louis). Dean, School of Philos. and Sc., St. Louis Univ., St. Louis, Mo.

SHAUB, H. C., A.M. (Dartmouth). Washington and Jefferson Coll., Washington, Pa. *440 E. Bean St.*

SHAW, R. S., B.S. (Bates). Grad. School, Univ. of Chicago, Chicago, Ill. *5461 Univ. Ave.*

SHEETS, R. A., A.M. (Denison). 811 15th St., Rock Island, Ill.

SHELDON, Prof. E. W., Ph.D. (Yale). Univ. of Alberta, Edmonton, Alb., Can.

SHELLENBERGER, R. C., A.M. (Michigan). Head of Dept., Jr. Coll., Bay City, Mich. *248 N. Grant St.*

SHENTON, Prof. W. F., Ph.D. (Hopkins). American Univ., Washington, D. C. *3605 Porter St. N. W.*

SHERK, Prof. W. H., A.M. (Chicago). Univ. of Buffalo, Buffalo, N. Y.

SHERWOOD, Prof. G. E. F., Ph.D. (Chicago). Univ. of California at Los Angeles, Calif.

SHEWHART, W. A., Ph.D. (California). Appl. Math., Bell Tel. Laboratories, New York, N. Y. *463 West St.*

SHIBLI, Asst. Prof. J., A.M. (Wisconsin). Penn. State Coll., State College, Pa. *505 Pugh St.*

SHIRK, Prof. J. A. G., M.S. (Kansas). Math. and Applied Mech., Kansas State Teachers Coll., Pittsburg, Kans.

SHIRLEY, PAULINE, A.B. (Baylor). Teacher, Jr. Coll., Wichita Falls, Tex. *2141 Ave. G.*
SHIVELY, Prof. C. S., Ph.D. (Denver). Juniata Coll., Huntington, Pa. *1805 Moore St.*
SHIVELY, Prof. L. S., Ph.D. (Chicago). Mount Morris Coll., Mt. Morris, Ill.
SHOHAT, Asst. Prof. J. A., Magister of Pure Math. (Petrograd). Univ. of Michigan, Ann Arbor, Mich. *508 Thompson St.*
SHOOK, Asst. Prof. C. A., Ph.D. (Johns Hopkins). Yale Univ. *108 First Ave., West Haven, Conn.*
SHORR, ROSE, A.B. (Hunter). Teacher, Manual Training High School, Brooklyn, N. Y.
SHORT, Supt. C. A., M.S. (Delaware). Public School, Laurel, Del.
SHORT, Asst. Prof. JESSIE M., A.M. (Carleton). Math. and Astr., Reed Coll., Portland, Ore.
SHORT, Prof. W. T., A.B. (Okla. Baptist Coll.). Oklahoma Baptist Univ., Shawnee, Okla.
SHOWMAN, Asst. Prof. H. M., A.M. (Harvard). Univ. of California at Los Angeles, Calif. *912 Hyperion Ave.*
SHULER, C. EUCEBIA, A.M. (Peabody). Head of Dept., Flora Macdonald Coll., Red Springs, N. C.
SHULL, Prof. C. E., A.M. (Bridgewater). Bridgewater Coll., Bridgewater, Va.
SHUMAN, J. W., B.S. (Mass. Inst. of Tech.). Secy.-Treas., Power Eng. Co., Consulting Eng., Minneapolis, Minn. *621 Metropolitan Life Bldg.*
SHUMWAY, Asso. Prof. R. R., A.B. (Minnesota). Univ. of Minnesota, Minneapolis, Minnesota.
SICELOFF, Asso. Prof. L. P., Ph.D. (Columbia). Columbia Univ., New York, N. Y.
SILBERSTEIN, Ludwik, Ph.D. (Berlin). Math. Physicist, Research Lab., Eastman Kodak Co., Rochester, N. Y.
SILVERMAN, Prof. L. L., Ph.D. (Missouri). Dartmouth Coll., Hanover, N. H.
SIMESTER, J. H., M.A. (Toronto). Instr., Univ. of Texas, Austin, Tex.
SIMMONS, Asst. Prof. H. A., Ph.D. (Chicago). Northwestern Univ., Evanston, Ill. *2125½ Ridge Ave.*
SIMON, Prof. W. G., Ph.D. (Chicago). Adelbert Coll., Western Reserve Univ., Cleveland, Ohio.
SIMONS, Asso. Prof. LAO G., Ph.D. (Columbia). Hunter Coll., New York, N. Y. *180 W. 88th St.*
SIMPSON, Prof. C. G., A.M. (Columbia). Math. and Mech., Coll. of Elec. Eng., Milwaukee, Wis. *692 Fifth-eighth St.*
SIMPSON, Prof. T. M., Ph.D. (Wisconsin). Univ. of Florida, Gainesville, Fla.
SIMPSON, Prof. T. McN., Jr., Ph.D. (Chicago). Randolph-Macon Coll., Ashland, Va.
SINCLAIR, Prof. MARY EMILY, Ph.D. (Chicago). Oberlin Coll., Oberlin, Ohio. *On leave of absence 1927-1928, Univ. of Miami, Coral Gables, Fla.*
SINGER, Prof. S. A., A.M. (Capital). Capital Univ., Columbus, Ohio. *2322 E. Main St.*
SISAM, Prof. C. H., Ph.D. (Cornell). Colorado Coll., Colorado Springs, Colo. *816 N. Weber St.*
SKARSTEDT, Prof. MARCUS, Ph.D. (California). Whittier Coll., Whittier, Calif.
SKILES, Dean W. V., A.M. (Harvard). Georgia School of Tech., Atlanta, Ga.
SKINNER, Prof. E. B., Ph.D. (Chicago). Univ. of Wisconsin, Madison, Wis. *210 Lathrop St.*
SKIRROW, W. A., M.A. (Queen's). Head of Dept., Humberside C. I., Toronto, Can.
SLAUGHT, Prof. H. E., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
SLEIGHT, Prof. E. R., A.M. (Albion). Albion Coll., Albion, Mich.
SLICHTER, Prof. C. S., M.S., Sc.D. (Northwestern). Univ. of Wisconsin, Madison, Wis. *636 Frances St.*
SLOBIN, Prof. H. L., Ph.D. (Clark). Univ. of New Hampshire, Durham, N. H.
SMAIL, Asso. Prof. L. L., Ph.D. (Columbia). Lehigh Univ., Bethlehem, Pa.
SMINK, R. D., M.S. (Bucknell). Teacher, High School, Lock Haven, Pa. *201 S. Fairview St.*
SMITH, Prof. A. W., Ph.D. (Chicago). Colgate Univ., Hamilton, N. Y.
SMITH, Prof. C. D., M.S. (Iowa). Louisiana Coll., Pineville, La.
SMITH, Prof. CLARA E., Ph.D. (Yale). Wellesley Coll., Wellesley, Mass. *14 Waban St.*
SMITH, C. W., A.B. (Minnesota). State Normal School, Superior, Wis.
SMITH, Prof. DAVID EUGENE, Ph.D., LL.D. (Syracuse). Emeritus, Teachers Coll., Columbia Univ., New York, N. Y. *501 W. 120th St.*
SMITH, Asso. Prof. D. M., Ph.D. (Chicago). Ga. School of Tech., Atlanta, Ga.
SMITH, Prof. E. R., Ph.D. (Munich). Iowa State Coll., Ames, Ia. *113 Clark Ave.*
SMITH, ELMER R., A.M. (Vanderbilt). Florida State Coll., Tallahassee, Fla. *79 College Ave.*
SMITH, Prof. E. S., Ph.D. (Virginia). Univ. of Cincinnati, Cincinnati, Ohio
SMITH, Asst. Prof. G. W., Ph.D. (Illinois). Univ. of Kansas, Lawrence, Kans. *917 Maine.*
SMITH, Asst Prof. H. L., Ph.D. (Chicago). La. State Univ., Baton Rouge, La.
SMITH, H. W., M.S. (Chicago). Oklahoma A. and M. Coll., Stillwater, Okla.
SMITH, Ida K., Diploma (M. S. C. W.). Head of Dept., High School, Meridian, Miss.
SMITH, Rev. J. P., A.M. (Woodstock). St. Andrew-on-Hudson, Poughkeepsie, N. Y.
SMITH, Prof. L. W., Ph.D. (Washington and Lee). Washington and Lee Univ., Lexington, Va.
SMITH, Pres. M. G., Ph.D. (Illinois). Spring Arbor Sem. and Jr. Coll., Spring Arbor, Mich.
SMITH, MARGARETHE M., A.M. (Radcliffe). 3 Wyoming St., Grove Hall, Boston, Mass.
SMITH, P. K., M.S. (Chicago). Head of Dept., Miss. Woman's Coll., Hattiesburg, Miss. *Sta. B.*
SMITH, Asst. Prof. R. F., M.S. (New York Univ.). Coll. of the City of New York, New York, N. Y.
SMITH, R. R., A.B. (Yale). Mgr., College Dept., Macmillan Co., New York, N. Y. *60 Fifth Ave.*

SMITH, Prof. SARAH E., B.S. (Mount Holyoke). Mount Holyoke Coll., South Hadley, Mass.
SMITH, Prof. W. M., Ph.D. (Columbia). Lafayette Coll., Easton, Pa.
SMYLIE, NATHANIEL, A.M. (S. W. P. Univ.). Head of Dept. of Math., Belhaven Coll., Jackson, Miss.
SNEDECOR, Asso. Prof. G. W., A.M. (Michigan). State Coll., Ames, Ia. *807 Hodge Ave.*
SNYDER, Asst. Prof. A. D., A.M. (Wisconsin). Union Coll., Schenectady, N. Y. *1527 Eastern Pkwy.*
SNYDER, E. H., B.S. (Iowa). Industrial Eng., La Crosse, Wis. *202 N. 8th St.*
SNYDER, Prof. VIRGIL, Ph.D. (Göttingen). Cornell Univ., Ithaca, N. Y. *214 Univ. Ave.*
SOLUM, A. K., A.B. (St. Olaf). Instr., St. Olaf Coll., Northfield, Minn. *1304 St. Olaf Ave.*
SOUSLEY, Prof. C. P., Ph.D. (Johns Hopkins). Rose Poly. Inst., Terre Haute, Ind. *92 Potomac St.*
SPANN, Asst. Prof. J. T., B.S. (Mississippi). Univ. of Maryland, College Park, Md.
SPARKS, Asso. Prof. F. W., M.S. (Chicago). Texas Tech. Coll., Lubbock, Tex.
SPARROW, Prof. C. M., Ph.D. (Johns Hopkins). Physics, Univ. of Virginia, University, Va.
SPAULDING, Ruth L., A.B. (Smith). Teacher, Public High School, Hartford, Conn. *164 Beacon St.*
SPEAR, Prof. JOSEPH, A.B. (Harvard). Chairman of Dept., School of Eng., Northeastern Univ., Boston, Mass.
SPEEKER, Asso. Prof. G. G., A.M. (Indiana). Michigan State Coll., East Lansing, Mich.
SPENCELEY, Asst. Prof. G. W., A.M. (Harvard). Miami Univ., Oxford, Ohio. *100 E. Walnut St.*
SPENCER, Prof. MARY C., M.S. (Cornell). Sophie Newcomb Memorial Coll., New Orleans, La.
SPERRY, MAY, A.M. (Brown). Instr., Syracuse Univ., Syracuse, N. Y.
SPERRY, Asst. Prof. PAULINE, Ph.D. (Chicago). Univ. of California, Berkeley, Calif. *1194 Cragmont Ave.*
SPIES, Prof. ANTOINETTE, Ph.D. (Fordham). Head of Dept. of Math., Manhattanville Coll., New York, N. Y.
SPINKS, M. J. Chief Eng., Champion Bridge Co., Wilmington, Ohio. *Box 594.*
SPOONER, Prof. C. C., A.M. (Amherst). State Normal School, Marquette, Mich. *210 E. Prospect St.*
STAFFORD, Adj. Prof. ELIZABETH T., M.S. (Brown). W. Tex. Coll. of Tech., Lubbock, Tex.
STAFFORD, W. A., A.M. (Stratford). Head of Dept., Oakland High School, Oakland, Calif. *31 Park Way, Piedmont, Calif.*
STAGER, H. W., Ph.D. (California). P. O. Box 245, Los Gatos, Calif.
STALEY, Asst. Prof. R. C., A.M. (Colorado). Univ. of North Dakota, Grand Forks, N. Dak.
STANWICK, C. A., B.S. in E.E. (Washington). 33 Beach St., Baltimore, Md.
STARK, Asst. Prof. MARION E., Ph.D. (Chicago). Wellesley Coll., Wellesley, Mass. *Wilder Hall.*
STAUFFER, J. R. K., A.B. (Franklin and Marshall). Instr., Purdue Univ., LaFayette, Ind. *135 Sheetz St.*
STEED, Asst. Prof. D. V., Ph.D. (California). Univ. of Southern California, Los Angeles, Calif.
STEIMLEY, L. L., Ph.D. (Illinois). Univ. of Illinois, Urbana, Ill. *Box 156, Univ. Sta.*
STEININGER, EDITH, A.M. (Kansas). Jr. Coll., Coffeyville, Kans. *912 Willow St.*
STEIRNAGLE, W. M., A.B. (Indiana). 111 Michigan Ave., Sturgis, Mich.
STEPHENS, Asst. Prof. EUGENE. Washington Univ., St. Louis, Mo.
STEPHENS, Prof. R. P., Ph.D. (Johns Hopkins). Univ. of Georgia, Athens, Ga.
STETSON, J. M., Ph.D. (Princeton). 210 E. Nittany Ave., State College, Pa.
STEVENS, W. R. Meteorologist, U. S. Weather Bureau, Washington, D. C.
STEVENSON, Asst. Prof. GUY, Ph.D. (Illinois). Univ. of Louisville, Louisville, Ky.
STEVENSON, Dean P. L., A.M. (Colorado). Westminster Jr. Coll., Salt Lake City, Utah. *1424 Kensington Ave.*
STOKES, ELLEN C., A.M. (Brown). Instr., N. Y. State Coll. for Teachers, Albany, N. Y. *21 Hampton Ave., Schenectady, N. Y.*
STONE, JOSEPHINE, A.M. (Peabody). Teacher, Athens Coll., Athens, Ala.
STONE, Prof. ORMOND, A.M. (Old Univ. of Chicago). Formerly Prof. of Astr., Univ. of Virginia, Charlottesville, Va. *Clifton Station, Fairfax Co., Va.*
STONE, Asso. Prof. R. B., A.M. (Harvard). Purdue Univ., W. LaFayette, Ind. *615 Russell St.*
STONER, P. W., M.S. (California). Teacher, High School, Pasadena, Calif. *2131 Spaulding Pl.*
STOUFFER, Prof. E. B., Ph.D. (Illinois). Univ. of Kansas, Lawrence, Kans.
STOUT, C. E., A.M. (Wisconsin). Instr., Purdue Univ., W. LaFayette, Ind. *318 N. Grant St.*
STOWELL, Prof. C. J., Ph.D. (Illinois). McKendree Coll., Lebanon, Ill.
STRATTON, Prof. W. T., A.M. (Indiana). Kansas State Agric. Coll., Manhattan, Kans.
STROM, C. W., A.M. (Iowa). B.A. (Oxon.). Instr., Luther Coll., Decorah, Ia.
STRONG, Prof. CORA, A.B. (Cornell). North Carolina Coll. for Women, Greensboro, N. C.
STURDIVANT, J. H., A.M. (Texas). Grad. Student and Teaching Fellow, Calif. Inst. of Tech., Pasadena, Calif. *Gates Chem. Lab.*
SUESMAN, W. P., LL.B. (Lake Forest). 237 Oaklawn Ave., Meshanticut, R. I.
SUFFA, Prof. MARY C., A.M. (Brown). Elmira Coll., Elmira, N. Y.
SWARTZEL, Prof. K. D., M.S. (Ohio State). Univ. of Pittsburgh, Pittsburgh, Pa. *4360 Center Ave.*

SWEAZEY, Dean G. B., A.M. (Wabash). Westminster Coll., Fulton, Mo.
SWEET, H. L. Instr., Phillips Exeter Acad., Exeter, N. H.
SWIFT, Prof. ELIJAH, Ph.D. (Göttingen). Univ. of Vermont, Burlington, Vt. *415 S. Willard St.*

TABER, G. H. P. O. Box 1214, Pittsburgh, Pa.
TALIAFERRO, CARRIE B., B.S. (Columbia). State Teachers Coll., Farmville, Va.
TALIAFERRO, Prof. T. H., Ph.D. (Johns Hopkins). Actg. Dean, Coll. of Arts and Sciences, Univ. of Maryland, College Park, Md.
TAMARKIN, Asst. Prof. JACOB, Ph.D., M.A.M. (Petrograd). Brown Univ., Providence, R. I.
TAN, Prof. V. A., Ph.D. (Chicago). Univ. of the Philippines, Manila, P. I.
TANZOLA, J. J., A.M. (Columbia). Instr., Cooper Union. *2041 Watson Ave., Bronx, N. Y.*
TAPPAN, Prof. A. HELEN, Ph.D. (Cornell). Western Coll., Oxford, Ohio.
TATE, JENNIE L., A.M. (Wisconsin). Head of Dept., McMurry Coll., Abilene, Tex.
TAYLOR, Prof. EUGENE, A.M. (DePauw). Univ. of Idaho, Moscow, Ida.
TAYLOR, Prof. E. H., Ph.D. (Harvard). Eastern Illinois State Teachers Coll., Charleston, Ill.
TAYLOR, Prof. F. J., A.B. (St. Thomas). St. Thomas Coll., St. Paul, Minn.
TAYLOR, Asso. Prof. J. S., Ph.D. (California). Univ. of Pittsburgh, Pittsburgh, Pa.
TAYLOR, Prof. MARY S., A.B. (Cornell). Asst. Teacher, N. Y. Training School for Teachers, New York, N. Y. *100 Oak Tree Pl., Leonia, N. J.*
TAYLOR, MILDRED E., A.M. (Illinois). Instr., Knox Coll., Galesburg, Ill.
TAYLOR, Prof. W. E., Ph.D. (Syracuse). Syracuse Univ., Syracuse, N. Y.
THECLA, Sister MARY, Ph.D. (Fordham). Teacher, St. Agnes Sem., Brooklyn, N. Y. *287 Union St.*
THEOBALD, Prof. JOHN, A. B. (Columbia Coll.), S.T.B. (Catholic Univ.). Columbia Coll., Dubuque, Ia.
THOM, ADELA M., A.B. (Kansas). Teacher, High School, Okmulgee, Okla. *606 N. Morton St.*
THOMAS, Asso. Prof. C. F., A.B. (Amherst). Case School of Appl. Sc., Cleveland, Ohio.
THOMAS, Prof. EVAN. Coll. of Eng., Univ. of Vermont, Burlington, Vt.
THOMAS, Asst. Prof. R. W., M.S. (Washington and Jefferson). Washington and Jefferson Coll., Washington, Pa. *333 Wilson Ave.*
THOME, W. J., C.E. (Mich. State). Brown-Hutchinson Iron Wks., Detroit, Mich. *14911 Fairfield Ave.*
THOMPSON, E. L. 6032 Ingleside Ave., Chicago, Ill.
THOMPSON, EVELYN R., A.M. (Columbia). Teacher, Western High School, Washington, D. C. *1857 Lamont St.*
THOMPSON, HELEN, Ph.D. (Columbia). Research Assoc., Anthropometry and Statistics, Yale Psycho-Clinic, Yale Univ., New Haven, Conn. *52 Hillside Ave.*
THOMPSON, J. E., A.M. (Columbia). Instr., Pratt Inst., Brooklyn, N. Y. *510 W. 113th St., New York, N. Y.*
THOMPSON, RUTH, B.S. (New Jersey Coll. for Women). Instr., N. J. Coll. for Women, New Brunswick, N. J.
THOMSEN, H. I., Ph.D. 1928 Mt. Royal Terr., Baltimore, Md.
THORNE, Asst. Prof. P. L., M.S. (New York Univ.). New York Univ., New York, N. Y.
THORNTON, Prof. W. M., LL.D. (Hampden-Sidney). Appl. Math., Univ. of Virginia, University, Va.
THORP, ELLA, A.B. (Minnesota). Instr., Univ. of Minnesota, Minneapolis, Minn. *656 Jefferson St. N. E.*
TIENZO, Asst. Prof. TELESFORO, M.S. (Chicago). Univ of the Philippines, Manila, P. I. *Dept. of Math.*
TILLEY, Asst. Prof. ARTHUR, M. S., M.E. (New York Univ.). Washington Square Coll., New York Univ., New York, N. Y.
TINNER, Prof. J. C., M.S. (Chicago). Wilberforce Univ., Wilberforce, Ohio.
TITSWORTH, Prof. A. A., M.S., C.E., D.Sc. (Rutgers). Rutgers Coll., New Brunswick, N. J.
TITSWORTH, Mrs. H. L., A.B. (Tulane). Fellow, Newcomb Coll., New Orleans, La. *2320 Nashville Ave.*
TITSWORTH, Prof. W. A., M.S. (Wisconsin). Alfred Univ., Alfred, N. Y.
TITT, Prof. H. G., A.M. (Michigan). Huron Coll., Huron, S. Dak.
TITUS, C. M., A.M. (Stanford). Instr., Univ. Farm School, Davis, Calif.
DE TOLEDO, Prof. L. O. Univ. of Madrid, Madrid, Spain. *Velasques 28–3°, Facultad de Ciencias.*
TOMELDEN, A. A., B.S. (Catholic Univ.). Jr. Mathematician, Coast and Geod. Surv., Washington, D. C. *1438 V St. N. W.*
TORRANCE, C. C., M.E., A.M. (Cornell). Instr., Cornell Univ., Ithaca, N. Y. *Dept. of Math.*
TORREY, MARIAN M., Ph.D. (Cornell). Goucher Coll., Baltimore, Md.
TOUCHSTONE, NORMA E., A.B. (La. State Univ.) Polk St., Alexandria, La.
TOUTON, Prof. F. C., Ph.D. (Columbia). Dir. of Educ. Research, Univ. of Southern Calif., Los Angeles, Calif.

TOWNSEND, Prof. E. J., Ph.D. (Göttingen). Univ. of Illinois, Urbana, Ill. *294 Nat. Hist. Bldg.*
TRACEY, Asso. Prof. J. I., Ph.D. (Johns Hopkins). Yale Univ., New Haven, Conn. *84 McKinley Ave.*
TRACY, SARAH E., B.L. (Swarthmore). Teacher, John Burroughs School, Clayton, Mo. *6188 McPherson Ave., St. Louis, Mo.*
TREFETHEN, Prof. H. E. Colby Coll., Waterville, Me. *4 West Court.*
TREMBLAY, Prof. ALTHÉOD. Higher Math., Laval Univ., Quebec, Can.
TRIPP, Prof. M. O., Ph.D. (Columbia). Wittenberg Coll., Springfield, Ohio.
TROFIMOV, L. A., Grad. Imperial Naval Acad., Petrograd. Development Eng., Electric Controller and Mfg. Co., Cleveland, Ohio. *2700 E. 79th St.*
TUCKER, Asso. Prof. E. R., B.S. (Milit. Coll. of S. C.), A.B. (T. C. U.). Texas Christian Univ., Fort Worth, Tex.
TURNER, Prof. A. B., Ph.D. (Penna.). Coll. of the City of New York. *245 N. Mountain Ave., Montclair, N. J.*
TURNER, Asso. Prof. BIRD M., Ph.D. (Bryn Mawr). Univ. of West Virginia, Morgantown, W. Va. *107 High St.*
TURNER, Asso. Prof. J. S., Ph.D. (Chicago). Iowa State Coll., Ames, Ia. *509 Welch Ave.*
TWISS, J. F., M.A. (McMaster). Instr., Royal Milit. Coll., Kingston, Ont., Can.
TYLER, Prof. H. W., Ph.D. (Erlangen). Mass. Inst. of Tech., Cambridge, Mass.
TYLER, Asst. Prof. JOHN, U. S. Naval Acad., Annapolis, Md. *3 Southgate Ave.*

UDINSKI, W. P., Ph.D. (Illinois). Jr. Coll., San Antonio, Tex.
UHLER, Asso. Prof. H. S., Ph.D. (Johns Hopkins). Physics, Yale Univ., New Haven, Conn. *140 Cottage St.*
ULRICH, F. E., B.S. in E.E. (Union). Instr., Union Coll., Schenectady, N. Y. *37 Ray St.*
UNDERHILL, Asso. Prof. A. L., Ph.D. (Chicago). Univ. of Minnesota, Minneapolis, Minn.
UNDERWOOD, P. H. Ball High School, Galveston, Tex. *2527 Ave. I.*
UNDERWOOD, Asso. Prof. R. S., A.M. (Minnesota). Texas Tech. Coll., Lubbock, Tex.
UNSELD, G. P., A.M. (Colorado). Instr., West High School, Salt Lake City, Utah. *128 I St.*
UPTON, Prof. C. B., A.M. (Columbia). Teachers Coll., Columbia Univ., New York, N. Y.

VAN BUSKIRK, Prof. H. C., Ph. B. (Cornell). Calif. Inst. of Tech., Pasadena, Calif. *3480 E. Colorado St.*
VANCE, B. B., B.S. (Louisville). Instr., Univ. of Louisville, Louisville, Ky. *Belknap Campus.*
VANDIVER, Asso. Prof. H. S. Univ. of Texas, Austin, Tex. Guggenheim Fellow 1927–1928.
VAN FLEET, G. S., B.S. (Montana State). Instr., Oregon State Coll., Corvallis, Ore. *135 S. 9th St.*
VAN HEE, LOUIS, Jesuit Father. Teacher of Math., Liége, Belgium. *66 Quai de Longdoz, 66.*
VAN HORNE, R. N., Ph.B. (Morningside). Morningside Coll., Sioux City, Ia.
VAN ORSTRAND, C. E., M.S. (Michigan). Geophysicist, U. S. Geol. Surv., Washington, D. C.
VAN VELZER, C. A., Ph.D. (Hillsdale). Carthage Coll., Carthage, Ill.
VAN VLECK, Prof. E. B., Ph.D. (Göttingen). Univ. of Wisconsin, Madison, Wis. *519 N. Pinckney St.*
VASS, J. I., A.M. (Northwestern). Part-time Instr., Univ. of Wisconsin, Madison, Wis. *1314 St James Ct.*
VEATCH, R. W., A.M. (Northwestern). Ursinus Coll., Collegeville, Pa.
VEBLEN, Prof. OSWALD, Ph.D. (Chicago). Princeton Univ., Princeton, N. J.
VEDDER, Prof. J. N., A.M. Union Coll., Schenectady, N. Y.
VEHSE, C. H., M.S. (Brown). Asst., Brown Univ. *Apponaug, R. I.*
VIVIAN, Prof. ROXANA, Ph.D. (Chicago). Wellesley Coll., Wellesley, Mass.

WAGNER, P. S., A.M. (Johns Hopkins). Lebanon Valley Coll., Annville, Pa.
WAHLIN, Prof. G. E., Ph.D. (Yale). Univ. of Missouri, Columbia, Mo. *1401 Anthony St.*
WAIT, Adj. Prof. A. H., A.M. (Wisconsin). Texas Tech. Coll., Lubbock, Tex.
WALDEN, Prof. E. E., A.M. (Colorado). Lambuth Coll., Jackson, Tenn.
WALKER, Pres. B. M., Ph.D. (Chicago). Miss. A and M. Coll., A. and M. College, Miss.
WALKER, Asst. Prof. EVELYN, A.M. (Columbia). Hunter Coll., New York, N. Y. *695 Park Ave.*
WALKER, HELEN M., A.M. (Columbia). Part-time Instr., and Fellow, Teachers Coll., Columbia Univ., New York, N. Y.
WALKER, L. C., A.M. (Stanford). Ceresco, Nebr.
WALLACE, W. W., A.M. (Stanford). Jr. Coll., Sacramento, Calif.
WALLIS, W. J., A.M. (Columbian Univ.), LL.B. (George Washington). Head of Dept. of Math., High Schools, Washington, D. C. *Central High School.*
WALSH, Asst. Prof. J. L., Ph.D. (Harvard). Harvard Univ., Cambridge, Mass *547 Widener Libr.*
WALTER, ARTHUR, A.M. (Stanford). City Supt. of Schools, Salinas, Calif.
WALTON, A. R. Head of Dept., High School, Waycross, Ga. *Box 401.*
WALTON, Asst. Prof. T. O., M.S. (Chicago). Kalamazoo Coll., Kalamazoo, Mich.

WALTZ, A. K., Ph.D. (Cornell). Instr., Coll. Math., Univ. of Pittsburgh, Pittsburgh, Pa.
WAPPLE, Prof. A. R., A.M. (California). Southwestern Univ., Georgetown, Tex.
WARD, L. E., Ph.D. (Harvard). Assoc., Univ. of Iowa, Iowa City, Ia. *127 Physics Bldg.*
WARD, MORGAN, A.B. (California). Teaching Fellow, Calif. Inst. of Tech., Pasadena, Calif.
WARNE, W. R., Ph.B., Pd.B. (Syracuse). Instr., Syracuse Univ., Syracuse, N. Y. *1016 Lancaster Ave.*
WARNER, I. N., B.S. (Chicago). State Normal School, Platteville, Wis.
WARREN, Prof. L.A.H., Ph.D. (Chicago). Univ. of Manitoba, Winnipeg, Can.
WASHBURN, A. D., A.B. (Harvard). Math. and Sc., The Rivers School, Brookline, Mass.
WASHBURNE, A. C., Grad. U. S. Milit. Acad. Actuary, Berkshire Life Ins. Co., Pittsfield, Mass.
WATKEYS, Prof. C. W., A.M. (Harvard). Univ. of Rochester, Rochester, N. Y. *287 Dartmouth St.*
WATKINS, EMILY H., A.M. (Teachers Coll., Columbia). Instr., North Carolina Coll. for Women, Greensboro, N. C.
WATT, Asst. Prof. MARTHA W., A.M. (Columbia). Wheaton Coll., Norton, Mass. *2144 Broad St., Providence, R. I.*
WATT, MILDRED, A.M. (Cornell). Teacher, Buffalo Sem., Bidwell Pkwy., Buffalo, N. Y.
WATTS, C. B., A.B. (Indiana). Asso. Astr., Naval Observatory, Washington, D. C.
WATTS, Prof. C. W., C.E. (Va. Milit. Inst.). Virginia Milit. Inst., Lexington, Va. *P. O. Box 713.*
WATTS, E. VIRGINIA, A.M. (Chicago). Emanuel Community House, 1308 Race St., Cincinnati, Ohio.
WEAR, Asso. Prof. L. E., Ph.D. (Johns Hopkins). Calif. Inst. of Tech., Pasadena, Calif. *68 S. Grand Oaks Ave.*
WEAVER, ESTHER M., M.S. (Chicago). 110 Jackson St., Joplin, Mo.
WEAVER, Asst. Prof. J. H., Ph.D. (Penna.). Ohio State Univ., Columbus, Ohio. *Hilliard, Ohio.*
WEAVER, Asso. Prof. WARREN, Ph.D. (Wisconsin). Univ. of Wisconsin, Madison, Wis. *1852 Summit Ave.*
WEBBER, Prof. W. P., Ph.D. (Cincinnati). Louisiana State Univ., Baton Rouge, La.
WEBER, W. W., A.M. (Georgia). Dean and Prof. of Math., Lander Coll., Greenwood, S. C. *P. O. Box 111.*
WECHSLER, A. L., A.M. (Columbia). Instr., Univ. Extension, Columbia Univ., New York, N. Y. *Apt. No. 712, 27 W. 72nd St.*
WEDDERBURN, Asso. Prof. J. H. M., D.Sc. (Edinburgh). Princeton Univ., Princeton, N. J. *P. O. Box 53.*
WEIDA, Asst. Prof. F. M., Ph.D. (Iowa). Lehigh Univ., Bethlehem, Pa. *1016 N. New St.*
WEINBERG, Prof. E. F., C.E. (Manhattan). Rollins Coll., Winter Park, Fla.
WEISNER, Asst. Prof. LOUIS, Ph. D. (Columbia). Hunter Coll., New York, N. Y. *2067 Davidson Ave., Bronx, N. Y.*
WELCH, Asso. Prof. J. F., B.S. (Ohio State). La. State Univ., Baton Rouge, La. *P. O. Box 275.*
WELLING, W. C., A.B. (Yale). Dir., Bureau of Vital Stat., State Dept. of Health, Hartford, Conn. *Drawer K, Sta. A.*
WELLS, E. D., A.M. (Minnesota). Instr., Coll. of Eng., Univ. of Minnesota, Minneapolis, Minn. *212 Walnut St. S. E.*
WELLS, F. A., B.S. (Virginia). Univ. of Virginia, Charlottesville, Va. *414 Park St.*
WELLS, Asso. Prof. MARY EVELYN, Ph.D. (Chicago). Vassar Coll., Poughkeepsie, N. Y.
WELLS, Prof. R. A., A.M. (Franklin). Park Coll., Parkville, Mo.
WELLS, Asst. Prof. V. H., Ph.D. (Michigan). Williams Coll., Williamstown, Mass. *3 Chapin Ct.*
WELTON, P. L., A.M. (Michigan). Teacher, East High School, Rochester, N. Y. *89 Deerfield Dr.*
WERKMAN, MARY R., M.S. (Chicago). Instr., St. Xavier Coll., Chicago, Ill. *2629 N. Spaulding Ave., Logan Square Sta.*
WERNER, Asst. Prof. MIRIAM. Hunter Coll., New York, N. Y. *Care Mrs. E. D. Younghem, 27 W. 96th St.*
WERNICKE, PAUL, Ph.D. (Göttingen). Patent Examiner, U. S. Patent Office, Washington, D. C. *3600 S. Dak. Ave. N. E.*
WESCOTT, M. E., B.S. (Northwestern). Instr., Northwestern Univ., Evanston, Ill. *Acacia House.*
WEST, Mrs. GRACE R., A.M. (Texas), Teacher, High School, Tulsa, Okla. *Mary Brockman Apts.*
WEST, Asst. Prof. J. M., A.M. (Hamilton). Penn. State Coll., State College, Pa. *111 N. Burrowes St.*
WESTER, Prof. C. W. Iowa State Teachers Coll., Cedar Falls, Ia. *823 W. 6th St.*
WESTFALL, Prof. W. D. A., Ph.D. (Göttingen). Univ. of Missouri, Columbia, Mo.
WHALEY, MARY RUTH, A.B. (Smith). Phys. and Math., The Faulkner School, Chicago, Ill. *4546 Oakenwald Ave.*
WHEELER, A. H., A.M. (Clark). Teacher, North High School, Worcester, Mass.; Lecturer, Wellesley Coll. *16 Bellevue St., Worcester, Mass.*
WHEELER, Mrs. ANNA PELL, Ph.D. (Chicago). 109 Broadmead, Princeton, N. J.
WHEELER, Asst. Prof. J. J., A.M. (Kansas). Univ. of Kansas, Lawrence, Kans. *1024 Ala. St.*
WHITE, Prof. A. E., M.S. (Purdue). State Agric. Coll., Manhattan, Kans.
WHITE, Prof. C. E., A.M. (Indiana). Muskingum Coll., New Concord, Ohio.

WHITE, Prof. E. W., A.M. (Columbia). Head of Dept., Carson and Newman Coll., Jefferson City, Tenn.
WHITE, IDA, A.B. (Coll. of Idaho). Teacher, High School, Caldwell, Ida.
WHITE, Asst. Prof. MARION B., Ph.D. (Chicago). Carleton Coll., Northfield, Minn.
WHITED, Willis, M.E. (Ames). Advisory Bridge Engineer, State Dept. of Highways, Harrisburg, Pa. *202 Reily St.*
WHITFORD, Pres. A. E., A.M. (Wisconsin). Milton Coll., Milton, Wis.
WHITFORD, D. E., A.M. (Brown). Instr., Harvard Univ., Cambridge, Mass. *34 Ames St., Phenix, R. I.*
WHITFORD, Asso. Prof. E. E., Ph.D. (Columbia). Coll. of the City of New York, New York, N. Y. *3681 Broadway.*
WHITING, MABEL G., A.M. (Oberlin). Teacher, Jr. Coll., Santa Ana, Calif. *508 E. Chestnut.*
WHITMAN, Asst. Prof. E. A., A.M. (Pittsburgh). Carnegie Inst. of Tech., Pittsburgh, Pa. *623 Kirtland St., Homewood Sta.*
WHITNEY, ANNA M., A.M. (Columbia). Head of Dept., High School, Yakima, Wash. *203 S. 8th Ave.*
WHITNEY, Prof. C. S., A.M. (Oklahoma). Northeastern Okla. State Jr. Coll., Miami, Okla.
WHITTED, Prof. J. A., A.M. (Southwestern). Ohio Northern Univ., Ada, Ohio. *927 S. Main St.*
WHITTEMORE, Asso. Prof. J. K., A.M. (Harvard). Yale Univ., New Haven, Conn. *45 Lincoln St.*
WHYBURN, Prof. W. M., Ph.D. (Texas). Texas Tech. Coll., Lubbock, Tex. *Natl. Research Fellow 1927–, Harvard Univ. and Univ. of Chicago.*
WIBLE, Prof. W. M., A.M. (Indiana). Intermountain Union Coll., Helena, Mont. *1735 Winne Ave., Lenox Add.*
WIELEITNER, HEINRICH, Ph.D. (Munich). Dir., Neues Realgymnasium, Munich, Germany. *Müllerstr. 7, SO 2, Munich.*
WIFVAT, S. J., B.S. (Drake). Instr., Drake Univ., Des Moines, Ia. *Box 97, Univ. Pl. Sta.*
WILBUR, W. E., M.S. (Maine). Instr., Univ. of New Hampshire, Durham, N. H.
WILCOX, Asst. Prof. H. B., M.S. (Minnesota). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn. *3400 E. 50th St.*
WILCZEWSKI, Prof. JOSEPH, A.M. (St. Louis Univ.). St. Xavier Coll., Cincinnati, Ohio.
WILDER, Asst. Prof. C. E., Ph.D. (Harvard). Dartmouth Univ., Hanover, N. H.
WILDER, G. F. Head of Dept., Textile High School, New York, N. Y. *Textile H. S., 211 E. 32nd St.*
WILDER, Asst. Prof. R. L., Ph.D. (Texas). Univ. of Michigan, Ann Arbor, Mich. *1508 E. Park Pl.*
WILDERMUTH, Asso. Prof. R. B., A.M. (Ohio State). Capital Univ., Columbus, Ohio.
WILEY, Asst. Prof. ELLEN E., A.B. (St. Lawrence). Middlebury Coll., Middlebury, Vt. *120 Main St.*
WILEY, Prof. F. B., Ph.D. (Chicago). Denison Univ., Granville, Ohio.
WILKINS, Asst. Prof. P. D., M.S. (Case). Bates Coll., Lewiston, Me.
WILLETT, Prof. H. C., A.M. (So. Calif.). Univ. of Southern California, Los Angeles, Calif.
WILLIAMS, A. R., Ph.D. (California). Instr., Univ. of California, Berkeley, Calif. *416 Y. M. C. A.*
WILLIAMS, Prof. F. B., Ph.D. (Clark). Clark Univ., Worcester, Mass.
WILLIAMS, F. G., A.M. (Penn. State). Instr., Cornell Univ., Ithaca, N. Y. *138 Linden Ave.*
WILLIAMS, Asst. Prof. G. A., A.M. (California). Oregon State Coll., Corvallis, Ore.
WILLIAMS, Prof. J. E., Ph.D. (Virginia). Dean, Virginia Polytech. Inst., Blacksburg, Va.
WILLIAMS, Prof. K. P., Ph.D. (Princeton). Indiana Univ., Bloomington, Ind. *523 E. 3rd St.*
WILLIAMS, Prof. S. W., A.B., B.E. (Vanderbilt). Arkansas Coll., Batesville, Ark.
WILLIAMS, Adj. Prof. W. L., A.M. (South Carolina). Univ. of South Carolina, Columbia, S. C. *722 Meadow St.*
WILLIAMSON, Prof. C. O., M.S. (Ohio Univ.). Appl. Math., Coll. of Wooster, Wooster, Ohio. *1141 Beall Ave.*
WILLIS, J. E. Asst. Astronomer, U. S. Naval Observ., Washington, D. C.
WILLIS, Miss RUBY, A.B. (Wellesley). Head of Dept., Walnut Hill School, Natick, Mass.
WILLSON, Prof. F. N., C.E. (Rensselaer). Emeritus, Princeton Univ., Princeton, N. J.
WILMER, F. L. 1104 N. 16th St., Omaha, Nebr.
WILSON, Asso. Prof. A. H., Ph.D. (Chicago). Haverford Coll., Haverford, Pa.
WILSON, C. R., M.S. (Iowa). Instr., Rutgers Coll., New Brunswick, N. J.
WILSON, Prof. E. B., Ph.D. (Yale). Vital Statistics, Harvard School of Public Health, Boston 17, Mass.
WILSON, ELIZABETH W., A.M. (Radcliffe). Teacher, Central High School, Washington, D. C. *1431 Meridian St. N. W.*
WILSON, Prof. N. R., Ph.D. (Chicago). Univ. of Manitoba, Winnipeg, Can. *989 Grosvenor Ave.*
WILSON, Prof. W. A., Ph.D. (Yale). Yale Univ., New Haven, Conn. *1960 Chapel St.*
WINBIGLER, Prof. ALICE, A.M. (Monmouth). Monmouth Coll., Monmouth, Ill.
WINGER, Prof. R. M., Ph.D. (Johns Hopkins). Univ. of Washington, Seattle, Wash.
WINKELMANN, Rev. G. L., M.S. (Chicago). Head of Dept., St. John's Univ., Collegeville, Minn.

WINSLOW, J. B., A.M. (Michigan). Instr., Physics and Math., Toledo Univ., Toledo, Ohio. *Temperance, Mich.*
WITT, R. E., C.E. (W. and L.). Instr., Washington and Lee Univ., Lexington, Va. *Box 674.*
WOLEVER, FRANCES E., A.M. (Illinois). Teacher, High School, Monticello, Ill.
WOLFE, Asst. Prof. CLYDE, Ph.D. (California). Calif. Inst. of Tech., Pasadena, Calif.
WOLFE, Asst. Prof. H. E., Ph.D. (Indiana). Indiana Univ., Bloomington, Ind. *314 N. Washington Ave.*
WONG, Asst. Prof. B. C., Ph.D. (California). Univ. of California, Berkeley, Calif. *1933 Grant St.*
WOOD, Prof. FREDRICK, Ph.D. (Wisconsin). Wesleyan Coll., Macon, Ga.
WOOD, Asst. Prof. F. E., Ph.D. (Chicago). Northwestern Univ., Evanston, Ill.
WOOD, META A., A.M. (Wisconsin). Instr., Synodical Coll., Fulton, Mo.
WOOD, P. W., M.A. (Cambridge Univ.). Fellow, Tutor, Libr. and Math. Lecturer, Emmanuel Coll., Cambridge, Eng.
WOOD, ROSE B., A.B. (Barnard). Greenville Woman's Coll., Greenville, S. C.
WOOD, Prof. RUTH G., Ph.D. (Yale). Smith Coll., Northampton, Mass. *249 Crescent St.*
WOOD, R. R., B.S. (Haverford). 272 W. Main St., Moorestown, N. J.
WOODARD, Prof. D. W., M.S. (Chicago). Howard Univ., Washington, D. C.
WOODMANSEE, Prof. W. R., A.M. (Wisconsin). Ripon Coll., Ripon, Wis.
WOODS, Prof. F. S., Ph.D. (Göttingen). Mass. Inst. of Tech., Cambridge, Mass.
WOODS, Asst. Prof. ROSCOE, Ph.D. (Illinois). Univ. of Iowa, Iowa City, Ia. *221 Physics Bldg.*
WOODSON, G. F., Jr., A.M. (Ohio State). Head of Dept., Johnson C. Smith Univ., Charlotte, N. C.
WOODYARD, ELLA, A.B., B.Ped.(Baker). Teachers Coll., Columbia Univ., New York, N. Y.
WOOLARD, E. W., A.M. (George Washington). Meteorologist, U. S. Weather Bureau, Washington, D. C.
WORTHINGTON, Asst. Prof. EUPHEMIA R., Ph.D. (Yale). Univ. of California at Los Angeles, Calif. *855 N. Vermont Ave.*
WRAY, C. L., A.B. (Nebraska; York Coll.) Teacher, North High School, Denver, Colo. *North H. S.*
WREN, Asso. Prof. F. L., A.M. (Peabody). George Peabody Coll. for Teachers, Nashville, Tenn.
WRIGHT, FRANCES M., A.M. (Brown). Elmira Coll., Elmira, N.Y.
WRIGHT, FRANCES W., A.M. (Brown). Grad. Student, Radcliffe Coll., Cambridge, Mass. *Apt. 2, Rindge Apts., 1699 Cambridge St.*
WRIGHT, Dean H. N., Ph.D. (California). Earlham Coll., Richmond, Ind. *606 S. W. A St.*
WRIGHT, VERA L., A.M. (Minnesota). Instr., Martha Washington Coll., Abingdon, Va.
WRIGHT, Prof. W. L., A.M. (Princeton). Lincoln Univ., Lincoln University, Pa.
WUNDER, C. N., Ph.D. (Virginia). Head of Dept., Univ. of Mississippi, University, Miss.
WYANT, E. KATHRYN, A.M. (Missouri). Instr., Univ. of Missouri, Columbia, Mo. *213 Eng. Bldg.*
WYLIE, Asst. Prof. C. C., Ph.D. (Illinois). Astr., Univ. of Iowa, Iowa City, Ia. *Univ. Observ.*

YANNEY, Prof. B.F., Ph.D. (Chicago). The Coll. of Wooster, Wooster, Ohio. *666 N. Bever St.*
YANOSIK, G. A., C.E. (New York Univ.) Instr., New York Univ. *52 Greenvale Ave., Yonkers, N.Y.*
YEATON, Prof. C. H., Ph.D. (Chicago). Oberlin Coll., Oberlin, Ohio. *106 S. Cedar Ave.*
YOTHERS, J. L., A.M. (Otterbein). Head of Dept., Coe Coll., Cedar Rapids, Ia.
YOUNG, A. L., M.S. (Iowa State). Farm Mechanics, Univ. of Illinois, Urbana, Ill.
YOUNG, F. G., M.A. (Alberta). Instr., Provincial Inst. of Tech. and Art, Calgary, Alt. Can. *1301 9th St. N.W.*
YOUNG, Asst. Prof. JESSICA M., Ph.D. (California). Math. and Astr., Washington Univ., St. Louis, Mo. *4511 N. 20th St.*
YOUNG, Prof. J. W., Ph.D. (Cornell). Dartmouth Coll., Hanover, N.H.
YOUNG, Asso. Prof. J.W.A., Ph.D. (Clark). Univ. of Chicago, Chicago, Ill. *5422 Blackstone Ave.*
YOUNG, Asso. Prof. MABEL M., Ph.D. (Johns Hopkins). Wellesley Coll., Wellesley, Mass. *6 Norfolk Terr.*
YOWELL, E. I., C.E., Ph.D. (Cincinnati). Astronomer, Univ. of Cincinnati, Cincinnati, Ohio. *3127 Griest Ave.*

ZEHRING, Asso. Prof. W. A., A.M. (Otterbein). Purdue Univ., W. LaFayette, Ind. *303 Russell St.*
ZELDIN, S. D., Ph.D. Instr., Mass. Inst. of Tech., Cambridge, Mass.
ZIMMERMAN, Prof. JOHN, B.S. (Princeton). Univ. of Dubuque, Dubuque, Ia. *1089 Glen Oak Ave.*
ZINSZER, Prof. H. A., Ph.D. (Indiana). Prof. of Physics and Acting Prof. of Math., Hanover Coll., Hanover, Ind.
ZIWET, Prof. ALEXANDER, C. E. (Karlsruhe). Emeritus, Univ. of Michigan, Ann Arbor, Mich. *532 Packard St.*

INSTITUTIONAL MEMBERS.

Arkansas

University of Arkansas, Fayetteville, Ark.

California

Pomona College, Claremont, Calif.
Occidental College, Los Angeles, Calif.
University of California at Los Angeles, Los Angeles, Calif.
University of Southern California, Los Angeles, Calif.
California Institute of Technology, Pasadena, Calif.
St. Ignatius College, San Francisco, Calif.

Canada

University of Western Ontario, London, Ont., Canada.
University of Montreal, Montreal, Canada.

Colorado

University of Colorado, Boulder, Colo.

Connecticut

Wesleyan University, Middletown, Conn.

Delaware

The University of Delaware, Newark, Del.

District of Columbia

George Washington University, Washington, D. C.

Georgia

University of Georgia, Athens, Ga.
Georgia School of Technology, Atlanta, Ga.
Wesleyan College, Macon, Ga.

Illinois

Illinois Wesleyan University, Bloomington, Ill.
Armour Institute of Technology, Chicago, Ill.
De Paul University, Chicago, Ill.
St. Xavier College, Chicago, Ill.
University of Chicago, Chicago, Ill.
Northwestern University, Evanston, Ill.
Knox College, Galesburg, Ill.
University of Illinois, Urbana, Ill.

Indiana

Indiana University, Bloomington, Ind.
Butler University, Indianapolis, Ind.
Purdue University, LaFayette, Ind.
University of Notre Dame, Notre Dame, Ind.
St. Mary-of-the-Woods College, St. Mary-of-the-Woods, Ind.

Iowa

Iowa State College, Ames, Ia.
St. Ambrose College, Davenport, Ia.
Drake University, Des Moines, Ia.
Columbia College, Dubuque, Ia.
State University of Iowa, Iowa City, Ia.

Kansas

St. Benedict's College, Atchison, Kans.
University of Kansas, Lawrence, Kans.
State Agricultural College, Manhattan, Kans.

Louisiana

Louisiana State Normal College, Natchitoches, La.
Loyola University, New Orleans, La.
Tulane University of Louisiana, New Orleans, La.
Louisiana Polytechnic Institute, Ruston, La.

INSTITUTIONAL MEMBERS (*continued*).

MASSACHUSETTS

AMHERST COLLEGE, Amherst, Mass.
BOSTON UNIVERSITY, Boston, Mass.
NORTHEASTERN UNIVERSITY, Boston, Mass.
HARVARD COLLEGE, Cambridge, Mass.
WELLESLEY COLLEGE, Wellesley, Mass.
WORCESTER POLYTECHNIC INSTITUTE, Worcester, Mass.

MICHIGAN

ALBION COLLEGE, Albion, Mich.
UNIVERSITY OF MICHIGAN, Ann Arbor, Mich.
COLLEGE OF THE CITY OF DETROIT, Detroit, Mich.
MICHIGAN STATE COLLEGE, East Lansing, Mich.
KALAMAZOO COLLEGE, Kalamazoo, Mich.
MICHIGAN STATE NORMAL COLLEGE, Ypsilanti, Mich.

MINNESOTA

CARLETON COLLEGE, Northfield, Minn.
UNIVERSITY OF MINNESOTA, Minneapolis, Minn.
ST. BENEDICT'S COLLEGE, St. Joseph, Minn.
COLLEGE OF ST. TERESA, Winona, Minn.

MISSOURI

CULVER-STOCKTON COLLEGE, Canton, Mo.
UNIVERSITY OF MISSOURI, Columbia, Mo.
NORTHWEST MISSOURI STATE TEACHERS COLLEGE, Maryville, Mo.
MISSOURI SCHOOL OF MINES AND METALLURGY, Rolla, Mo.
THE PRINCIPIA, St. Louis, Mo.
WASHINGTON UNIVERSITY, St. Louis, Mo.

MONTANA

UNIVERSITY OF MONTANA, Missoula, Mont.

NEBRASKA

UNIVERSITY OF NEBRASKA, Lincoln, Nebr.
CREIGHTON UNIVERSITY, Omaha, Nebr.

NEW HAMPSHIRE

UNIVERSITY OF NEW HAMPSHIRE, Durham, N. H.
DARTMOUTH COLLEGE, Hanover, N. H.

NEW JERSEY

THE NEW JERSEY COLLEGE FOR WOMEN, New Brunswick, N. J.
RUTGERS UNIVERSITY, New Brunswick, N. J.
PRINCETON UNIVERSITY, Princeton, N. J.

NEW MEXICO

NEW MEXICO NORMAL UNIVERSITY, Les Vegas, N. M.

NEW YORK

NEW YORK STATE COLLEGE FOR TEACHERS, Albany, N. Y.
WELLS COLLEGE, Aurora, N. Y.
ADELPHI COLLEGE, Brooklyn, N. Y.
UNIVERSITY OF BUFFALO, Buffalo, N. Y.
HAMILTON COLLEGE, Clinton, N. Y.
ELMIRA COLLEGE, Elmira, N. Y.
COLLEGE OF MOUNT ST. VINCENT, Mount St. Vincent-on-Hudson, N. Y.
COLLEGE OF THE CITY OF NEW YORK, New York, N. Y.
COLLEGE OF THE SACRED HEART, New York, N. Y.
COLUMBIA UNIVERSITY, New York, N. Y.
THE COOPER UNION, New York, N. Y.
HUNTER COLLEGE OF THE CITY OF NEW YORK, New York, N. Y.
NEW YORK UNIVERSITY, New York, N. Y.
UNIVERSITY OF ROCHESTER, Rochester, N. Y.
UNION COLLEGE, Schenectady, N. Y.
UNITED STATES MILITARY ACADEMY, West Point, N. Y.

INSTITUTIONAL MEMBERS (*continued*)

Ohio

University of Akron, Akron, Ohio.
University of Cincinnati, Cincinnati, Ohio.
Case School of Applied Science, Cleveland, Ohio.
Western Reserve University, Cleveland, Ohio.
Ohio Wesleyan University, Delaware, Ohio.
Kenyon College, Gambier, Ohio.
Denison University, Granville, Ohio.
Hiram College, Hiram, Ohio.
Oberlin College, Oberlin, Ohio.
Otterbein College, Westerville, Ohio.
The College of Wooster, Wooster, Ohio.
Antioch College, Yellow Springs, Ohio.

Oklahoma

University of Oklahoma, Norman, Okla.

Oregon

University of Oregon, Eugene, Ore.

Pennsylvania

Bryn Mawr College, Bryn Mawr, Pa.
Lafayette College, Easton, Pa.
Seton Hill College, Greensburg, Pa.
Allegheny College, Meadville, Pa.
Albright College, Myerstown, Pa.
The Drexel Institute, Philadelphia, Pa.
Swarthmore College, Swarthmore, Pa.
Washington and Jefferson College, Washington, Pa.

Rhode Island

Brown University, Providence, R. I.
Rhode Island College of Education, Providence, R. I.

South Carolina

University of South Carolina, Columbia, S. C.

South Dakota

Northern Normal and Industrial School, Aberdeen, S. D.

Tennessee

Vanderbilt University, Nashville, Tenn.

Texas

West Texas State Teachers College, Canyon, Tex.
Southern Methodist University, Dallas, Tex.
Rice Institute, Houston, Tex.
Baylor University, Waco, Tex.

Utah

University of Utah, Salt Lake City, Utah.

Vermont

Middlebury College, Middlebury, Vt.

Virginia

Virginia Polytechnic Institute, Blacksburg, Va.
University of Virginia, University, Va.
College of William and Mary, Williamsburg, Va.

Washington

University of Washington, Seattle, Wash.

West Virginia

Bethany College, Bethany, W. Va.

Wyoming

University of Wyoming, Laramie, Wyo.

GEOGRAPHICAL DISTRIBUTION OF INDIVIDUAL MEMBERS

UNITED STATES AND CANADA

ALABAMA. (16)

ATHENS. J. Stone.
AUBURN. Crenshaw, Killebrew.
BIRMINGHAM. Balof, Eagles, Hess, W. A. Moore, Sewell.
FLORENCE. Culmer.
MONTEVALLO. Mears.
ROANOKE, Mickle.
TUSCALOOSA. Nixon.
UNION SPRINGS. Lee.
UNIVERSITY. Dahlene, F. A. Lewis, Ott.

ARIZONA. (6)

FLAGSTAFF. Lampland.
TUCSON. Atkinson, Cresse, Graesser, H. B. Leonard, Purcell.

ARKANSAS. (9)

ARKADELPHIA. Garretson.
BATESVILLE. S. W. Williams.
FAYETTEVILLE. Droke, A. M. Harding, J. C. Hughes, G. D. Nichols.
LITTLE ROCK. Bigbee, d'Unger.
SEARCY. B. R. Allen.

CALIFORNIA. (94)

ATASCADERO. H. E. Anderson, Schmiedel.
BAKERSFIELD. Powelson, Sagen.
BERKELEY. Bernstein, Cajori, M. W. Haskell, Irwin, Lehmer, S. H. Levy, McCarty, McDonald, Noble, T. M. Putnam, P. Sperry, A. R. Williams, Wong.
CHICO. Iloff.
CLAREMONT. G. E. Berry, W. P. Russell.
DAVIS. Titus.
FRESNO. F. R. Morris.
FULLERTON. L. E. Reynolds.
GLENDALE. Bolton, Prater.
HOLLYWOOD. Entz.
LA JOLLA. G. F. McEwen.
LONG BEACH. Lodwick.
LOS ANGELES. E. E. Allen, Ames, C. Bell, J. R. Campbell, Collier, Daus, Gaver, Glazier, Hedrick, G. H. Hunt, James, McClellan, W. E. Mason, Reinert, Sherwood, Showman, D. V. Steed, Touton, Willett, Worthington.
LOS GATOS. Stager.
MARE ISLAND. See.
MILLS COLLEGE. Alderton.
MODESTO. Pobanz.
MT. HAMILTON. Jeffers.
OAKLAND. G. E. Allen, F. M. Gibson.
ORLAND. Fenner.
PALO ALTO. Biggerstaff, Hoskins, R. L. Jackson.
PASADENA. Basoco, Bateman, E. T. Bell, Birchby, Hicks, Meek, C. B. Millikan, W. A. Newlin, Podolsky, Stoner, Sturdivant, Van Buskirk, M. Ward, Wear, C. Wolfe.
PIEDMONT. Stafford.
POINT LOMA. Dick.
REDLANDS. O. W. Albert, Keith.
RIVERSIDE. Linhart.
SACRAMENTO. Wallace.
SALINAS. Walter.
SAN DIEGO. Klauber, Livingston.
SAN FRANCISCO. Libby.
SANTA ANA. Whiting.
SANTA MARIA. Funk.
STANFORD UNIVERSITY. Blichfeldt, W. L. Crum, Eells, Green, Moreno.
STOCKTON. Corbin.
VENICE. W. A. Austin.
WHITTIER. Skarstedt.

CANADA. (37)

CALGARY. F. G. Young.
EDMONTON SOUTH. J. W. Campbell, Sheldon.
HALIFAX. F. H. Murray.
KINGSTON. Gummer, K. P. Johnston, Matheson, N. Miller, Twiss.
LENOXVILLE. A. V. Richardson.
LONDON. Kingston.
MONTREAL. Beaupré, D. A. Murray, Pelletier.
OTTAWA. Dube, Henroteau.
POINT GREY. E. E. Jordan.
PORT COLBORNE. Kinnear.
QUEBEC. Savary, Tremblay.
RED DEER. A. A. Grant.
REGINA. McEwen.
SASKATOON. Ling, Pyke.
TORONTO. Beatty, De Lury, Fields, Findlay, Pounder, Rosebrugh, Skirrow.
VANCOUVER. D. Buchanan, G. E. Robinson.
VICTORIA. Gage.
WINNIPEG. T. H. Milne, Warren, N. R. Wilson.

COLORADO. (29)

BOULDER. De Long, Hutchinson, Jekel, Kempner, Kendall, Lester, Light, McGinley, W. K. Nelson.
CANON CITY. McNatt.
COLORADO SPRINGS. Albright, Lovitt, Sisam.
DENVER. F. L. Carmichael, E. A. Cummings, Fenton, P. Fitch, Gorrell, Odell, Sabin, Wray.
FORT COLLINS. A. G. Clark, S. L. Macdonald.
GOLDEN. J. R. Everett, Martinson, Risley.
GREELEY. Finley.

GUNNISON. Ankebrant.
WHEATRIDGE. A. J. Lewis.

CONNECTICUT. (32)

BRIDGEPORT. M. G. Haseman.
GREENWICH. R. H. Hall.
HARTFORD. A. H. Andrews, Burkett, Dadourian, C. H. Davis, Elston, Flynn, E. B. Morris, Spaulding, Welling.
MIDDLETOWN. H. E. Arnold, B. H. Camp, Howland.
MILFORD. Rosenbaum.
NEW HAVEN. Barney, E. W. Brown, H. T. Burgess, Fisher, Gehman, Kovarik, Longley, Miles, L. T. Moore, H. Thompson, Tracey, Uhler, Whittemore, W. A. Wilson.
NEW LONDON. Leib.
SALES FERRY. Dimick.
WEST HAVEN. Shook.

DELAWARE. (7)

LAUREL. C. A. Short.
NEWARK. Boeder, Harter, Lerch, C. J. Rees.
WILMINGTON. Gardner, R. W. Jones.

DISTRICT OF COLUMBIA. (46)

BROOKLAND. Landry.
WASHINGTON. O. S. Adams, Arnaud, Ashmun, Avers, L. A. Bauer, W. J. Berry, Bixby, Breit, Burley, Cromwell, Dantzig, Darling, Duerksen, H. English, Erwin, Federico, J. F. Fox, Garner, Goldberg, W. M. Hamilton, Hodgkins, Kaufman, Keulegan, W. D. Lambert, Lemon, Mangold, McKnight, E. C. Phillips, Ramler, J. N. Rice, Richmond, Roeser, Ross, Shenton, Stevens, E. R. Thompson, Tomelden, Van Orstrand, Wallis, C. B. Watts, Wernicke, J. E. Willis, E. W. Wilson, Woodard, Woolard.

FLORIDA. (12)

APOPKA. Larson.
CENTER HILL. Frary.
GAINESVILLE. Chandler, Dostal, Higgins, Kokomoor, Kusner, C. A. Messick, T. M. Simpson.
MIAMI. Searle.
TALLAHASSEE. Elmer R. Smith.
WINTER PARK. Weinberg.

GEORGIA. (27)

ATHENS. Barrow, W. S. Beckwith, Cumming, P. R. Hill, R. P. Stephens.
ATLANTA. Benander, Fulmer, G. K. Howe, Morton, Patton, M. R. Richardson, Skiles, D. M. Smith.
DECATUR. F. Field, L. J. Gaylord, Parker, H. A. Robinson.
DEMOREST. J. C. Rogers.
EMORY UNIVERSITY. Messick, Rumble.
GAINESVILLE. Allison.
LA GRANGE. E. A. Bailey.
MACON. Day, Holder, Plymale, F. Wood.
ROME. Hightower.
WAYCROSS. A. R. Walton.

HAWAII. (1)

HONOLULU. Donaghho.

IDAHO. (4)

CALDWELL. J. M. Rankin, I. White.
MOSCOW. E. Taylor.
POCATELLO. Galloway.

ILLINOIS. (130)

ALTON. Denman.
ATWOOD. Hobart (?).
BELLWOOD. M. Hildebrandt.
BLOOMINGTON. M. Hunt.
BLUFFS. C. C. Carter.
CARLINVILLE. Renner.
CARTHAGE. Van Velzer.
CHAMPAIGN. H. W. Bailey, G. E. Moore.
CHARLESTON. Modesitt, E. H. Taylor.
CHICAGO. C. W. Andrews, Barnard, Bartky, Beenken, Bibb, Bliss, Burroughs, Cobb, Cope, Dickson, H. S. Everett, Feltges, Z. Ferguson, Georges, Ghormley, Granville, L. M. Graves, Jarrett, Jenkins, B. W. Jones, Kinney, Krathwohl, Kurzin, Lane, Lange, Laves, Logsdon, Lunn, MacMillan, E. H. Moore, F. R. Moulton, Muehlman, G. W. Myers, Nicolet, Nyberg, C. I. Palmer, Pettersen, Plapp, C. A. Rupp, Scharf, Schottenfels, Schweitzer, Shaw, Slaught, E. L. Thompson, Werkman, Whaley, J. W. A. Young.
CICERO. B. A. Rogers.
DECATUR. Kiefer.
DE KALB. G. H. Freeman, Parson.
EUREKA. Newson.
EVANSTON. V. M. Andrews, O. E. Brown, D. F. Campbell, Curtiss, H. L. Garabedian, Holgate, E. J. Moulton, Newell, Olson, Simmons, Westcott, F. E. Wood.
FREEPORT. Eichelberger, M. C. Martin, Mensenkamp.
GALESBURG. Heren, Sellew, M. E. Taylor.
GODFREY. Freas.
GURNEE. N. M. Johnston.
JACKSONVILLE. M. Anderson, E. B. Miller.
LAKE FOREST. Curtis.
LA SALLE. L. H. Brown, Carus.
LEBANON. Stowell.
LINCOLN. Denny.
LISLE. Fleisig.
MACOMB. Ginnings.
MONMOUTH. Winbigler.
MONTICELLO. Wolever.
MOUNT MORRIS. L. S. Shively.
NAPERVILLE. Coultrap.
NORMAL. Atkin, Flagg, C. N. Mills.
OAK PARK. Escott, Richards.
ORLEANS. J. T. Holmes.
PEORIA. Comstock, Gaut.
RIVER FOREST. Dobbin.
ROCKFORD. McGavock, B. I. Miller.
ROCK ISLAND. Cederberg, Sheets.
ROCKTON. Bigelow.
TAYLORVILLE. Dappert.

URBANA. B. M. Armstrong, Bender, T. L. Bennett, R. D. Carmichael, Crathorne, Emch, F. Harshbarger, Hazlett, F. E. Johnston, H. Levy, Libman, Lytle, G. A. Miller, Steimley, Townsend, A. L. Young.
WHEATON. Garlough, Hillard.

INDIANA. (52)

BLOOMINGTON. H. T. Davis, Davisson, Hanna, Hennel, Rothrock, K. P. Williams, H. E. Wolfe.
COLUMBIA CITY. Knisely.
CRAWFORDSVILLE. Carscallen, Cragwall.
CULVER. R. E. Peterson.
DANVILLE. Cole.
EARLHAM. F. Long.
ELWOOD. R. N. Haskell.
GOSHEN. Lehman.
GREENCASTLE. Arnold, Greenleaf.
HANOVER. Zinszer.
INDIANAPOLIS. Aley, Banes, Diederich, E. N. Johnson, J. M. Lutz, Mathias.
LAFAYETTE. Marshall, Mason.
MT. VERNON. Blackburn.
MUNCIE. Edwards.
NORTH MANCHESTER. Dotterer.
NOTRE DAME. Caparo, Hull, McCue, Maurus.
OAKLAND CITY. W. Jordan.
RICHMOND. E. D. Grant, H. N. Wright.
TERRE HAUTE. Kennedy, Sousley.
WESTFIELD. E. H. Johnson.
WEST LAFAYETTE. Bolks, Copp, Doan, Edington, G. H. Graves, Hadley, Hazard, Hodge, Robbins, Stauffer, R. B. Stone, Stout, Zehring.

IOWA. (49)

AMES. Brandner, Colpitts, Daniells, Fleming, Gouwens, J. V. McKelvey, M. M. McKelvey, M. M. Roberts, E. R. Smith, Snedecor, J. S. Turner.
CEDAR FALLS. Condit, Kearney, Wester.
CEDAR RAPIDS. L. M. Coffin, Yothers.
DECORAH. Strom.
DES MOINES. Blom, Corey, Neff, Wifvat.
DUBUQUE. Theobald, Zimmerman.
FAYETTE. Deming.
GRINNELL. McClenon, Rusk.
HOPKINTON. Earhart.
INDIANOLA. C. W. Emmons.
IOWA CITY. F. E. Baker, R. P. Baker, Chittenden, Conkwright, Y. V. Maizlish, Reilly, Rietz, L. E. Ward, R. Woods, Wylie.
LE MARS. Blue.
MOUNT VERNON. McGaw, Moots.
OSKALOOSA. S. M. Hadley.
SILVER CITY. Paul.
SIOUX CITY. Graber, Gwinn, Van Horne.
STORM LAKE. Reusser.
WEBSTER CITY. H. L. Rice.
WELLMAN. Kreth.

KANSAS. (49)

BALDWIN CITY. P. L. Evans, Garrett.
COFFEYVILLE. Steininger.
EMPORIA. Phillips, Rumney.
HAYS. Colyer.
HESSTON. Driver.
HIGHLAND. T. B. Henry.
INDEPENDENCE. A. M. Austin.
IOLA. R. H. Carpenter.
KANSAS CITY. Dougherty, Helwig.
LAWRENCE. Ashton, Babcock, Black, H. E. Jordan, Marm, U. G. Mitchell, G. W. Smith, Stouffer, J. J. Wheeler.
MANHATTAN. W. H. Andrews, Holroyd, Hyde, Janes, R. E. Leonard, C. F. Lewis, Lyons, Mossman, T. I. Porter, Remick, Stratton, A. E. White.
NEWTON. Richert.
OTTAWA. J. L. Bennett, Loewen.
PARSONS. Farner.
PITTSBURG. R. W. Hart, W. H. Hill, Shirk.
STERLING. T. Bell.
TOPEKA. Croom, Harshbarger, McLatchey.
WICHITA. Hoare, Longenecker, Mendenhall, Reagan.
WINFIELD. H. S. Myers.

KENTUCKY. (18)

BEREA. Hutcherson, T. A. Martin.
DANVILLE. Fehn,
GEORGETOWN. Clayton, C. H. Richardson.
LEXINGTON. Boyd, M. E. Clarke, J. M. Davis, Downing, Latimer, Le Stourgeon, Maney, E. L. Rees.
LOUISVILLE. Bullitt, Stevenson, Vance.
PARIS. F. A. Scott.
WINCHESTER. Dearman.

LOUISIANA. (32)

ALEXANDRIA. Kilpatrick, Longmire, Touchstone.
BATON ROUGE. Daspit, Guilbeau, Nichols, O'Quinn, S. T. Sanders, H. L. Smith, Webber, Welch.
CLINTON. Petty.
NATCHITOCHES. L. Blair, Killen, Maddox.
NEW ORLEANS. H. E. Buchanan, Dinwiddie, Duren, Frankenbush, A. M. Howe, Many, McShane, Menuet, J. H. Meyer, Rothfuss, Spencer, H. L. Titsworth.
PINEVILLE. C. D. Smith.
SHREVEPORT. Hardin, I. Maizlish, Mauldin.
SLIDELL. Gaulden.

MAINE. (11)

BRUNSWICK. Hammond, Moody.
LEWISTON. Ramsdell, Wilkins.
ORONO. Bryan, Hadlock, J. N. Hart, M. F. Jordan, W. P. Morse.
WATERVILLE. Ashcraft, Trefethen.

MARYLAND. (51)

ABERDEEN. Dederick.
ANNAPOLIS. Bingley, Bramble, Bullard, Capron, Clements, Dillingham, Eppes, Garrison, Hemke, Kells, Leiper, Mayer, Rawlins, Robert, Root, J. B. Scarborough, J. Tyler.

BALTIMORE,. Aitchison, Bacon, Bassler, Blumenthal, Cary, Coble, Cohen, C. W. R. Crum, Gwinner, Hallam, Harry, Hulburt, F. P. Lewis, A. K. Mitchell, W. K. Morrill, T. W. Moore, F. Morley, Murnaghan, Musselman, Peters, L. J. Reed, Richeson, Thomsen, Torrey.
CHELTENHAM. Hartnell.
CHESTERTOWN. J. S. W. Jones.
COLLEGE PARK. Spann, T. H. Taliaferro.
EMMITSBURG. Burke.
FREDERICK. K. S. Arnold, L. O. Brown.
ROLAND PARK. Morrow.
WESTMINSTER. B. I. Hart.

MASSACHUSETTS. (103)

AMHERST. Esty, F. C. Moore, Olds, C. S. Porter.
ANDOVER. F. E. Newton.
BELMONT. Douglass, Downey, Rutledge.
BOSTON. Brigham, Bruce, H. D. Gaylord, Gould, Holbrook, Laurentine, Leary, Lucey, Marsh, Mode, Osborne, M. M. Smith, Spear, Edwin B. Wilson.
BROOKLINE. Andrew, A. L. Miller, Washburn.
CAMBRIDGE. F. H. Bailey, Beatley, Birkhoff, Bradley, T. H. Brown, Coolidge, A. H. Fox, Franklin, Graustein, Hollis, E. V. Huntington, Kellogg, Kennelly, Lubin, C. L. E. Moore, M. Morse, Osgood, Passano, F. W. Perkins, J. K. Peterson, H. B. Phillips, Poritsky, Price, L. H. Rice, H. W. Tyler, Walsh, D. E. Whitford, F. S. Woods, F. W. Wright, Zeldin.
DANVERS. Majella.
DORCHESTER. J. W. Davis, Quigley.
EAST NORTHFIELD. Daboll.
EVERETT. Bryant.
FRAMINGHAM. Bickford.
GLOUCESTER. MacNutt.
HOLYOKE. Moriarty.
LAWRENCE. Lord.
LYNN. G. W. Evans.
MEDFORD. Cheney.
NATICK. Nutt, R. Willis.
NORTHAMPTON. Anderton, S. R. Benedict, Eversull, Munroe, Rambo, R. G. Wood.
PITTSFIELD. Washburne.
SHEFFIELD. Eipper.
SOUTH HADLEY. Doak, E. N. Martin, S. E. Smith.
SPRINGFIELD. S. E. Cook, Hickox.
TUFTS COLLEGE. Mergendahl, Ransom.
WELLESLEY. Copeland, Doughty, Merrill, C. E. Smith, Stark, Vivian, M. M. Young.
WESTON. O'Donnell.
WILLIAMSTOWN. Agard, Hardy, V. H. Wells.
WOLLASTON. Dennison. Gardner.
WORCESTER. E. C. Brown, Gay, Melville, R. K. Morley, H. Rice, A. H. Wheeler, F. B. Williams.

MICHIGAN. (66)

ALBION. Evers, F. E. Field, Sleight.
ALMA. Clack.
ANN ARBOR. Anning, Bradshaw, Coe, Craig, Denton, P. Field, S. E. Field, W. B. Ford, J. W. Glover, T. H. Hildebrandt, Hopkins, H. K. Hughes, Karpinski, Kazarinoff, Love, Markley, Newsom, Nyswander, O. J. Peterson, Poor, Raiford, Rainich, Rood, Rouse, Running, Schorling, Schreiber, Shohat, R. L. Wilder, Ziwet.
BAY CITY. Shellenberger.
DETROIT. Baldwin, Borgman, Chalmers, Darnell, Frumveller, Mullen, A. L. Nelson, Paula, Thome.
EAST LANSING. Crowe, L. C. Emmons, V. G. Grove, H. L. Olson, Plant, Powell, Speeker.
HILLSDALE. Herron, Penrod.
HOLLAND. Lampen.
KALAMAZOO. H. Blair, J. P. Everett, T. O. Walton.
MARQUETTE. Spooner.
MOUNT PLEASANT. Pearce.
ROYAL OAK. Schoonover.
SPRING ARBOR. M. G. Smith.
STURGIS. Steirnagle.
YPSILANTI. Barnhill, Lindquist, Lyman, Matteson.

MINNESOTA. (41)

COLLEGEVILLE. Winklemann.
DULUTH. Morin.
MANKATO. Chapman, A. V. Robbins.
MINNEAPOLIS. Beal, Brink, Brooke, Bussey, S. E. Carlson, Dalaker, Earl, Gibbens, W. L. Hart, Hartig, D. Jackson, Jensen, Kirchner, Ness, Priester, Risselman, Scammon, Shuman, Shumway, Thorp, Underhill, E. D. Wells, Wilcox.
MOOREHEAD. K. Leonard.
NORTHFIELD. Gingrich, Nordgaard, Solum, M. B. White.
ST. PAUL. Alice Irene, Kingery, Kyes, Moench, W. D. Morgan, Reuterdahl, F. J. Taylor.
ST. PETER. Rundstrom.
VIRGINIA. C. L. Hancock.
WINONA. Bogard.

MISSISSIPPI. (14)

A. AND M. COLLEGE. H. Fox, B. M. Walker.
BLUE MOUNTAIN. Hutchins.
CLEVELAND. Dale.
GRENADA. M. E. Harris.
HATTIESBURG. P. K. Smith.
JACKSON. Babbitt, B. E. Mitchell, Smylie.
MERIDIAN. I. K. Smith.
STARKVILLE. A Edmondson.
UNIVERSITY. Hume, Wunder.
VICKSBURG. M. L. Newton.

MISSOURI. (46)

AXTELL. Epperson.
CANTON. B. Ingold.
CAPE GIRARDEAU. B. F. Johnson, Knepper.
CARTHAGE. Murto, Edna Robinson.
CLAYTON. Haertter.
COLUMBIA. E. F. Allen, Callaway, Ingold, Jaeger, Wahlin, W. D. A. Westfall, Wyant.
FULTON. Sweazey, M. A. Wood.

JOPLIN. E. M. Weaver.
KANSAS CITY. Cutting, Luby, F. L. Marshall, Pierson.
KIRKSVILLE. Cosby, Jamison.
LIBERTY. Fleet.
MAYSVILLE. Saunders.
PARKVILLE. R. A. Wells.
ROLLA. Hinsch.
SPRINGFIELD. Finkel, E. M. Gibson.
ST. CHARLES. Karr.
ST. JOSEPH. Burney.
ST. LOUIS. Ammerman, Dunkel, Gerst, A. H. Huntington, King, Nauer, Osborn, Rider, Roever, Shannon, E. Stephens, Tracy, J. M. Young.
TARKIO. Jenison.
WARRENSBURG. J. H. Scarborough.
WEBSTER GROVES. M. B. Clarke.

MONTANA. (7)

BUTTE. Bowersox.
GLASGOW. Calderwood.
HELENA. Canning, Wible.
MISSOULA. Carey, Lennes, A. S. Merrill.

NEBRASKA. (22)

BURWELL. Opp.
CERESCO. L. C. Walker.
CRETE. J. M. Hawkes.
GRAND ISLAND. H. Anderson.
HASTINGS. McDill.
KEARNEY. Hanthorn.
LINCOLN. Brenke, C. C. Camp, Candy, Congdon, Gaba, Gossard, Howie, M. F. Jackson, T. A. Pierce, Runge.
OMAHA. Bettinger, H. A. Campbell, Frankish, Wilmer.
PERU. A. L. Hill.
YORK. Feemster.

NEVADA. (1)

RENO. C. Haseman.

NEW HAMPSHIRE. (15)

CONCORD. G. M. Conwell.
DURHAM. G. N. Bauer, Slobin, Wilbur.
EXETER. Barber, Sweet.
HANOVER. Beetle, Bill, B. H. Brown, Forsyth, Mathewson, Morgan, Silverman, C. E. Wilder, J. W. Young.

NEW JERSEY. (42)

BELLEPLAIN. Durell.
BRIDGETON. J. M. Rice.
CONVENT STATION. Maskell.
CRANFORD. Knedler.
EAST ORANGE. Koch, Mallory, F. H. Robinson, Stanwick.
LAWRENCEVILLE. Kimball, Mikesh.
LEONIA. Gafafer, M. S. Taylor.
MOORESTOWN. R. R. Wood.
MORRIS PLAINS. A. E. Johnson.
NEWARK. Conkling.
NEW BRUNSWICK. F. C. Hall, Huber, Meder, R. Morris, C. A. Nelson, R. Thompson, A. A. Titsworth, C. R. Wilson.
PATERSON. Caster.
PRINCETON. E. P. Adams, Alexander, Cramlet, Eisenhart, Fine, D. A. Flanders, W. Gillespie, Hille, Knebelman, Lefschetz, P. M. Morse, Veblen, Wedderburn, A. P. Wheeler, Willson.
RUTHERFORD. McMackin.
SOMERVILLE. Moyle.
TRENTON. Colliton.

NEW MEXICO. (5)

ALBUQUERQUE. Barnhart.
EAST LAS VEGAS. Rodgers.
MONTEZUMA. Barrick.
SILVER CITY. Hunter.
SOCORRO. Reece.

NEW YORK. (190)

ALBANY. Birchenough, Do Bell.
ALFRED. Seidlin, W. A. Titsworth.
ANNANDALE-ON-HUDSON. Phalen.
AURORA. Hollcroft.
BALDWIN. C. C. Grove.
BRONX. Tanzola, Weisner.
BROOKLYN. Angelica, Bergstresser, W. J. Berry, Bowden, Emery, Fleisher, Kreines, Langman, Lehmann, Locke, Schuyler, Shorr, Thecla, J. E. Thompson.
BUFFALO. Archer, Cusick, Harrington, Pound, Sherk, M. Watt.
CLINTON. H. S. Brown, Carruth, Ferry, A. L. Fitch, Patterson.
CORONA. Hanson.
ELMIRA. Suffa, F. M. Wright.
FLUSHING. P. H. Graham, Oglesby.
FRIENDSHIP. O'Kean.
GENEVA. W. H. Durfee, W. P. Durfee, Hubbs.
HAMILTON. Aude, W. B. Campbell, A. W. Smith.
ITHACA. Agnew, Beisel, Boothroyd, Carver, H. A. Davis, Dye, D. C. Gillespie, Hurwitz, Jeffery, Karapetoff, Lowenstein, Ranum, Schoonmaker, V. Snyder, Torrance, F. G. Williams.
JAMAICA. Barrett.
MOUNT VERNON. Breckenridge.
NEWBURGH. C. W. Miller.
NEW YORK. J. Allen, Auerbach, Beiler, Berger, Berkeley, V. Blair, W. M. Bond, Brahdy, Brewster, Burdick, R. W. Burgess, G. A. Campbell, J. R. Clark, R. F. Clark, J. G. Coffin, Cooley, Eckersley, Edmonson, Eisele, O. Everett, Farnum, Fiske, Fite, Foster, Frankel, Fry, Gill, Harper, Hawkes, R. Henderson, Himwich, Hirsch, Hodgdon, Hoyt, Jablonower, Joffe, M. I. Johnson, R. A. Johnson, P. C. Jones, Kasner, Kunte, Larkin, Linehan, Littauer, H. L. Lutz, MacGregor, Maiden, Mirick, Molina, Mullins, Paaswell, Packer, Payne, Pedersen, Penn, Pooler, Post, Pride, Quilty, Reddick, Reeve, Ritt, Rosanoff, Sanford, Saurel, Schmall, Schub, Shewhart, Siceloff, Simons, D. E. Smith, R. F. Smith, R. R. Smith, Spies, Thorne, Tilley, A. B. Turner, Upton, E. Walker, H. M. Walker, Wechsler, Werner, E. E. Whitford, Woodyard.

NISKAYUNA. Male.
OLEAN. Lowry.
PARISH. Church.
POTSDAM. Isham.
POUGHKEEPSIE. Cowley, L. D. Cummings, J. P. Smith, M. E. Wells.
RICHMOND HILL. F. H. Miller.
ROCHESTER. Betz, Gale, Garver, H. Harding, T. R. Long, Silberstein, Watkeys, Welton.
SCARSDALE. MacNeish.
SCHENECTADY. Caruthers, Hussey, D. S. Morse, Newkirk, A. D. Snyder, Stokes, Ulrich, Vedder.
SYRACUSE. A. D. Campbell, Carroll, Decker, Lindsey, Secy. Pi Mu Epsilon Frat., Roe, M. Sperry, W. E. Taylor, Warne.
TARRYTOWN. R. G. Putnam.
TROY. Crockett.
WEST POINT. C. P. Echols.
YONKERS. Hubert, John, Yanosik.

NORTH CAROLINA. (24)

CHAPEL HILL. Browne, Cain, A. Henderson, M. A. Hill, Hobbs, Lasley, Mackie.
CHARLOTTE. O. M. Jones, Woodson.
DAVIDSON. J. L. Douglas.
DURHAM. Elliott, W. W. Rankin, Robison.
ELIZABETH CITY. C. F. Holmes.
ELON COLLEGE. Amick.
GREENSBORO. Barton, Pegram, Strong, Watkins.
GREENVILLE. M. D. Graham.
JAMESTOWN. Ragsdale.
MARS HILL. W. F. Robinson.
MURFREESBORO. Caldwell.
RED SPRINGS. Shuler.

NORTH DAKOTA (4).

FARGO. Householder.
GRAND FORKS. Staley.
UNIVERSITY. Hitchcock.
VALLEY CITY. J. B. Meyer.

OHIO. (104)

ADA. Fairchild, Whitted.
AKRON. Reger.
ATHENS. Borger, F. W. Reed.
BEREA. Baur, Dustheimer.
BOWLING GREEN. Overman.
BLUFFTON. Hirschler.
CINCINNATI. I. A. Barnett, Brand, C. A. Garabedian, H. Hancock, Justice, Kersten, Kindle, C. N. Moore, Mullings, C. E. Rhodes, Salkover, E. S. Smith, E. V. Watts, Wilczewski, Yowell.
CLEVELAND. Boyce, Burington, Focke, W. W. Johnson, Justin, J. E. Merrill, M. Morris, Nassau, Palmié, Simon, C. F. Thomas, Trofimov.
COLUMBUS. C. L. Arnold, Atwater, Bareis, Beatty, H. Blumberg, Bumer, V. B. Caris, Harmount, Horn, R. S. Kimball, Kuhn, MacDuffee, Manson, Michal, C. C. Morris, Preston, Rasor, Rickard, B. Saunders, Singer, Wildermuth.
DAYTON. Hartwick.
DEFIANCE. A. G. Caris.
DELAWARE. Crane, R. L. Newlin, Rowland.
GAMBIER. R. B. Allen, Redditt.
GRANVILLE. E. C. Rupp, F. B. Wiley.
HILLIARD. J. H. Weaver.
HIRAM. E. H. Clarke, Jerome.
KENT. N. B. Freeman, Manchester.
MACEDONIA. Burwell.
MARIETTA. Coar, Rea.
MOUNT ST. JOSEPH. Molloy.
NEW CONCORD. C. E. White.
OBERLIN. Battig, R. L. Brown, Cairns, Carr, M. M. Johnson, Sinclair, Yeaton.
OXFORD. J. P. Albert, W. E. Anderson, Baudin, Erickson, Pollard, Spenceley, Tappan.
PAINESVILLE. A. D. Lewis.
ROSS. Haldeman.
SPRINGFIELD. Tripp.
TIFFIN. J. Pierce.
TOLEDO. Brandeberry, Mercedes, Winslow.
WESTERVILLE. B. C. Glover.
WILBERFORCE. Tinner.
WILMINGTON. Spinks.
WOOSTER. L. C. Knight, Williamson, Yanney.
ZANESVILLE. Riesbeck.

OKLAHOMA. (25)

ADA. K. K. Knight.
ALVA. H. L. Hall.
CHICKASHA. J. J. Miller.
MIAMI. C. S. Whitney.
NORMAN. Brixey, Court, Hampton, Hassler, D. McFarland, E. McFarland, Meacham, Raynor, S. W. Reaves.
OKMULGEE. Thom.
SHAWNEE. W. T. Short.
STILLWATER. R. L. Flanders, Gundersen, H. W. Smith.
TULSA. Byrd, W. E. Howard, H. F. Mitchell, Roman, Rutherford, G. R. West.
WEATHERFORD. McCormick.

OREGON (14)

ALBANY. Butler.
CORVALLIS. Beaty, C. L. Johnson, Van Fleet, G. A. Williams.
EUGENE. D. R. Davis, De Cou, McAlister, W. E. Milne.
McMINNVILLE. M. Ramsey.
PENDLETON, J. A. Carlson.
PORTLAND. Griffin, Merriss, J. M. Short.

PENNSYLVANIA. (105)

ALLENTOWN. Bauman.
ANNVILLE. P. S. Wagner.
ARDMORE. M. H. Macdonald.
BALA. Gummere.
BEAVER FALLS. Cleland.
BETHLEHEM. L. S. Barnes, Fort, Lamson, McDonough, Paradiso, Rau, J. B. Reynolds, Smail, Weida.
CAMP HILL. Foberg.
CARLISLE. Landis.
CHESTER. Holt.
COLLEGEVILLE. Clawson, Veatch.

CYNWYD. Sensenig.
DEVON. J. A. Clarke.
EASTON. Doushkess, W. S. Hall, Hatch, Marquard, W. M. Smith.
GROVE CITY. A. Ramsey.
HARRISBURG. Whited.
HAVERFORD. L. W. Reid, A. H. Wilson.
HUNTINGTON. C. S. Shively.
IRWIN. A. A. Jones.
LANCASTER. R. L. Charles, W. F. Long.
LANSDOWNE. Chambers, Glenn.
LATROBE. Seubert.
LEWISBURG. Gold, Lindemann, MacCreadie.
LINCOLN UNIVERSITY. W. L. Wright.
LOCK HAVEN. High, Smink.
MARCUS HOOK. H. M. Manning.
MEADVILLE. Akers.
MILLERSVILLE. Seiverling.
NEW WILMINGTON. McCain.
PHILADELPHIA. Adkisson, P. A. Caris, Crawley, J. E. Davis, Eshleman, H. B. Evans, F. H. Jackson, Kline, Linton, Lufkin, H. H. Mitchell, Oergel, Partridge, Rittenhouse, W. Roberts, Rosengarten, Rothermel, Safford.
PITTSBURGH. Baird, Bishop, Geckeler, R. P. Johnson, J. H. Mathews, Minister, Moskowitz, Neelley, E. G. Olds, Riggs, Rosenbach, Swartzel, Taber, J. S. Taylor, Waltz, Whitman.
SEWICKLEY. W. I. Miller.
SHIPPENSBURG. Kieffer.
STATE COLLEGE. Bushyager, T. Cohen, Gravatt, L. S. Johnston, E. D. McCarthy, F. W. Owens, H. B. Owens, Shibli, Stetson, J. M. West.
SWARTHMORE. Dresden, Marriott, J. A. Miller, Pitman.
SWISSVALE. Foraker.
WASHINGTON. Atchison, Bert, Rasel, Shaub, R. W. Thomas.
WEST PHILADELPHIA. Latshaw.
WILLIAMSPORT. Hoshauer.

PANAMA. (1)

PANAMA CITY. Linares.

PHILLIPPINE ISLANDS. (5)

LAGUNA. Salvosa.
MANILA. Jimenez, V. Mills, Tan, Tienzo.

PORTO RICO. (3)

MAYAGUEZ. Sanchez.
RIO PEDRAS. Horne.
SAN JUAN. Betancourt del Valle.

RHODE ISLAND. (15)

APPONAUG. Vehse.
NEWPORT. Chase.
PROVIDENCE. C. R. Adams, Archibald, A. A. Bennett, Carlen, Chace, Currier, Gilman, Hickson, H. P. Manning, R. G. D. Richardson, Suesman, Tamarkin, M. W. Watt.

SOUTH CAROLINA. (14)

CHARLESTON. O. J. Bond, R. H. Coleman.
COLUMBIA. Coker, J. B. Coleman, J. B. Jackson, Moorefield, Peele, W. L. Williams.
GREENVILLE. Bowen, Earle, R. B. Wood.
GREENWOOD. Weber.
HARTSVILLE. C. M. Reaves.
ROCK HILL. G. T. Pugh.
SALUDA. Ramage.

SOUTH DAKOTA. (8)

BROOKINGS. Gore, I. L. Miller.
HURON. Titt.
MITCHELL. Knox.
RAPID CITY. Bowles.
SPEARFISH. Hesseltine.
VERMILLION. McKinney.
YANKTON. Faught.

TENNESSEE. (17)

CHATTANOOGA. Perry.
CLEVELAND. Ryno.
HARROGATE. Hatfield.
JACKSON. E. L. Carr, Walden.
JEFFERSON CITY. E. W. White.
KNOXVILLE. Baten, J. D. Bond, Brezler.
MARYVILLE. Knapp.
NASHVILLE. R. V. Blair, S. I. Jones, N. P. Miser, W. L. Miser, Wren.
NORMAL. P. L. Armstrong.
PULASKI. Mize.

TEXAS. (87)

ABILENE. Burnam, Tate.
ALPINE. Gilley.
ARANSAS PASS. W. N. Barnes.
AUSTIN. W. L. Ayres, Batchelder, H. Y. Benedict, Blau, Cooper, Decherd, Dodd, Dorroh, Duncan, Ettlinger, H. L. Holmes, Horton, Jacobs, Lubben, D. E. Mitchell, R. L. Moore, M. B. Porter, W. T. Reid, J. H. Roberts, Simester, Vandiver.
BENAVIDES. Pickett.
BOERNE. Hathaway.
BROWNSVILLE. de la Garza.
BROWNWOOD. Gayden.
BRYAN. W. A. Rees.
CAMERON. Newton.
CANYON. C. A. Murray.
COLLEGE STATION. F. Ayres, Binney, A. A. Blumberg, Halperin, McKee, W. L. Porter.
DALLAS. Dice, Hartsfield, Hosford, E. H. Jones, Mahoney, Reinsch.
DENTON. M. C. Brown.
EDINBURG. A. B. Horton.
FORT WORTH. Estes, Howard, E. R. Tucker.
GALVESTON. P. H. Underwood.
GEORGETOWN. Wapple.
HOUSTON. Bray, Dean, G. C. Evans, Ewing, L. R. Ford, Hickey, E. O. Lovett.
LUBBOCK. Chaney, Lyle, Michie, Moursund, P. K. Rees, L. V. Robinson, Sparks, E. T. Stafford, R. S. Underwood, Wait, Whyburn.
MILFORD. Durham.
NACOGDOCHES. C. E. Ferguson.
PENELOPE. Christian.
RANGER. E. C. Kennedy.
SAN ANTONIO. Klipple, McNelly, J. E. Nelson, Udinski.

SAN MARCOS. Sewell.
SEMINARY HILL. Hendricks.
SHERMAN. May.
STEPHENVILLE. McSweeny, Marrs, Redden.
TYLER. W. A. Nelson.
WACO. Harrell.
WICHITA FALLS. B. T. Adams, Shirley.

UTAH. (5)

SALT LAKE CITY. J. L. Gibson, Horsfall, Pehrson, Stevenson, Unseld.

VERMONT. (8)

BURLINGTON. Butterfield, Donahue, Millington, Swift, E. Thomas.
MIDDLEBURY. Hazeltine, L. R. Perkins, E. E. Wiley.

VIRGINIA. (37)

ABINGDON. V. L. Wright.
ASHLAND. Blincoe, T. McN. Simpson.
BLACKSBURG. Brodie, Hatcher, O'Shaughnessy, J. E. Williams.
BRIDGEWATER. Shull.
CHARLOTTESVILLE. F. A. Wells.
CLIFTON STATION. O. Stone.
EAST RADFORD. Bowers.
EMORY. J. S. Miller.
FARMVILLE. C. B. Taliaferro.
HOLLINS. Dickinson.
LEXINGTON. Funkhouser, Paxton, L. W. Smith, C. W. Watts, Witt.
LYNCHBURG. E. M. Berry, Larew, Mauch, Pattillo.
MONTEREY. Colaw.
RICHMOND. Amig.
RICHMOND COLLEGE. Gaines.
SALEM. Carpenter.
SWEET BRIAR. Calkins, Morenus.
UNIVERSITY. W. H. Echols, Linfield, Luck, Sparrow, Thornton.
UNIVERSITY OF RICHMOND. I. Harris.
WILLIAMSBURG. Rowe, B. Russell.

WASHINGTON. (12)

EVERETT. Robb.
PULLMAN. Issacs.
SEATTLE. Ballantine, Jerbert, Moritz, Mullemeister, Neikirk, Winger.
SPOKANE. Buxton.
TACOMA. Hanawalt.
WALLA WALLA. Bratton.
YAKIMA. A. M. Whitney.

WEST VIRGINIA. (12)

FAIRMONT. M. E. McCarty.
HARPERS FERRY. Drew.
HUNTINGTON. Hackney.
INSTITUTE. Cox.
KENOVA. E. M. Henry.
MORGANTOWN. M. Buchanan, Colwell, Eiesland, R. M. Mathews, C. N. Reynolds, B. M. Turner.
WHEELING. Bagby.

WISCONSIN. (38)

BELOIT. H. H. Conwell, Fischer, Huffer.
LA CROSSE. Adkins, E. H. Snyder.
MADISON. F. E. Allen, Dancer, Dowling, Fowlkes, W. W. Hart, F. Hopkins, Ingraham, Langer, Mickelson, Roth, Skinner, Slichter, Van Vleck, Vass, W. Weaver.
MILTON. A. E. Whitford.
MILWAUKEE. Bear, E. R. Beckwith, Bunyan, Ericson, P. H. Evans, Knight, Levandowski, Parkinson, Pettit, Quarles, C. G. Simpson.
PLATTEVILLE. Warner.
RIPON. Woodmansee.
SUPERIOR. C. W. Smith.
WAUPUN. Hein.
WEST DE PERE. DeCleene.
WISCONSIN RAPIDS. McMillan.

WYOMING. (5)

LARAMIE. Barr, Bellamy, Fitterer, Neubauer, Rechard.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINE. (1)

BUENOS AIRES. Baidaff.

BELGIUM. (2)

BRUSSELS. Sauté.
LIEGE. Van Hee.

CHINA. (7)

CANTON. W. E. MacDonald.
CHANGSHA. Lao, Leavens.
NANSIANG. Loh.
PEKING. Konantz.
SHANGHAI. Ely.
TANGSHAN. Patten.

FRANCE. (4)

NANCY. Gérardin.
PARIS. Borel, Hadamard.
STRASBOURG. Fréchet.

GERMANY. (1)

MUNICH. Wieleitner.

GREAT BRITAIN. (5)

CAMBRIDGE. P. W. Wood.
EDINBURGH. Horsburgh.
HOVE. Chepmell.
OXFORD. Frecheville, Hardy.

INDIA. (2)

ALLAHABAD CITY. Mitra.
MADURA. Lockwood.

ITALY. (6)

BOLOGNA. Bortolotti, Enriques, Pincherle.
PISA. Bianchi.
ROME. Laboccetta.
TURIN. Fubini.

JAPAN. (2)

SENDAI. Hayashi.
TOKYO. Mikami.

NEW ZEALAND. (1)

DUNEDIN. Martyn.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

SOUTH AFRICA. (3)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.
RONDEBOSCH. Muir.

SPAIN. (1)

MADRID. de Toledo.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHATEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (1)

CONSTANTINOPLE. Mourad.

UKRAINE. (1)

KIEFF. Kryloff.

RECAPITULATION OF MEMBERSHIP.

Individual members November 15, 1927	1,849	
Institutional members November 15, 1927	127	
Total membership November 15, 1927		1,976
Total membership November 15, 1925		1,874

CHARTER MEMBERSHIP

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	804	
Net gain in institutional members	75	
Total net gain over charter membership		879
Total net gain since November 15, 1925		102

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

Article I—Name, Purpose and Corporate Seal.

1. This organization shall be known as

The Mathematical Association of America (Incorporated).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

Article III—Board of Trustees and Officers.

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members

for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificate shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings.

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections.

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues.

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws.

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION.

Presidents.

E. R. Hedrick	1916
Florian Cajori	1917
E. V. Huntington	1918
H. E. Slaught	1919
D. E. Smith	1920
G. A. Miller	1921
R. C. Archibald	1922
R. D. Carmichael	1923
H. L. Rietz	1924
J. L. Coolidge	1925
Dunham Jackson	1926
W. B. Ford	1927–1928

Vice-Presidents.

E. V. Huntington	1916
G. A. Miller	1916
D. N. Lehmer	1917, 1918
Oswald Veblen	1917
J. W. Young	1918, 1926
R. G. D. Richardson	1919
H. L. Rietz	1919
Helen A. Merrill	1920
E. J. Wilczynski	1920
R. C. Archibald	1921
R. D. Carmichael	1921, 1922
B. F. Finkel	1922
A. B. Chace	1923
L. P. Eisenhart	1923
J. L. Coolidge	1924
Dunham Jackson	1924, 1925
A. A. Bennett	1925
W. B. Ford	1926
A. J. Kempner	1927
Clara E. Smith	1927

Secretary-Treasurer.

(Appointed by the Trustees after 1918)

W. D. Cairns 1916–

Committee on Publications.

(Appointed by the Trustees.)

H. E. Slaught	1916–
R. D. Carmichael	1916–1918
W. H. Bussey	1916–1918
R. C. Archibald	1919–1921
W. A. Hurwitz	1919–1921
A. A. Bennett	1922
H. P. Manning	1921–1922
W. B. Ford	1923–1925
J. L. Coolidge	1923
A. J. Kempner	1924–
W. H. Bussey	1926–

Elected Members of the Council or Board.

D. N. Lehmer	1916–1918, 1922–1924
R. E. Moritz	1916–1918
K. D. Swartzel	1916
Oswald Veblen	1916, 1920–1922, 1926–
R. C. Archibald	1916–1917, 1923–
Florian Cajori	1916, 1918–1923, 1926–
M. B. Porter	1916–1917
J. W. Young	1916–1917, 1920–1922
B. F. Finkel	1916–1921
E. H. Moore	1916–1921, 1923–
Alexander Ziwet	1916–1918
E. R. Hedrick	1917–1922, 1924–
J. N. Van der Vries	1916–1918
Helen A. Merrill	1917–1919
D. E. Smith	1917–1919, 1921–1926
Elizabeth B. Cowley	1918–1920
G. A. Miller	1918–1920, 1922–1924
E. J. Wilczynski	1918–1919, 1922–1926
L. P. Eisenhart	1919–1922, 1925–
E. V. Huntington	1917, 1919–
E. L. Dodd	1920
R. D. Carmichael	1920, 1924–
A. A. Bennett	1921
H. L. Rietz	1921–1923, 1925–
C. F. Gummer	1921–1925
Dunham Jackson	1923–
Clara E. Smith	1923–1925
A. B. Chace	1924–1925
J. L. Coolidge	1926–
E. T. Bell	1927–

The Carus Mathematical Monographs

The CARUS MONOGRAPHS are already fulfilling their mission as intended by the generous donor, MRS. MARY HEGELER CARUS and her son, DR. EDWARD H. CARUS.

Somewhat more than one-half the members of the Association have taken advantage of the distribution at cost of the first three Monographs already published. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. Each member is entitled to one copy of each Monograph at this special price.

It would be a great tribute to the donor and an honor to the Association if a large majority of the members would subscribe for the complete series.

It is believed that the Association is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

MONOGRAPHS THUS FAR PUBLISHED

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression in 1925, Second Impression in 1927.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L RIETZ. (First Impression, March, 1927.)

Nos. 4, 5 in preparation.

The Rhind Mathematical Papyrus

CHANCELLOR ARNOLD BUFFUM CHACE, of Brown University, is rendering signal honor to the Association by publishing under its auspices his RHIND MATHEMATICAL PAPYRUS. The entire receipts from the sale of this great work will be used to start an endowment fund of the Association to be known as the ARNOLD BUFFUM CHACE FUND

Volume I, over 200 pages, contains the free Translation, Commentary, and Bibliography of Egyptian Mathematics.

Volume II, 140 plates in two colors with Text and Introductions, contains the Photographic Facsimile, Hieroglyphic Transcription, Transliteration, and Literal Translation.

This exposition of the oldest mathematical document in the world will be of great value, not only to students of mathematics but to any one interested in the work of a civilization of nearly 4,000 years ago.

A special price of $15.00 per set, far below cost, has been made for individual and institutional members of the Association on application to the SECRETARY. To all others the price will be $20.00 per set, obtainable only through the OPEN COURT PUBLISHING COMPANY, 339 East Chicago Avenue, Chicago, Illinois.